Lectures on Calabi–Yau and special Lagrangian geometry

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1 Introduction

Calabi–Yau m-folds are compact Ricci-flat Kähler manifolds (M,J,g) of complex dimension m, with trivial canonical bundle K_M. Taken together, the complex structure J, Kähler metric g, and a holomorphic section Ω of K_M make up a rich, fairly rigid geometrical structure with very interesting properties — for instance, Calabi–Yau m-folds occur in smooth, finite-dimensional moduli spaces with known dimension.

Using Algebraic Geometry and Yau's solution of the Calabi Conjecture, one can construct Calabi–Yau m-folds in huge numbers. String Theorists (a species of theoretical physicist) are very interested in Calabi–Yau 3-folds, and have made some extraordinary conjectures about them, in the subject known as Mirror Symmetry.

Special Lagrangian m-folds (SL m-folds) are a distinguished class of real m-dimensional minimal submanifolds N in Calabi–Yau m-folds M. They are fairly rigid and well-behaved, so that compact SL m-folds occur in smooth moduli spaces of dimension $b^1(N)$, for instance.

The existence of compact special Lagrangian m-folds in general Calabi–Yau m-folds is not well understood, but there are reasons to believe they are very abundant. They are important in String Theory, and are expected to play a rôle in the eventual explanation of Mirror Symmetry.

In this paper we aim to do three things. Firstly, we introduce Calabi–Yau manifolds, going via Riemannian holonomy groups and Kähler geometry. Secondly, we introduce special Lagrangian submanifolds, going via calibrated geometry. Finally, we survey an area of current research on the singularities of SL m-folds, and its application to Mirror Symmetry and the SYZ Conjecture. Exercises are given at the end of each section.

I hope the paper will be useful to graduate students in Geometry, String Theorists, and others who wish to learn the subject. Apart from the last two sections, the paper is intended as a straightforward exposition of standard material, to take the reader from a starting point of a good background in Differential Geometry as far as the boundaries of current research in an exciting area.

Our approach to Calabi–Yau m-folds is resolutely Differential Geometric.

That is, we regard them as smooth real manifolds equipped with a geometric structure. The alternative is to use Algebraic Geometry, regard them as complex algebraic varieties, and mostly forget about the metric g. Though very important, this side of the story will barely enter these notes, mainly because SL m-folds are not algebraic objects, and Algebraic Geometry has (so far) little to say about them.

I have chosen to introduce Calabi–Yau m-folds in the context of Riemannian holonomy groups, and special Lagrangian m-folds in the context of calibrated geometry. Though these are perhaps unnecessary diversions, I hope the reader will gain something through understanding the wider horizon into which Calabi–Yau and special Lagrangian geometry fits.

Also, it is my strong conviction that holonomy groups and calibrated geometry belong together as partner subjects, and I want to take the opportunity to teach them together. Though the field of Riemannian holonomy is now mature, the subject of calibrated submanifolds of Riemannian manifolds with special holonomy is really only beginning to be explored.

We begin in §2 with some background from Differential Geometry, and define holonomy groups of connections and of Riemannian metrics. Section 3 explains Berger's classification of holonomy groups of Riemannian manifolds. Section 4 discusses Kähler geometry and Ricci curvature of Kähler manifolds and defines Calabi–Yau manifolds, and §5 sketches the proof of the Calabi Conjecture, and how it is used to construct examples of Calabi–Yau manifolds via Algebraic Geometry.

The second part of the paper begins in §6 with an introduction to calibrated geometry. Section 7 covers general properties of special Lagrangian m-folds in $\mathbb{C}^m$, and §8 construction of examples. Section 9 discusses compact SL m-folds in Calabi–Yau m-folds, and §10 the singularities of SL m-folds. Finally, §11 briefly introduces String Theory and Mirror Symmetry, explains the *SYZ Conjecture*, and summarizes some research on the singularities of special Lagrangian fibrations.

Readers are warned that sections 8, 10 and 11 are unashamedly biased in favour of presenting the author's ideas and opinions; this is not intended as an even-handed survey of the whole field. Further, sections 10 and 11 are fairly speculative, as I am setting out what I think the interesting problems in the field are, and where I would like it to go in the next few years.

The author's paper [23] is a much shortened version of this paper, containing only the special Lagrangian material, roughly §7–§11 below, slightly rewritten. People already well-informed about Calabi–Yau geometry may prefer to read [23].

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2 Introduction to holonomy groups

We begin by giving some background from differential and Riemannian geometry, principally to establish notation, and move on to discuss connections on vector bundles, parallel transport, and the definition of holonomy groups. Some suitable reading for this section is my book [17, §2 & §3], and Kobayashi and Nomizu [29, §I–§IV].

2.1 Tensors and forms

Let M be a smooth n-dimensional manifold, with tangent bundle TM and cotangent bundle T^*M. Then TM and T^*M are *vector bundles* over M. If E is a vector bundle over M, we use the notation $C^\infty(E)$ for the vector space of smooth sections of E. Elements of $C^\infty(TM)$ are called *vector fields*, and elements of $C^\infty(T^*M)$ are called 1-*forms*. By taking tensor products of the vector bundles TM and T^*M we obtain the bundles of *tensors* on M. A *tensor* T on M is a smooth section of a bundle $\bigotimes^k TM \otimes \bigotimes^l T^*M$ for some $k, l \in \mathbb{N}$.

It is convenient to write tensors using the *index notation*. Let U be an open set in M, and $(x^1, \ldots, x^n)$ coordinates on U. Then at each point $x \in U$, $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ are a basis for T_xU. Hence, any vector field v on U may be uniquely written $v = \sum_{a=1}^n v^a \frac{\partial}{\partial x^a}$ for some smooth functions $v^1, \ldots, v^n : U \to \mathbb{R}$. We denote v by v^a, which is understood to mean the collection of n functions $v^1, \ldots, v^n$, so that a runs from 1 to n.

Similarly, at each $x \in U$, $\mathrm{d}x^1, \ldots, \mathrm{d}x^n$ are a basis for T_x^*U. Hence, any 1-form α on U may be uniquely written $\alpha = \sum_{b=1}^n \alpha_b \mathrm{d}x^b$ for some smooth functions $\alpha_1, \ldots, \alpha_n : U \to \mathbb{R}$. We denote α by α_b, where b runs from 1 to n. In the same way, a general tensor T in $C^\infty(\bigotimes^k TM \otimes \bigotimes^l T^*M)$ is written $T^{a_1 \ldots a_k}_{b_1 \ldots b_l}$, where

$$T = \sum_{\substack{1 \leqslant a_i \leqslant n,\ 1 \leqslant i \leqslant k \\ 1 \leqslant b_j \leqslant n,\ 1 \leqslant j \leqslant l}} T^{a_1 \ldots a_k}_{b_1 \ldots b_l} \frac{\partial}{\partial x^{a_1}} \otimes \cdots \frac{\partial}{\partial x^{a_k}} \otimes \mathrm{d}x^{b_1} \otimes \cdots \otimes \mathrm{d}x^{b_l}.$$

The k^{th} exterior power of the cotangent bundle T^*M is written $\Lambda^k T^*M$. Smooth sections of $\Lambda^k T^*M$ are called k-*forms*, and the vector space of k-forms is written $C^\infty(\Lambda^k T^*M)$. They are examples of tensors. In the index notation they are written $T_{b_1 \ldots b_k}$, and are antisymmetric in the indices $b_1, \ldots, b_k$. The *exterior product* $\wedge$ and the *exterior derivative* d are important natural operations on forms. If α is a k-form and β an l-form then $\alpha \wedge \beta$ is a $(k+l)$-form and $\mathrm{d}\alpha$ a

$(k+1)$-form, which are given in index notation by

$$(\alpha \wedge \beta)_{a_1 \dots a_{k+l}} = \alpha_{[a_1 \dots a_k} \beta_{a_{k+1} \dots a_{k+l}]} \quad \text{and} \quad (\mathrm{d}\alpha)_{a_1 \dots a_{k+1}} = \frac{\partial}{\partial x^{[a_1}} \alpha_{a_2 \dots a_{k+1}]},$$

where $[\cdots]$ denotes antisymmetrization over the enclosed group of indices.

Let v, w be vector fields on M. The *Lie bracket* $[v, w]$ of v and w is another vector field on M, given in index notation by

$$[v, w]^a = v^b \frac{\partial w^a}{\partial x^b} - w^b \frac{\partial v^a}{\partial x^b}. \tag{1}$$

Here we have used the *Einstein summation convention*, that is, the repeated index b on the right hand side is summed from 1 to n. The important thing about this definition is that it is independent of choice of coordinates $(x^1, \dots, x^n)$.

2.2 Connections on vector bundles and curvature

Let M be a manifold, and $E \to M$ a vector bundle. A *connection* ∇^E on E is a linear map $\nabla^E : C^\infty(E) \to C^\infty(E \otimes T^*M)$ satisfying the condition

$$\nabla^E(\alpha\, e) = \alpha \nabla^E e + e \otimes \mathrm{d}\alpha,$$

whenever $e \in C^\infty(E)$ is a smooth section of E and α is a smooth function on M.

If ∇^E is such a connection, $e \in C^\infty(E)$, and $v \in C^\infty(TM)$ is a vector field, then we write $\nabla^E_v e = v \cdot \nabla^E e \in C^\infty(E)$, where '$\cdot$' contracts together the TM and T^*M factors in v and $\nabla^E e$. Then if $v \in C^\infty(TM)$ and $e \in C^\infty(E)$ and α, β are smooth functions on M, we have

$$\nabla^E_{\alpha v}(\beta e) = \alpha\beta \nabla^E_v e + \alpha(v \cdot \beta) e.$$

Here $v \cdot \beta$ is the Lie derivative of β by v. It is a smooth function on M, and could also be written $v \cdot \mathrm{d}\beta$.

There exists a unique, smooth section $R(\nabla^E) \in C^\infty\big(\mathrm{End}(E) \otimes \Lambda^2 T^*M\big)$ called the *curvature* of ∇^E, that satisfies the equation

$$R(\nabla^E) \cdot (e \otimes v \wedge w) = \nabla^E_v \nabla^E_w e - \nabla^E_w \nabla^E_v e - \nabla^E_{[v,w]} e \tag{2}$$

for all $v, w \in C^\infty(TM)$ and $e \in C^\infty(E)$, where $[v, w]$ is the Lie bracket of v, w.

Here is one way to understand the curvature of ∇^E. Define $v_i = \partial/\partial x^i$ for $i = 1, \dots, n$. Then v_i is a vector field on U, and $[v_i, v_j] = 0$. Let e be a smooth section of E. Then we may interpret $\nabla^E_{v_i} e$ as a kind of *partial derivative* $\partial e/\partial x^i$ of e. Equation (2) then implies that

$$R(\nabla^E) \cdot (e \otimes v_i \wedge v_j) = \frac{\partial^2 e}{\partial x^i \partial x^j} - \frac{\partial^2 e}{\partial x^j \partial x^i}. \tag{3}$$

Thus, *the curvature $R(\nabla^E)$ measures how much partial derivatives in E fail to commute.*

Now let ∇ be a connection on the tangent bundle TM of M, rather than a general vector bundle E. Then there is a unique tensor $T = T^a_{bc}$ in $C^\infty\bigl(TM \otimes \Lambda^2 T^*M\bigr)$ called the *torsion* of ∇, satisfying

$$T \cdot (v \wedge w) = \nabla_v w - \nabla_w v - [v, w] \quad \text{for all } v, w \in C^\infty(TM).$$

A connection ∇ with zero torsion is called *torsion-free*. Torsion-free connections have various useful properties, so we usually restrict attention to torsion-free connections on TM.

A connection ∇ on TM extends naturally to connections on all the bundles of tensors $\bigotimes^k TM \otimes \bigotimes^l T^*M$ for $k, l \in \mathbb{N}$, which we will also write ∇. That is, we can use ∇ to differentiate not just vector fields, but any tensor on M.

2.3 Parallel transport and holonomy groups

Let M be a manifold, $E \to M$ a vector bundle over M, and ∇^E a connection on E. Let $\gamma : [0,1] \to M$ be a smooth curve in M. Then the pull-back $\gamma^*(E)$ of E to $[0,1]$ is a vector bundle over $[0,1]$ with fibre $E_{\gamma(t)}$ over $t \in [0,1]$, where E_x is the fibre of E over $x \in M$. The connection ∇^E pulls back under γ to give a connection on $\gamma^*(E)$ over $[0,1]$.

Definition 2.1 Let M be a manifold, E a vector bundle over M, and ∇^E a connection on E. Suppose $\gamma : [0,1] \to M$ is (piecewise) smooth, with $\gamma(0) = x$ and $\gamma(1) = y$, where $x, y \in M$. Then for each $e \in E_x$, there exists a unique smooth section s of $\gamma^*(E)$ satisfying $\nabla^E_{\dot\gamma(t)} s(t) = 0$ for $t \in [0,1]$, with $s(0) = e$. Define $P_\gamma(e) = s(1)$. Then $P_\gamma : E_x \to E_y$ is a well-defined linear map, called the *parallel transport map*.

We use parallel transport to define the *holonomy group* of ∇^E.

Definition 2.2 Let M be a manifold, E a vector bundle over M, and ∇^E a connection on E. Fix a point $x \in M$. We say that γ is a *loop based at* x if $\gamma : [0,1] \to M$ is a piecewise-smooth path with $\gamma(0) = \gamma(1) = x$. The parallel transport map $P_\gamma : E_x \to E_x$ is an invertible linear map, so that P_γ lies in $\mathrm{GL}(E_x)$, the group of invertible linear transformations of E_x. Define the *holonomy group* $\mathrm{Hol}_x(\nabla^E)$ of ∇^E based at x to be

$$\mathrm{Hol}_x(\nabla^E) = \bigl\{P_\gamma : \gamma \text{ is a loop based at } x\bigr\} \subset \mathrm{GL}(E_x). \tag{4}$$

The holonomy group has the following important properties.

- It is a *Lie subgroup* of $\mathrm{GL}(E_x)$. To show that $\mathrm{Hol}_x(\nabla^E)$ is a subgroup of $\mathrm{GL}(E_x)$, let γ, δ be loops based at x, and define loops $\gamma\delta$ and γ^{-1} by

$$\gamma\delta(t) = \begin{cases} \delta(2t) & t \in [0, \frac{1}{2}] \\ \gamma(2t-1) & t \in [\frac{1}{2}, 1] \end{cases} \quad \text{and} \quad \gamma^{-1}(t) = \gamma(1-t) \quad \text{for } t \in [0,1].$$

Then $P_{\gamma\delta} = P_\gamma \circ P_\delta$ and $P_{\gamma^{-1}} = P_\gamma^{-1}$, so $\mathrm{Hol}_x(\nabla^E)$ is closed under products and inverses.

- It is *independent of basepoint* $x \in M$, in the following sense. Let $x, y \in M$, and let $\gamma : [0,1] \to M$ be a smooth path from x to y. Then $P_\gamma : E_x \to E_y$, and $\mathrm{Hol}_x(\nabla^E)$ and $\mathrm{Hol}_y(\nabla^E)$ satisfy $\mathrm{Hol}_y(\nabla^E) = P_\gamma \mathrm{Hol}_x(\nabla^E) P_\gamma^{-1}$.

  Suppose E has fibre $\mathbb{R}^k$, so that $\mathrm{GL}(E_x) \cong \mathrm{GL}(k, \mathbb{R})$. Then we may regard $\mathrm{Hol}_x(\nabla^E)$ as a subgroup of $\mathrm{GL}(k, \mathbb{R})$ defined up to conjugation, and it is then independent of basepoint x.

- If M is simply-connected, then $\mathrm{Hol}_x(\nabla^E)$ is connected. To see this, note that any loop γ based at x can be continuously shrunk to the constant loop at x. The corresponding family of parallel transports is a continuous path in $\mathrm{Hol}_x(\nabla^E)$ joining P_γ to the identity.

The holonomy group of a connection is closely related to its curvature. Here is one such relationship. As $\mathrm{Hol}_x(\nabla^E)$ is a Lie subgroup of $\mathrm{GL}(E_x)$, it has a *Lie algebra* $\mathfrak{hol}_x(\nabla^E)$, which is a Lie subalgebra of $\mathrm{End}(E_x)$. It can be shown that the curvature $R(\nabla^E)_x$ at x lies in the linear subspace $\mathfrak{hol}_x(\nabla^E) \otimes \Lambda^2 T_x^* M$ of $\mathrm{End}(E_x) \otimes \Lambda^2 T_x^* M$. Thus, *the holonomy group of a connection places a linear restriction upon its curvature.*

Now let ∇ be a connection on TM. Then from §2.2, ∇ extends to connections on all the tensor bundles $\bigotimes^k TM \otimes \bigotimes^l T^*M$. We call a tensor S on M *constant* if $\nabla S = 0$. The constant tensors on M are determined by the holonomy group $\mathrm{Hol}(\nabla)$.

Theorem 2.3 *Let M be a manifold, and ∇ a connection on TM. Fix $x \in M$, and let $H = \mathrm{Hol}_x(\nabla)$. Then H acts naturally on the tensor powers $\bigotimes^k T_xM \otimes \bigotimes^l T_x^*M$. Suppose $S \in C^\infty\bigl(\bigotimes^k TM \otimes \bigotimes^l T^*M\bigr)$ is a constant tensor. Then $S|_x$ is fixed by the action of H on $\bigotimes^k T_xM \otimes \bigotimes^l T_x^*M$. Conversely, if $S|_x \in \bigotimes^k T_xM \otimes \bigotimes^l T_x^*M$ is fixed by H, it extends to a unique constant tensor $S \in C^\infty\bigl(\bigotimes^k TM \otimes \bigotimes^l T^*M\bigr)$.*

The main idea in the proof is that if S is a constant tensor and $\gamma : [0,1] \to M$ is a path from x to y, then $P_\gamma(S|_x) = S|_y$. Thus, constant tensors are invariant under parallel transport.

2.4 Riemannian metrics and the Levi-Civita connection

Let g be a *Riemannian metric* on M. We refer to the pair (M, g) as a *Riemannian manifold.* Here g is a tensor in $C^\infty(S^2T^*M)$, so that $g = g_{ab}$ in index notation with $g_{ab} = g_{ba}$. There exists a unique, torsion-free connection ∇ on TM with $\nabla g = 0$, called the *Levi-Civita connection*, which satisfies

$$\begin{aligned} 2g(\nabla_u v, w) = {} & u \cdot g(v,w) + v \cdot g(u,w) - w \cdot g(u,v) \\ & + g([u,v],w) - g([v,w],u) - g([u,w],v) \end{aligned}$$

for all $u,v,w \in C^\infty(TM)$. This result is known as the *fundamental theorem of Riemannian geometry.*

The curvature $R(\nabla)$ of the Levi-Civita connection is a tensor $R^a{}_{bcd}$ on M. Define $R_{abcd} = g_{ae}R^e{}_{bcd}$. We shall refer to both $R^a{}_{bcd}$ and R_{abcd} as the *Riemann curvature* of g. The following theorem gives a number of symmetries of R_{abcd}. Equations (6) and (7) are known as the *first* and *second Bianchi identities*, respectively.

Theorem 2.4 *Let (M,g) be a Riemannian manifold, ∇ the Levi-Civita connection of g, and R_{abcd} the Riemann curvature of g. Then R_{abcd} and $\nabla_e R_{abcd}$ satisfy the equations*

$$R_{abcd} = -R_{abdc} = -R_{bacd} = R_{cdab}, \tag{5}$$

$$R_{abcd} + R_{adbc} + R_{acdb} = 0, \tag{6}$$

$$\textit{and} \qquad \nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec} = 0. \tag{7}$$

Let (M,g) be a Riemannian manifold, with Riemann curvature $R^a{}_{bcd}$. The *Ricci curvature* of g is $R_{ab} = R^c{}_{acb}$. It is a component of the full Riemann curvature, and satisfies $R_{ab} = R_{ba}$. We say that g is *Einstein* if $R_{ab} = \lambda g_{ab}$ for some constant $\lambda \in \mathbb{R}$, and *Ricci-flat* if $R_{ab} = 0$. Einstein and Ricci-flat metrics are of great importance in mathematics and physics.

2.5 Riemannian holonomy groups

Let (M,g) be a Riemannian manifold. We define the *holonomy group* $\mathrm{Hol}_x(g)$ of g to be the holonomy group $\mathrm{Hol}_x(\nabla)$ of the Levi-Civita connection ∇ of g, as in §2.3. Holonomy groups of Riemannian metrics, or *Riemannian holonomy groups*, have stronger properties than holonomy groups of connections on arbitrary vector bundles. We shall explore some of these.

Firstly, note that g is a *constant tensor* as $\nabla g = 0$, so g is invariant under $\mathrm{Hol}(g)$ by Theorem 2.3. That is, $\mathrm{Hol}_x(g)$ lies in the subgroup of $\mathrm{GL}(T_xM)$ which preserves $g|_x$. This subgroup is isomorphic to $\mathrm{O}(n)$. Thus, $\mathrm{Hol}_x(g)$ may be regarded as a *subgroup of* $\mathrm{O}(n)$ *defined up to conjugation*, and it is then independent of $x \in M$, so we will often write it as $\mathrm{Hol}(g)$, dropping the basepoint x.

Secondly, the holonomy group $\mathrm{Hol}(g)$ constrains the Riemann curvature of g, in the following way. The Lie algebra $\mathfrak{hol}_x(\nabla)$ of $\mathrm{Hol}_x(\nabla)$ is a vector subspace of $T_xM \otimes T_x^*M$. From §2.3, we have $R^a{}_{bcd}|_x \in \mathfrak{hol}_x(\nabla) \otimes \Lambda^2 T_x^*M$.

Use the metric g to identify $T_xM \otimes T_x^*M$ and $\otimes^2 T_x^*M$, by equating $T^a{}_b$ with $T_{ab} = g_{ac}T^c{}_b$. This identifies $\mathfrak{hol}_x(\nabla)$ with a vector subspace of $\otimes^2 T_x^*M$ that we will write as $\mathfrak{hol}_x(g)$. Then $\mathfrak{hol}_x(g)$ lies in $\Lambda^2 T_x^*M$, and $R_{abcd}|_x \in \mathfrak{hol}_x(g) \otimes \Lambda^2 T_x^*M$. Applying the symmetries (5) of R_{abcd}, we have:

Theorem 2.5 *Let (M,g) be a Riemannian manifold with Riemann curvature R_{abcd}. Then R_{abcd} lies in the vector subspace $S^2\mathfrak{hol}_x(g)$ in $\Lambda^2 T_x^*M \otimes \Lambda^2 T_x^*M$ at each $x \in M$.*

Combining this theorem with the Bianchi identities, (6) and (7), gives strong restrictions on the curvature tensor R_{abcd} of a Riemannian metric g with a prescribed holonomy group $\mathrm{Hol}(g)$. These restrictions are the basis of the classification of Riemannian holonomy groups, which will be explained in §3.

2.6 Exercises

2.1 Let M be a manifold and u, v, w be vector fields on M. The *Jacobi identity* for the Lie bracket of vector fields is

$$[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0.$$

Prove the Jacobi identity in coordinates $(x^1, \dots, x^n)$ on a coordinate patch U. Use the coordinate expression (1) for the Lie bracket of vector fields.

2.2 In §2.3 we explained that if M is a manifold, $E \to M$ a vector bundle and ∇^E a connection, then $\mathrm{Hol}(\nabla^E)$ is connected when M is simply-connected. If M is not simply-connected, what is the relationship between the fundamental group $\pi_1(M)$ and $\mathrm{Hol}(\nabla^E)$?

2.3 Work out your own proof of Theorem 2.3.

3 Berger's classification of holonomy groups

Next we describe Berger's classification of Riemannian holonomy groups, and briefly discuss the possibilities in the classification. Some references for the material of this section are my book [17, §3] and Kobayashi and Nomizu [30, §XI]. Berger's original paper is [2], but owing to language and notation most will now find it difficult to read.

3.1 Reducible Riemannian manifolds

Let (P, g) and (Q, h) be Riemannian manifolds with positive dimension, and $P \times Q$ the product manifold. Then at each (p, q) in $P \times Q$ we have $T_{(p,q)}(P \times Q) \cong T_pP \oplus T_qQ$. Define the *product metric* $g \times h$ on $P \times Q$ by $g \times h|_{(p,q)} = g|_p + h|_q$ for all $p \in P$ and $q \in Q$. We call $(P \times Q, g \times h)$ a *Riemannian product.*

A Riemannian manifold (M, g') is said to be *(locally) reducible* if every point has an open neighbourhood isometric to a Riemannian product $(P \times Q, g \times h)$, and *irreducible* if it is not locally reducible. It is easy to show that the holonomy of a product metric $g \times h$ is the product of the holonomies of g and h.

Proposition 3.1 *If (P, g) and (Q, h) are Riemannian manifolds, then* $\mathrm{Hol}(g \times h) = \mathrm{Hol}(g) \times \mathrm{Hol}(h)$.

Here is a kind of converse to this.

Theorem 3.2 *Let M be an n-manifold, and g an irreducible Riemannian metric on M. Then the representation of* $\mathrm{Hol}(g)$ *on* $\mathbb{R}^n$ *is irreducible.*

To prove the theorem, suppose $\mathrm{Hol}(g)$ acts reducibly on $\mathbb{R}^n$, so that $\mathbb{R}^n$ is the direct sum of representations $\mathbb{R}^k$, $\mathbb{R}^l$ of $\mathrm{Hol}(g)$ with $k, l > 0$. Using parallel transport, one can define a splitting $TM = E \oplus F$, where E, F are vector subbundles with fibres $\mathbb{R}^k, \mathbb{R}^l$. These vector subbundles are *integrable*, so locally $M \cong P \times Q$ with $E = TP$ and $F = TQ$. One can then show that the metric on M is the product of metrics on P and Q, so that g is locally reducible.

3.2 Symmetric spaces

Next we discuss Riemannian symmetric spaces.

Definition 3.3 A Riemannian manifold (M, g) is said to be a *symmetric space* if for every point $p \in M$ there exists an isometry $s_p : M \to M$ that is an involution (that is, s_p^2 is the identity), such that p is an isolated fixed point of s_p.

Examples include $\mathbb{R}^n$, spheres $\mathcal{S}^n$, projective spaces $\mathbb{CP}^m$ with the Fubini–Study metric, and so on. Symmetric spaces have a transitive group of isometries.

Proposition 3.4 *Let (M, g) be a connected, simply-connected symmetric space. Then g is complete. Let G be the group of isometries of (M, g) generated by elements of the form $s_q \circ s_r$ for $q, r \in M$. Then G is a connected Lie group acting transitively on M. Choose $p \in M$, and let H be the subgroup of G fixing p. Then H is a closed, connected Lie subgroup of G, and M is the homogeneous space G/H.*

Because of this, symmetric spaces can be classified completely using the theory of Lie groups. This was done in 1925 by Élie Cartan. From Cartan's classification one can quickly deduce the list of holonomy groups of symmetric spaces.

A Riemannian manifold (M, g) is called *locally symmetric* if every point has an open neighbourhood isometric to an open set in a symmetric space, and *nonsymmetric* if it is not locally symmetric. It is a surprising fact that Riemannian manifolds are locally symmetric if and only if they have *constant curvature.*

Theorem 3.5 *Let (M, g) be a Riemannian manifold, with Levi-Civita connection ∇ and Riemann curvature R. Then (M, g) is locally symmetric if and only if $\nabla R = 0$.*

3.3 Berger's classification

In 1955, Berger proved the following result.

Theorem 3.6 (Berger) *Suppose M is a simply-connected manifold of dimension n, and that g is a Riemannian metric on M, that is irreducible and nonsymmetric. Then exactly one of the following seven cases holds.*

(i) $\mathrm{Hol}(g) = \mathrm{SO}(n)$,

(ii) $n = 2m$ *with* $m \geqslant 2$, *and* $\mathrm{Hol}(g) = \mathrm{U}(m)$ *in* $\mathrm{SO}(2m)$,

(iii) $n = 2m$ *with* $m \geqslant 2$, *and* $\mathrm{Hol}(g) = \mathrm{SU}(m)$ *in* $\mathrm{SO}(2m)$,

(iv) $n = 4m$ *with* $m \geqslant 2$, *and* $\mathrm{Hol}(g) = \mathrm{Sp}(m)$ *in* $\mathrm{SO}(4m)$,

(v) $n = 4m$ *with* $m \geqslant 2$, *and* $\mathrm{Hol}(g) = \mathrm{Sp}(m)\,\mathrm{Sp}(1)$ *in* $\mathrm{SO}(4m)$,

(vi) $n = 7$ *and* $\mathrm{Hol}(g) = G_2$ *in* $\mathrm{SO}(7)$, *or*

(vii) $n = 8$ *and* $\mathrm{Hol}(g) = \mathrm{Spin}(7)$ *in* $\mathrm{SO}(8)$.

Notice the three simplifying assumptions on M and g: that M is simply-connected, and g is irreducible and nonsymmetric. Each condition has consequences for the holonomy group $\mathrm{Hol}(g)$.

- As M is simply-connected, $\mathrm{Hol}(g)$ is connected, from §2.3.
- As g is irreducible, $\mathrm{Hol}(g)$ acts irreducibly on $\mathbb{R}^n$ by Theorem 3.2.
- As g is nonsymmetric, $\nabla R \not\equiv 0$ by Theorem 3.5.

The point of the third condition is that there are some holonomy groups H which can *only* occur for metrics g with $\nabla R = 0$, and these holonomy groups are excluded from the theorem.

One can remove the three assumptions, at the cost of making the list of holonomy groups much longer. To allow g to be symmetric, we must include the holonomy groups of Riemannian symmetric spaces, which are known from Cartan's classification. To allow g to be reducible, we must include all products of holonomy groups already on the list. To allow M not simply-connected, we must include non-connected Lie groups whose identity components are already on the list.

Berger proved that the groups on his list were the only possibilities, but he did not show whether the groups actually do occur as holonomy groups. It is now known (but this took another thirty years to find out) that all of the groups on Berger's list do occur as the holonomy groups of irreducible, nonsymmetric metrics.

3.4 A sketch of the proof of Berger's Theorem

Let (M, g) be a Riemannian n-manifold with M simply-connected and g irreducible and nonsymmetric, and let $H = \mathrm{Hol}(g)$. Then it is known that H is a closed, connected Lie subgroup of $\mathrm{SO}(n)$. The classification of such subgroups follows from the classification of Lie groups. Berger's method was to take the list of all closed, connected Lie subgroups H of $\mathrm{SO}(n)$, and apply two tests to each possibility to find out if it could be a holonomy group. The only groups H which passed both tests are those in the Theorem 3.6.

Berger's tests are algebraic and involve the curvature tensor. Suppose R_{abcd} is the Riemann curvature of a metric g with $\mathrm{Hol}(g) = H$, and let $\mathfrak{h}$ be the Lie algebra of H. Then Theorem 2.4 shows that $R_{abcd} \in S^2\mathfrak{h}$, and the first Bianchi identity (6) applies.

If $\mathfrak{h}$ has large codimension in $\mathfrak{so}(n)$, then the vector space $\mathfrak{R}^H$ of elements of $S^2\mathfrak{h}$ satisfying (6) will be small, or even zero. But the *Ambrose–Singer Holonomy Theorem* shows that $\mathfrak{R}^H$ must be big enough to generate $\mathfrak{h}$, in a certain sense. For many of the candidate groups H this does not hold, and so H cannot be a holonomy group. This is the first test.

Now $\nabla_e R_{abcd}$ lies in $(\mathbb{R}^n)^* \otimes \mathfrak{R}^H$, and also satisfies the second Bianchi identity (7). Frequently these requirements imply that $\nabla R = 0$, so that g is locally symmetric. Therefore we may exclude such H, and this is Berger's second test.

3.5 The groups on Berger's list

Here are some brief remarks about each group on Berger's list.

(i) $\mathrm{SO}(n)$ is the holonomy group of generic Riemannian metrics.

(ii) Riemannian metrics g with $\mathrm{Hol}(g) \subseteq \mathrm{U}(m)$ are called *Kähler metrics*. Kähler metrics are a natural class of metrics on complex manifolds, and generic Kähler metrics on a given complex manifold have holonomy $\mathrm{U}(m)$.

(iii) Metrics g with $\mathrm{Hol}(g) \subseteq \mathrm{SU}(m)$ are called *Calabi–Yau metrics*. Since $\mathrm{SU}(m)$ is a subgroup of $\mathrm{U}(m)$, all Calabi–Yau metrics are Kähler. If g is Kähler and M is simply-connected, then $\mathrm{Hol}(g) \subseteq \mathrm{SU}(m)$ if and only if g is Ricci-flat. Thus Calabi–Yau metrics are locally the same as Ricci-flat Kähler metrics.

(iv) Metrics g with $\mathrm{Hol}(g) \subseteq \mathrm{Sp}(m)$ are called *hyperkähler*. As $\mathrm{Sp}(m) \subseteq \mathrm{SU}(2m) \subset \mathrm{U}(2m)$, hyperkähler metrics are Ricci-flat and Kähler.

(v) Metrics g with holonomy group $\mathrm{Sp}(m)\,\mathrm{Sp}(1)$ for $m \geqslant 2$ are called *quaternionic Kähler*. (Note that quaternionic Kähler metrics are not in fact Kähler.) They are Einstein, but not Ricci-flat.

(vi) and (vii) The holonomy groups G_2 and $\mathrm{Spin}(7)$ are called the *exceptional holonomy groups*. Metrics with these holonomy groups are Ricci-flat.

The groups can be understood in terms of the four *division algebras*: the *real numbers* $\mathbb{R}$, the *complex numbers* $\mathbb{C}$, the *quaternions* $\mathbb{H}$, and the *octonions* or *Cayley numbers* $\mathbb{O}$.

- $\mathrm{SO}(n)$ is a group of automorphisms of $\mathbb{R}^n$.
- $\mathrm{U}(m)$ and $\mathrm{SU}(m)$ are groups of automorphisms of $\mathbb{C}^m$
- $\mathrm{Sp}(m)$ and $\mathrm{Sp}(m)\,\mathrm{Sp}(1)$ are automorphism groups of $\mathbb{H}^m$.
- G_2 is the automorphism group of $\mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$. $\mathrm{Spin}(7)$ is a group of automorphisms of $\mathbb{O} \cong \mathbb{R}^8$.

Here are three ways in which we can gather together the holonomy groups on Berger's list into subsets with common features.

- The *Kähler holonomy groups* are $\mathrm{U}(m)$, $\mathrm{SU}(m)$ and $\mathrm{Sp}(m)$. Any Riemannian manifold with one of these holonomy groups is a Kähler manifold, and thus a complex manifold.
- The *Ricci-flat holonomy groups* are $\mathrm{SU}(m)$, $\mathrm{Sp}(m)$, G_2 and $\mathrm{Spin}(7)$. Any metric with one of these holonomy groups is Ricci-flat. This follows from the effect of holonomy on curvature discussed in §2.5 and §3.4: if H is one of these holonomy groups and R_{abcd} any curvature tensor lying in $S^2\mathfrak{h}$ and satisfying (6), then R_{abcd} has zero Ricci component.
- The *exceptional holonomy groups* are G_2 and $\mathrm{Spin}(7)$. They are the exceptional cases in Berger's classification, and they are rather different from the other holonomy groups.

3.6 Exercises

3.1 Work out your own proofs of Proposition 3.1 and (harder) Theorem 3.2.

3.2 Suppose that (M, g) is a simply-connected Ricci-flat Kähler manifold of complex dimension 4. What are the possibilities for $\mathrm{Hol}(g)$?
[You may use the fact that the only simply-connected Ricci-flat symmetric spaces are $\mathbb{R}^n$, $n \in \mathbb{N}$.]

4 Kähler geometry and holonomy

We now focus our attention on Kähler geometry, and the Ricci curvature of Kähler manifolds. This leads to the definition of *Calabi–Yau manifolds*, compact Ricci-flat Kähler manifolds with holonomy $\mathrm{SU}(m)$. A reference for this section is my book [17, §4, §6].

4.1 Complex manifolds

We begin by defining *complex manifolds* M. The usual definition of complex manifolds involves an atlas of complex coordinate patches covering M, whose transition functions are holomorphic. However, for our purposes we need a more differential geometric definition, involving a tensor J on M called a *complex structure*.

Let M be a real manifold of dimension $2m$. An *almost complex structure* J on M is a tensor J_a^b on M satisfying $J_a^b J_b^c = -\delta_a^c$. For each vector field v on M define Jv by $(Jv)^b = J_a^b v^a$. Then $J^2 = -1$, so J gives each tangent space T_pM the structure of a *complex vector space*.

We can associate a tensor $N = N^a_{bc}$ to J, called the *Nijenhuis tensor*, which satisfies

$$N^a_{bc} v^b w^c = \big([v,w] + J\big([Jv,w] + [v,Jw]\big) - [Jv,Jw]\big)^a$$

for all vector fields v, w on M, where $[\ ,\ ]$ is the Lie bracket of vector fields. The almost complex structure J is called a *complex structure* if $N \equiv 0$. A *complex manifold* (M, J) is a manifold M with a complex structure J.

Here is why this is equivalent to the usual definition. A smooth function $f : M \to \mathbb{C}$ is called *holomorphic* if $J^b_a(\mathrm{d}f)_b \equiv i(\mathrm{d}f)_a$ on M. These are called the *Cauchy–Riemann equations*. It turns out that the Nijenhuis tensor N is the obstruction to the existence of holomorphic functions. If $N \equiv 0$ there are many holomorphic functions locally, enough to form a set of holomorphic coordinates around every point.

4.2 Kähler manifolds

Let (M, J) be a complex manifold, and let g be a Riemannian metric on M. We call g a *Hermitian metric* if $g(v, w) = g(Jv, Jw)$ for all vector fields v, w on M, or $g_{ab} = J^c_a J^d_b g_{cd}$ in index notation. When g is Hermitian, define the *Hermitian form* ω of g by $\omega(v, w) = g(Jv, w)$ for all vector fields v, w on M, or $\omega_{ac} = J^b_a g_{bc}$ in index notation. Then ω is a (1,1)-form, and we may reconstruct g from ω by $g(v, w) = \omega(v, Jw)$.

A Hermitian metric g on a complex manifold (M, J) is called *Kähler* if one of the following three equivalent conditions holds:

(i) $\mathrm{d}\omega = 0$,

(ii) $\nabla J = 0$, or

(iii) $\nabla\omega = 0$,

where ∇ is the Levi-Civita connection of g. We then call (M, J, g) a *Kähler manifold*. Kähler metrics are a natural and important class of metrics on complex manifolds.

By parts (ii) and (iii), if g is Kähler then J and ω are *constant tensors* on M. Thus by Theorem 2.3, the holonomy group $\mathrm{Hol}(g)$ must preserve a complex structure J_0 and 2-form ω_0 on $\mathbb{R}^{2m}$. The subgroup of $\mathrm{O}(2m)$ preserving J_0 and ω_0 is $\mathrm{U}(m)$, so $\mathrm{Hol}(g) \subseteq \mathrm{U}(m)$. So we prove:

Proposition 4.1 *A metric g on a $2m$-manifold M is Kähler with respect to some complex structure J on M if and only if* $\mathrm{Hol}(g) \subseteq \mathrm{U}(m) \subset \mathrm{O}(2m)$.

4.3 Kähler potentials

Let (M, J) be a complex manifold. We have seen that to each Kähler metric g on M there is associated a closed real (1,1)-form ω, called the Kähler form. Conversely, if ω is a closed real (1,1)-form on M, then ω is the Kähler form of a Kähler metric if and only if ω is *positive*, that is, $\omega(v, Jv) > 0$ for all nonzero vectors v.

Now there is an easy way to manufacture closed real (1,1)-forms, using the ∂ and $\bar{\partial}$ operators on M. If $\phi : M \to \mathbb{R}$ is smooth, then $i\partial\bar{\partial}\phi$ is a closed real (1,1)-form, and every closed real (1,1)-form may be locally written in this way. Therefore, every Kähler metric g on M may be described locally by a function $\phi : M \to \mathbb{R}$ called a *Kähler potential*, such that the Kähler form ω satisfies $\omega = i\partial\bar{\partial}\phi$.

However, in general one cannot write $\omega = i\partial\bar{\partial}\phi$ globally on M, because $i\partial\bar{\partial}\phi$ is *exact*, but ω is usually not exact (never, if M is compact). Thus we are led to consider the *de Rham cohomology class* $[\omega]$ of ω in $H^2(M, \mathbb{R})$. We call $[\omega]$ the *Kähler class* of g. If two Kähler metrics g, g' on M lie in the same Kähler class, then they differ by a Kähler potential.

Proposition 4.2 *Let (M, J) be a compact complex manifold, and let g, g' be Kähler metrics on M with Kähler forms ω, ω'. Suppose that $[\omega] = [\omega'] \in H^2(M, \mathbb{R})$. Then there exists a smooth, real function ϕ on M such that $\omega' = \omega + i\partial\bar{\partial}\phi$. This function ϕ is unique up to the addition of a constant.*

Note also that if ω is the Kähler form of a fixed Kähler metric g and ϕ is sufficiently small in C^2, then $\omega' = \omega + i\partial\bar{\partial}\phi$ is the Kähler form of another Kähler metric g' on M, in the same Kähler class as g. This implies that if there exists one Kähler metric on M, then there exists an infinite-dimensional family — Kähler metrics are very abundant.

4.4 Ricci curvature and the Ricci form

Let (M, J, g) be a Kähler manifold, with Ricci curvature R_{ab}. Define the *Ricci form* ρ by $\rho_{ac} = J_a^b R_{bc}$. Then it turns out that $\rho_{ac} = -\rho_{ca}$, so that ρ is a 2-form. Furthermore, it is a remarkable fact that ρ is a *closed, real* $(1,1)$*-form.* Note also that the Ricci curvature can be recovered from ρ by the formula $R_{ab} = \rho_{ac} J_b^c$.

To explain this, we will give an explicit expression for the Ricci form. Let $(z_1, \ldots, z_m)$ be holomorphic coordinates on an open set U in M. Define a smooth function $f : U \to (0, \infty)$ by

$$\omega^m = f \cdot \frac{(-1)^{m(m-1)/2} i^m m!}{2^m} \cdot \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_m \wedge \mathrm{d}\bar{z}_1 \wedge \cdots \wedge \mathrm{d}\bar{z}_m. \tag{8}$$

Here the constant factor ensures that f is positive, and gives $f \equiv 1$ when ω is the standard Hermitian form on $\mathbb{C}^m$. Then it can be shown that

$$\rho = -i\partial\bar{\partial}(\log f) \quad \text{on } U, \tag{9}$$

so that ρ is indeed a closed real (1,1)-form.

Using some algebraic geometry, we can interpret this. The *canonical bundle* $K_M = \Lambda^{(m,0)} T^* M$ is a holomorphic line bundle over M. The Kähler metric g on M induces a metric on K_M, and the combination of metric and holomorphic structure induces a connection ∇^K on K_M. The curvature of this connection is a closed 2-form with values in the Lie algebra $\mathfrak{u}(1)$, and identifying $\mathfrak{u}(1) \cong \mathbb{R}$ we get a closed 2-form, which is the Ricci form.

Thus the Ricci form ρ may be understood as the curvature 2-form of a connection ∇^K on the canonical bundle K_M. So by characteristic class theory we may identify the de Rham cohomology class $[\rho]$ of ρ in $H^2(M,\mathbb{R})$: it satisfies

$$[\rho] = 2\pi\, c_1(K_M) = 2\pi\, c_1(M), \tag{10}$$

where $c_1(M)$ is the first Chern class of M in $H^2(M,\mathbb{Z})$. It is a topological invariant depending on the homotopy class of the (almost) complex structure J.

4.5 Calabi–Yau manifolds

Here is our definition of Calabi–Yau manifold.

Definition 4.3 Let $m \geqslant 2$. A *Calabi–Yau m-fold* is a quadruple (M,J,g,Ω) such that (M,J) is a compact m-dimensional complex manifold, g a Kähler metric on (M,J) with holonomy group $\mathrm{Hol}(g) = \mathrm{SU}(m)$, and Ω a nonzero constant $(m,0)$-form on M called the *holomorphic volume form*, which satisfies

$$\omega^m/m! = (-1)^{m(m-1)/2}(i/2)^m \Omega \wedge \bar{\Omega}, \tag{11}$$

where ω is the Kähler form of g. The constant factor in (11) is chosen to make $\mathrm{Re}\,\Omega$ a *calibration.*

Readers are warned that there are several *different* definitions of Calabi–Yau manifolds in use in the literature. Ours is unusual in regarding Ω as part of the given structure. Some authors define a Calabi–Yau m-fold to be a compact Kähler manifold (M,J,g) with holonomy $\mathrm{SU}(m)$. We shall show that one can associate a holomorphic volume form Ω to such (M,J,g) to make it Calabi–Yau in our sense, and Ω is unique up to phase.

Lemma 4.4 *Let (M,J,g) be a compact Kähler manifold with $\mathrm{Hol}(g) = \mathrm{SU}(m)$. Then M admits a holomorphic volume form Ω, unique up to change of phase $\Omega \mapsto \mathrm{e}^{i\theta}\Omega$, such that (M,J,g,Ω) is a Calabi–Yau manifold.*

Proof. Let (M,J,g) be compact and Kähler with $\mathrm{Hol}(g) = \mathrm{SU}(m)$. Now the holonomy group $\mathrm{SU}(m)$ preserves the standard metric g_0 and Kähler form ω_0 on $\mathbb{C}^m$, and an $(m,0)$-form Ω_0 given by

$$g_0 = |\mathrm{d}z_1|^2 + \cdots + |\mathrm{d}z_m|^2, \quad \omega_0 = \frac{i}{2}(\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \cdots + \mathrm{d}z_m \wedge \mathrm{d}\bar{z}_m),$$
$$\text{and} \quad \Omega_0 = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_m.$$

Thus, by Theorem 2.3 there exist corresponding constant tensors g, ω (the Kähler form), and Ω on (M,J,g). Since ω_0 and Ω_0 satisfy

$$\omega_0^m/m! = (-1)^{m(m-1)/2}(i/2)^m \Omega_0 \wedge \bar{\Omega}_0$$

on $\mathbb{C}^m$, it follows that ω and Ω satisfy (11) at each point, so (M, J, g, Ω) is Calabi–Yau. It is easy to see that Ω is unique up to change of phase. $\square$

Suppose (M, J, g, Ω) is a Calabi–Yau m-fold. Then Ω is a constant section of the canonical bundle K_M. As Ω is constant, it is holomorphic. Thus the canonical bundle K_M admits a nonvanishing holomorphic section, so (M, J) has *trivial canonical bundle*, which implies that $c_1(M) = 0$.

Further, the connection ∇^K on K_M must be *flat*. However, from §4.4 the curvature of ∇^K is the Ricci form ρ. Therefore $\rho \equiv 0$, and g is Ricci-flat. That is, Calabi–Yau m-folds are automatically *Ricci-flat*. More generally, the following proposition explains the relationship between the Ricci curvature and holonomy group of a Kähler metric.

Proposition 4.5 *Let (M, J, g) be a Kähler m-fold with $\mathrm{Hol}(g) \subseteq \mathrm{SU}(m)$. Then g is Ricci-flat. Conversely, let (M, J, g) be a Ricci-flat Kähler m-fold. If M is simply-connected or K_M is trivial, then $\mathrm{Hol}(g) \subseteq \mathrm{SU}(m)$.*

In the last part, M simply-connected implies that K_M is trivial for Ricci-flat Kähler manifolds, but not vice versa.

4.6 Exercises

4.1 Let U be a simply-connected subset of $\mathbb{C}^m$ with coordinates $(z_1, \ldots, z_m)$, and g a Ricci-flat Kähler metric on U with Kähler form ω. Use equations (8) and (9) to show that there exists a holomorphic $(m, 0)$-form Ω on U satisfying

$$\omega^m/m! = (-1)^{m(m-1)/2}(i/2)^m \Omega \wedge \bar{\Omega}.$$

Hint: Write $\Omega = F\, \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_m$ for some holomorphic function F. Use the fact that if f is a real function on a simply-connected subset U of $\mathbb{C}^m$ and $\partial\bar{\partial} f \equiv 0$, then f is the real part of a holomorphic function on U.

4.2 Let $\mathbb{C}^2$ have complex coordinates (z_1, z_2), and define $u = |z_1|^2 + |z_2|^2$. Let $f : [0, \infty) \to \mathbb{R}$ be a smooth function, and define a closed real (1,1)-form ω on $\mathbb{C}^2$ by $\omega = i\partial\bar{\partial} f(u)$.

(a) Calculate the conditions on f for ω to be the Kähler form of a Kähler metric g on $\mathbb{C}^2$.
(You can define g by $g(v, w) = \omega(v, Jw)$, and need to ensure that g is positive definite).

(b) Supposing g is a metric, calculate the conditions on f for g to be Ricci-flat. You should get an o.d.e. on f. If you can, solve this o.d.e., and write down the corresponding Kähler metrics in coordinates.

5 The Calabi Conjecture and constructions of Calabi–Yau m-folds

The *Calabi Conjecture* specifies which closed (1,1)-forms on a compact complex manifold can be the Ricci form of a Kähler metric. It was posed by Calabi in 1954, and proved by Yau in 1976. We shall explain the conjecture, and sketch its proof. An important application of the Calabi Conjecture is the construction of large numbers of *Calabi–Yau manifolds*. We explain some ways to do this, using algebraic geometry.

A good general reference for this section is my book [17, §§5, 6 & 7.3], which includes a proof of the Calabi Conjecture. Other references on the Calabi Conjecture are Aubin's book [1] and Yau [43], which is the original proof of the conjecture, but fairly hard going unless you know a lot of analysis.

5.1 The Calabi Conjecture

Let (M, J) be a compact, complex manifold, and g a Kähler metric on M, with Ricci form ρ. From §4.4, ρ is a closed real (1,1)-form and $[\rho] = 2\pi\, c_1(M) \in H^2(M, \mathbb{R})$. The Calabi Conjecture specifies which closed (1,1)-forms can be the Ricci forms of a Kähler metric on M.

The Calabi Conjecture *Let (M, J) be a compact, complex manifold, and g a Kähler metric on M, with Kähler form ω. Suppose that ρ' is a real, closed $(1,1)$-form on M with $[\rho'] = 2\pi\, c_1(M)$. Then there exists a unique Kähler metric g' on M with Kähler form ω', such that $[\omega'] = [\omega] \in H^2(M, \mathbb{R})$, and the Ricci form of g' is ρ'.*

Note that $[\omega'] = [\omega]$ says that g and g' are in the *same Kähler class*. The conjecture was posed by Calabi in 1954, and was eventually proved by Yau in 1976. Its importance to us is that when $c_1(M) = 0$ we can take $\rho' \equiv 0$, and then g' is Ricci-flat. Thus, assuming the Calabi Conjecture we prove:

Corollary 5.1 *Let (M, J) be a compact complex manifold with $c_1(M) = 0$ in $H^2(M, \mathbb{R})$. Then every Kähler class on M contains a unique Ricci-flat Kähler metric g.*

If in addition M is simply-connected or K_M is trivial, then Proposition 4.5 shows that $\mathrm{Hol}(g) \subseteq \mathrm{SU}(m)$. When $\mathrm{Hol}(g) = \mathrm{SU}(m)$, which will happen under certain fairly simple topological conditions on M, then by Lemma 4.4 we can make (M, J, g) into a *Calabi–Yau manifold* (M, J, g, Ω). So Yau's proof of the Calabi Conjecture gives a way to find examples of Calabi–Yau manifolds, which is how Calabi–Yau manifolds got their name.

Note that we know *almost nothing* about the Ricci-flat Kähler metric g except that it exists; we cannot write it down explicitly in coordinates, for instance. In fact, *no* explicit non-flat examples of Calabi–Yau metrics on compact manifolds are known at all.

5.2 Sketch of the proof of the Calabi Conjecture

The Calabi Conjecture is proved by rewriting it as a second-order nonlinear elliptic p.d.e. upon a real function ϕ on M, and then showing that this p.d.e. has a unique solution. We first explain how to rewrite the Calabi Conjecture as a p.d.e.

Let (M, J) be a compact, complex manifold, and let g, g' be two Kähler metrics on M with Kähler forms ω, ω' and Ricci forms ρ, ρ'. Suppose g, g' are in the same Kähler class, so that $[\omega'] = [\omega] \in H^2(M, \mathbb{R})$. Define a smooth function $f : M \to \mathbb{R}$ by $(\omega')^m = \mathrm{e}^f \omega^m$. Then from equations (8) and (9) of §4.4, we find that $\rho' = \rho - i\partial\bar{\partial} f$. Furthermore, as $[\omega'] = [\omega]$ in $H^2(M, \mathbb{R})$, we have $[\omega']^m = [\omega]^m$ in $H^{2m}(M, \mathbb{R})$, and thus $\int_M \mathrm{e}^f \omega^m = \int_M \omega^m$.

Now suppose that we are given the real, closed $(1,1)$-form ρ' with $[\rho'] = 2\pi\, c_1(M)$, and want to construct a metric g' with ρ' as its Ricci form. Since $[\rho] = [\rho'] = 2\pi\, c_1(M)$, $\rho - \rho'$ is an *exact* real (1,1)-form, and so by the $\partial\bar{\partial}$-Lemma there exists a smooth function $f : M \to \mathbb{R}$ with $\rho - \rho' = i\partial\bar{\partial} f$. This f is unique up to addition of a constant, but the constant is fixed by requiring that $\int_M \mathrm{e}^f \omega^m = \int_M \omega^m$. Thus we have proved:

Proposition 5.2 *Let (M, J) be a compact complex manifold, g a Kähler metric on M with Kähler form ω and Ricci form ρ, and ρ' a real, closed $(1,1)$-form on M with $[\rho'] = 2\pi\, c_1(M)$. Then there is a unique smooth function $f : M \to \mathbb{R}$ such that*

$$\rho' = \rho - i\partial\bar{\partial} f \quad \textit{and} \quad \int_M \mathrm{e}^f \omega^m = \int_M \omega^m, \tag{12}$$

and a Kähler metric g on M with Kähler form ω' satisfying $[\omega'] = [\omega]$ in $H^2(M, \mathbb{R})$ has Ricci form ρ' if and only if $(\omega')^m = \mathrm{e}^f \omega^m$.

Thus we have transformed the Calabi Conjecture from seeking a metric g' with *prescribed Ricci curvature* ρ' to seeking a metric g' with *prescribed volume form* $(\omega')^m$. This is an important simplification, because the Ricci curvature depends on the second derivatives of g', but the volume form depends only on g' and not on its derivatives.

Now by Proposition 4.2, as $[\omega'] = [\omega]$ we may write $\omega' = \omega + i\partial\bar{\partial}\phi$ for ϕ a smooth real function on M, unique up to addition of a constant. We can fix the constant by requiring that $\int_M \phi \, \mathrm{d}V_g = 0$. So, from Proposition 5.2 we deduce that the Calabi Conjecture is equivalent to:

The Calabi Conjecture (second version) *Let (M, J) be a compact, complex manifold, and g a Kähler metric on M, with Kähler form ω. Let f be a smooth real function on M satisfying $\int_M \mathrm{e}^f \omega^m = \int_M \omega^m$. Then there exists a unique smooth real function ϕ such that*

(i) *$\omega + i\partial\bar{\partial}\phi$ is a positive $(1,1)$-form, that is, it is the Kähler form of some Kähler metric g',*

(ii) *$\int_M \phi \, \mathrm{d}V_g = 0$, and*

(iii) $(\omega + i\partial\bar\partial\phi)^m = \mathrm{e}^f\omega^m$ *on* M.

This reduces the Calabi Conjecture to a problem in analysis, that of showing that the nonlinear p.d.e. $(\omega+i\partial\bar\partial\phi)^m = \mathrm{e}^f\omega^m$ has a solution ϕ for every suitable function f. To prove this second version of the Calabi Conjecture, Yau used the *continuity method*.

For each $t \in [0,1]$, define $f_t = tf + c_t$, where c_t is the unique real constant such that $\mathrm{e}^{c_t}\int_M \mathrm{e}^{tf}\omega^m = \int_M \omega^m$. Then f_t depends smoothly on t, with $f_0 \equiv 0$ and $f_1 \equiv f$. Define S to be the set of $t \in [0,1]$ such that there exists a smooth real function ϕ on M satisfying parts (i) and (ii) above, and also

(iii)$'$ $(\omega + i\partial\bar\partial\phi)^m = \mathrm{e}^{f_t}\omega^m$ on M.

The idea of the continuity method is to show that S is both *open* and *closed* in $[0,1]$. Thus, S is a connected subset of $[0,1]$, so $S = \emptyset$ or $S = [0,1]$. But $0 \in S$, since as $f_0 \equiv 0$ parts (i), (ii) and (iii)$'$ are satisfied by $\phi \equiv 0$. Thus $S = [0,1]$. In particular, (i), (ii) and (iii)$'$ admit a solution ϕ when $t = 1$. As $f_1 \equiv f$, this ϕ satisfies (iii), and the Calabi Conjecture is proved.

Showing that S is open is fairly easy, and was done by Calabi. It depends on the fact that (iii) is an *elliptic* p.d.e. — basically, the operator $\phi \mapsto (\omega+i\partial\bar\partial\phi)^m$ is rather like a nonlinear Laplacian — and uses only standard facts about elliptic operators.

However, showing that S is closed is much more difficult. One must prove that S contains its limit points. That is, if $(t_n)_{n=1}^\infty$ is a sequence in S converging to $t \in [0,1]$ then there exists a sequence $(\phi_n)_{n=1}^\infty$ satisfying (i), (ii) and $(\omega + i\partial\bar\partial\phi_n)^m = \mathrm{e}^{f_{t_n}}\omega^m$ for $n = 1, 2, \ldots$, and we need to show that $\phi_n \to \phi$ as $n \to \infty$ for some smooth real function ϕ satisfying (i), (ii) and (iii)$'$, so that $t \in S$.

The thing you have to worry about is that the sequence $(\phi_n)_{n=1}^\infty$ might converge to some horrible non-smooth function, or might not converge at all. To prove this doesn't happen you need *a priori estimates* on the ϕ_n and all their derivatives. In effect, you need upper bounds on $|\nabla^k\phi_n|$ for all n and k, bounds which are allowed to depend on M, J, g, k and f_{t_n}, but not on n or ϕ_n. These a priori estimates were difficult to find, because the nonlinearities in ϕ of $(\omega + i\partial\bar\partial\phi)^m = \mathrm{e}^f\omega^m$ are of a particularly nasty kind, and this is why it took so long to prove the Calabi Conjecture.

5.3 Calabi–Yau 2-folds and $K3$ surfaces

Recall from §3.5 that the *Kähler holonomy groups* are $\mathrm{U}(m)$, $\mathrm{SU}(m)$ and $\mathrm{Sp}(k)$. Calabi–Yau manifolds of complex dimension m have holonomy $\mathrm{SU}(m)$ for $m \geqslant 2$, and hyperkähler manifolds of complex dimension $2k$ have holonomy $\mathrm{Sp}(k)$ for $k \geqslant 1$. In complex dimension 2 these coincide, as $\mathrm{SU}(2) = \mathrm{Sp}(1)$. Because of this, Calabi–Yau 2-folds have special features which are not present in Calabi–Yau m-folds for $m \geqslant 3$.

Calabi–Yau 2-folds are very well understood, through the classification of compact complex surfaces. A $K3$ *surface* is defined to be a compact, complex surface (X, J) with $h^{1,0}(X) = 0$ and trivial canonical bundle. All Calabi–Yau

2-folds are $K3$ surfaces, and conversely, every $K3$ surface (X, J) admits a family of Kähler metrics g making it into a Calabi–Yau 2-fold. All $K3$ surfaces (X, J) are diffeomorphic, sharing the same smooth 4-manifold X, which is simply-connected, with Betti numbers $b^2 = 22$, $b^2_+ = 3$, and $b^2_- = 19$.

The moduli space $\mathcal{M}_{K3}$ of $K3$ surfaces is a connected 20-dimensional singular complex manifold, which can be described very precisely via the 'Torelli Theorems'. Some $K3$ surfaces are *algebraic*, that is, they can be embedded as complex submanifolds in $\mathbb{CP}^N$ for some N, and some are not. The set of algebraic $K3$ surfaces is a countable, dense union of 19-dimensional subvarieties in $\mathcal{M}_{K3}$. Each $K3$ surface (X, J) admits a real 20-dimensional family of Calabi–Yau metrics g, so the family of Calabi–Yau 2-folds (X, J, g) is a nonsingular 60-dimensional real manifold.

5.4 General properties of Calabi–Yau m-folds for $m \geqslant 3$

Using general facts about Ricci-flat manifolds (the *Cheeger–Gromoll Theorem*) one can show that every Calabi–Yau m-fold (M, J, g, Ω) has finite fundamental group. Also, using the 'Bochner argument' one can show that any closed $(p, 0)$-form ξ on M is constant under the Levi-Civita connection ∇ of g.

However, the set of constant tensors on M is determined by the holonomy group $\mathrm{Hol}(g)$ of g, which is $\mathrm{SU}(m)$ by definition. It is easy to show that the vector space of closed $(p, 0)$-forms on M is $\mathbb{C}$ if $p = 0, m$ and 0 otherwise. But the vector space of closed $(p, 0)$ forms is the *Dolbeault cohomology group* $H^{p,0}(M)$, whose dimension is the *Hodge number* $h^{p,0}$ of M. Thus we prove:

Proposition 5.3 *Let (M, J, g, Ω) be a Calabi–Yau m-fold with Hodge numbers $h^{p,q}$. Then M has finite fundamental group, $h^{0,0} = h^{m,0} = 1$ and $h^{p,0} = 0$ for $p \neq 0, m$.*

For $m \geqslant 3$ this gives $h^{2,0}(M) = 0$, and this has important consequences for the complex manifold (M, J). It can be shown that a complex line bundle L over a compact Kähler manifold (M, J, g) admits a holomorphic structure if and only if $c_1(L)$ lies in $H^{1,1}(M) \subseteq H^2(M, \mathbb{C})$. But $H^2(M, \mathbb{C}) = H^{2,0}(M) \oplus H^{1,1}(M) \oplus H^{0,2}(M)$, and $H^{2,0}(M) = H^{0,2}(M) = 0$ as $h^{2,0}(M) = 0$. Thus $H^{1,1}(M) = H^2(M, \mathbb{C})$, and so *every* complex line bundle L over M admits a holomorphic structure.

Thus, Calabi–Yau m-folds for $m \geqslant 3$ are richly endowed with holomorphic line bundles. Using the *Kodaira Embedding Theorem* one can show that some of these holomorphic line bundles admit many holomorphic sections. By taking a line bundle with enough holomorphic sections (a *very ample* line bundle) we can construct an embedding of M in $\mathbb{CP}^N$ as a complex submanifold. So we prove:

Theorem 5.4 *Let (M, J, g, Ω) be a Calabi–Yau manifold of dimension $m \geqslant 3$. Then M is projective. That is, (M, J) is isomorphic as a complex manifold to a complex submanifold of $\mathbb{CP}^N$, and is an algebraic variety.*

This shows that Calabi–Yau manifolds (or at least, the complex manifolds underlying them) can be studied using *complex algebraic geometry.*

5.5 Constructions of Calabi–Yau m-folds

The easiest way to find examples of Calabi–Yau m-folds for $m \geqslant 3$ is to choose a method of generating a large number of complex algebraic varieties, and then check the topological conditions to see which of them are Calabi–Yau. Here are some ways of doing this.

- **Hypersurfaces in** $\mathbb{CP}^{m+1}$. Suppose that X is a smooth degree d hypersurface in $\mathbb{CP}^{m+1}$. When is X a Calabi–Yau manifold? Well, using the *adjunction formula* one can show that the canonical bundle of X is given by $K_X = L^{d-m-2}|_X$, where $L \to \mathbb{CP}^{m+1}$ is the hyperplane line bundle on $\mathbb{CP}^{m+1}$.

  Therefore K_X is trivial if and only if $d = m+2$. It is not difficult to show that any smooth hypersurface of degree $m+2$ in $\mathbb{CP}^{m+1}$ is a Calabi–Yau m-fold. All such hypersurfaces are diffeomorphic, for fixed m. For instance, the *Fermat quintic*

  $$\left\{[z_0, \ldots, z_4] \in \mathbb{CP}^4 : z_0^5 + \cdots + z_4^5 = 0\right\}$$

  is a Calabi–Yau 3-fold, with Betti numbers $b^0 = 1$, $b^1 = 0$, $b^2 = 1$ and $b^3 = 204$.

- **Complete intersections in** $\mathbb{CP}^{m+k}$. In the same way, suppose X is a *complete intersection* of transverse hypersurfaces $H_1, \ldots, H_k$ in $\mathbb{CP}^{m+k}$ of degrees $d_1, \ldots, d_k$, with each $d_j \geqslant 2$. It can be shown that X is Calabi–Yau m-fold if and only if $d_1 + \cdots + d_k = m+k+1$. This yields a finite number of topological types in each dimension m.

- **Hypersurfaces in toric varieties**. A *toric variety* is a complex m-manifold X with a holomorphic action of $(\mathbb{C}^*)^m$ which is transitive and free upon a dense open set in X. Toric varieties can be constructed and studied using only a finite amount of combinatorial data.

  The conditions for a smooth hypersurface in a compact toric variety to be a Calabi–Yau m-fold can be calculated using this combinatorial data. Using a computer, one can generate a large (but finite) number of Calabi–Yau m-folds, at least when $m = 3$, and calculate their topological invariants such as Hodge numbers. This has been done by Candelas, and other authors.

- **Resolution of singularities**. Suppose you have some way of producing examples of *singular* Calabi–Yau m-folds Y. Often it is possible to find a *resolution* X of Y with holomorphic map $\pi : X \to Y$, such that X is a nonsingular Calabi–Yau m-fold. Basically, each singular point in Y is replaced by a finite union of complex submanifolds in X.

Resolutions which preserve the Calabi–Yau property are called *crepant resolutions*, and are well understood when $m = 3$. For certain classes of singularities, such as singularities of Calabi–Yau 3-orbifolds, a crepant resolution always exists.

This technique can be applied in a number of ways. For instance, you can start with a nonsingular Calabi–Yau m-fold X, deform it till you get a singular Calabi–Yau m-fold Y, and then resolve the singularities of Y to get a second nonsingular Calabi–Yau m-fold X' with different topology to X.

Another method is to start with a nonsingular Calabi–Yau m-fold X, divide by the action of a finite group G preserving the Calabi–Yau structure to get a singular Calabi–Yau manifold (orbifold) $Y = X/G$, and then resolve the singularities of Y to get a second nonsingular Calabi–Yau m-fold X' with different topology to X.

5.6 Exercises

5.1 The most well-known examples of Calabi–Yau 3-folds are *quintics* X in $\mathbb{CP}^4$, defined by

$$X = \big\{[z_0, \dots, z_4] \in \mathbb{CP}^4 : p(z_0, \dots, z_4) = 0\big\},$$

where $p(z_0, \dots, z_4)$ is a homogeneous quintic polynomial in its arguments. Every nonsingular quintic has Hodge numbers $h^{1,1} = h^{2,2} = 1$ and $h^{2,1} = h^{1,2} = 101$.

(i) Calculate the dimension of the vector space of homogeneous quintic polynomials $p(z_0, \dots, z_4)$. Hence find the dimension of the moduli space of nonsingular quintics in $\mathbb{CP}^4$. (A generic quintic is nonsingular).

(ii) Identify the group of complex automorphisms of $\mathbb{CP}^4$ and calculate its dimension.

(iii) Hence calculate the dimension of the moduli space of quintics in $\mathbb{CP}^4$ up to automorphisms of $\mathbb{CP}^4$.

It is a general fact that if (X, J, g) is a Calabi–Yau 3-fold, then the moduli space of complex deformations of (X, J) has dimension $h^{2,1}(X)$, and each nearby deformation is a Calabi–Yau 3-fold. In this case, $h^{2,1}(X) = 101$, and this should be your answer to (iii). That is, deformations of quintics in $\mathbb{CP}^4$ are also quintics in $\mathbb{CP}^4$.

5.2 One can also construct Calabi–Yau 3-folds as the complete intersection of two cubics in $\mathbb{CP}^5$,

$$X = \big\{[z_0, \dots, z_5] \in \mathbb{CP}^5 : p(z_0, \dots, z_5) = q(z_0, \dots, z_5) = 0\big\},$$

where p, q are linearly independent homogeneous cubic polynomials. Using the method of Question 5.1, calculate the dimension of the moduli space of such complete intersections up to automorphisms of $\mathbb{CP}^5$, and hence predict $h^{2,1}(X)$.

6 Introduction to calibrated geometry

The theory of *calibrated geometry* was invented by Harvey and Lawson [13]. It concerns *calibrated submanifolds*, a special kind of *minimal submanifold* of a Riemannian manifold M, which are defined using a closed form on M called a *calibration*. It is closely connected with the theory of Riemannian holonomy groups because Riemannian manifolds with special holonomy usually come equipped with one or more natural calibrations.

Some references for this section are Harvey and Lawson [13, §I, §II], Harvey [12] and the author [17, §3.7]. Some background reading on minimal submanifolds and Geometric Measure Theory is Lawson [33] and Morgan [37].

6.1 Minimal submanifolds

Let (M, g) be an n-dimensional Riemannian manifold, and N a compact k-dimensional submanifold of M. Regard N as an immersed submanifold (N, ι), with immersion $\iota : N \to M$. Using the metric g we can define the *volume* $\mathrm{Vol}(N)$ of N, by integration over N. We call N a *minimal submanifold* if its volume is stationary under small variations of the immersion $\iota : N \to M$. When $k = 1$, a curve in M is minimal if and only if it is a *geodesic*.

Let $\nu \to N$ be the normal bundle of N in M, so that $TM|_N = TN \oplus \nu$ is an orthogonal direct sum. The *second fundamental form* is a section B of $S^2T^*N \otimes \nu$ such that whenever v, w are vector fields on M with $v|_N, w|_N$ sections of TN over N, then $B \cdot \big(v|_N \otimes w|_N\big) = \pi_\nu\big(\nabla_v w|_N\big)$, where '$\cdot$' contracts S^2T^*N with $TN \otimes TN$, ∇ is the Levi-Civita connection of g, and π_ν is the projection to ν in the splitting $TM|_N = TN \oplus \nu$.

The *mean curvature vector* κ of N is the trace of the second fundamental form B taken using the metric g on N. It is a section of the normal bundle ν. It can be shown by the Euler–Lagrange method that a submanifold N is minimal if and only if its mean curvature vector κ is zero. Note that this is a local condition. Therefore we can also define noncompact submanifolds N in M to be minimal if they have zero mean curvature. This makes sense even when N has infinite volume.

If $\iota : N \to M$ is a immersed submanifold, then the mean curvature κ of N depends on ι and its first and second derivatives, so the condition that N be minimal is a *second-order* equation on ι. Note that minimal submanifolds may not have minimal area, even amongst nearby homologous submanifolds. For instance, the equator in $\mathcal{S}^2$ is minimal, but does not minimize length amongst lines of latitude.

The following argument is important in the study of minimal submanifolds. Let (M, g) be a compact Riemannian manifold, and α a nonzero homology class in $H_k(M, \mathbb{Z})$. We would like to find a compact, minimal immersed, k-dimensional submanifold N in M with homology class $[N] = \alpha$. To do this, we choose a minimizing sequence $(N_i)_{i=1}^{\infty}$ of compact submanifolds N_i with $[N_i] = \alpha$, such that $\mathrm{Vol}(N_i)$ approaches the infimum of volumes of submanifolds with homology class α as $i \to \infty$.

Pretend for the moment that the set of all closed k-dimensional submanifolds N with $\mathrm{Vol}(N) \leqslant C$ is a *compact* topological space. Then there exists a subsequence $(N_{i_j})_{j=1}^{\infty}$ which converges to some submanifold N, which is the minimal submanifold we want. In fact this does not work, because the set of submanifolds N does not have the compactness properties we need.

However, if we work instead with *rectifiable currents*, which are a measure-theoretic generalization of submanifolds, one can show that every integral homology class α in $H_k(M, \mathbb{Z})$ is represented by a minimal rectifiable current. One should think of rectifiable currents as a class of singular submanifolds, obtained by completing the set of nonsingular submanifolds with respect to some norm. They are studied in the subject of *Geometric Measure Theory*.

The question remains: how close are these minimal rectifiable currents to being submanifolds? For example, it is known that a k-dimensional minimal rectifiable current in a Riemannian n-manifold is an embedded submanifold except on a singular set of Hausdorff dimension at most $k-2$. When $k = 2$ or $k = n-1$ one can go further. In general, it is important to understand the possible singularities of such singular minimal submanifolds.

6.2 Calibrations and calibrated submanifolds

Let (M, g) be a Riemannian manifold. An *oriented tangent k-plane V* on M is a vector subspace V of some tangent space $T_x M$ to M with $\dim V = k$, equipped with an orientation. If V is an oriented tangent k-plane on M then $g|_V$ is a Euclidean metric on V, so combining $g|_V$ with the orientation on V gives a natural *volume form* vol_V on V, which is a k-form on V.

Now let φ be a closed k-form on M. We say that φ is a *calibration* on M if for every oriented k-plane V on M we have $\varphi|_V \leqslant \mathrm{vol}_V$. Here $\varphi|_V = \alpha \cdot \mathrm{vol}_V$ for some $\alpha \in \mathbb{R}$, and $\varphi|_V \leqslant \mathrm{vol}_V$ if $\alpha \leqslant 1$. Let N be an oriented submanifold of M with dimension k. Then each tangent space $T_x N$ for $x \in N$ is an oriented tangent k-plane. We say that N is a *calibrated submanifold* or *φ-submanifold* if $\varphi|_{T_x N} = \mathrm{vol}_{T_x N}$ for all $x \in N$.

All calibrated submanifolds are automatically *minimal submanifolds*. We prove this in the compact case, but it is true for noncompact submanifolds as well.

Proposition 6.1 *Let (M, g) be a Riemannian manifold, φ a calibration on M, and N a compact φ-submanifold in M. Then N is volume-minimizing in its homology class.*

Proof. Let $\dim N = k$, and let $[N] \in H_k(M, \mathbb{R})$ and $[\varphi] \in H^k(M, \mathbb{R})$ be the homology and cohomology classes of N and φ. Then

$$[\varphi] \cdot [N] = \int_{x \in N} \varphi\big|_{T_x N} = \int_{x \in N} \mathrm{vol}_{T_x N} = \mathrm{Vol}(N),$$

since $\varphi|_{T_x N} = \mathrm{vol}_{T_x N}$ for each $x \in N$, as N is a calibrated submanifold. If N' is any other compact k-submanifold of M with $[N'] = [N]$ in $H_k(M, \mathbb{R})$, then

$$[\varphi] \cdot [N] = [\varphi] \cdot [N'] = \int_{x \in N'} \varphi\big|_{T_x N'} \leqslant \int_{x \in N'} \mathrm{vol}_{T_x N'} = \mathrm{Vol}(N'),$$

since $\varphi|_{T_x N'} \leqslant \mathrm{vol}_{T_x N'}$ because φ is a calibration. The last two equations give $\mathrm{Vol}(N) \leqslant \mathrm{Vol}(N')$. Thus N is volume-minimizing in its homology class. $\square$

Now let (M, g) be a Riemannian manifold with a calibration φ, and let $\iota : N \to M$ be an immersed submanifold. Whether N is a φ-submanifold depends upon the tangent spaces of N. That is, it depends on ι and its first derivative. So, to be calibrated with respect to φ is a *first-order* equation on ι. But if N is calibrated then N is minimal, and we saw in §6.1 that to be minimal is a *second-order* equation on ι.

One moral is that the calibrated equations, being first-order, are often easier to solve than the minimal submanifold equations, which are second-order. So calibrated geometry is a fertile source of examples of minimal submanifolds.

6.3 Calibrated submanifolds of $\mathbb{R}^n$

One simple class of calibrations is to take (M, g) to be $\mathbb{R}^n$ with the Euclidean metric, and φ to be a constant k-form on $\mathbb{R}^n$, such that $\varphi|_V \leqslant \mathrm{vol}_V$ for every oriented k-dimensional vector subspace $V \subseteq \mathbb{R}^n$. Each such φ defines a class of minimal k-submanifolds in $\mathbb{R}^n$. However, this class may be very small, or even empty. For instance, $\varphi = 0$ is a calibration on $\mathbb{R}^n$, but has no calibrated submanifolds.

For each constant calibration k-form φ on $\mathbb{R}^n$, define $\mathcal{F}_\varphi$ to be the set of oriented k-dimensional vector subspaces V of $\mathbb{R}^n$ such that $\varphi|_V = \mathrm{vol}_V$. Then an oriented submanifold N of $\mathbb{R}^n$ is a φ-submanifold if and only if each tangent space $T_x N$ lies in $\mathcal{F}_\varphi$. To be interesting, a calibration φ should define a fairly abundant class of calibrated submanifolds, and this will only happen if $\mathcal{F}_\varphi$ is reasonably large.

Define a *partial order* $\preceq$ on the set of constant calibration k-forms φ on $\mathbb{R}^n$ by $\varphi \preceq \varphi'$ if $\mathcal{F}_\varphi \subseteq \mathcal{F}_{\varphi'}$. A calibration φ is *maximal* if it is maximal with respect to this partial order. A maximal calibration φ is one in which $\mathcal{F}_\varphi$ is as large as possible.

It is an interesting problem to determine the maximal calibrations φ on $\mathbb{R}^n$. The symmetry group $G \subset \mathrm{O}(n)$ of a maximal calibration is usually quite large. This is because if $V \in \mathcal{F}_\varphi$ and $\gamma \in G$ then $\gamma \cdot V \in \mathcal{F}_\varphi$, that is, G acts on $\mathcal{F}_\varphi$. So if G is big we expect $\mathcal{F}_\varphi$ to be big too. Symmetry groups of maximal calibrations are often possible holonomy groups of Riemannian metrics, and the classification problem for maximal calibrations can be seen as in some ways parallel to the classification problem for Riemannian holonomy groups.

6.4 Calibrated submanifolds and special holonomy

Next we explain the connection with Riemannian holonomy. Let $G \subset \mathrm{O}(n)$ be a possible holonomy group of a Riemannian metric. In particular, we can take G to be one of the holonomy groups $\mathrm{U}(m)$, $\mathrm{SU}(m)$, $\mathrm{Sp}(m)$, G_2 or $\mathrm{Spin}(7)$ from Berger's classification. Then G acts on the k-forms $\Lambda^k(\mathbb{R}^n)^*$ on $\mathbb{R}^n$, so we can look for G-invariant k-forms on $\mathbb{R}^n$.

Suppose φ_0 is a nonzero, G-invariant k-form on $\mathbb{R}^n$. By rescaling φ_0 we can arrange that for each oriented k-plane $U \subset \mathbb{R}^n$ we have $\varphi_0|_U \leqslant \mathrm{vol}_U$, and that $\varphi_0|_U = \mathrm{vol}_U$ for at least one such U. Thus $\mathcal{F}_{\varphi_0}$ is nonempty. Since φ_0 is G-invariant, if $U \in \mathcal{F}_{\varphi_0}$ then $\gamma \cdot U \in \mathcal{F}_{\varphi_0}$ for all $\gamma \in G$. Generally this means that $\mathcal{F}_{\varphi_0}$ is 'reasonably large'.

Let M be a manifold of dimension n, and g a metric on M with Levi-Civita connection ∇ and holonomy group G. Then by Theorem 2.3 there is a k-form φ on M with $\nabla\varphi = 0$, corresponding to φ_0. Hence $\mathrm{d}\varphi = 0$, and φ is closed. Also, the condition $\varphi_0|_U \leqslant \mathrm{vol}_U$ for all oriented k-planes U in $\mathbb{R}^n$ implies that $\varphi|_V \leqslant \mathrm{vol}_V$ for all oriented tangent k-planes V in M. Thus φ is a *calibration* on M.

At each point $x \in M$ the family of oriented tangent k-planes V with $\varphi|_V = \mathrm{vol}_V$ is isomorphic to $\mathcal{F}_{\varphi_0}$, which is 'reasonably large'. This suggests that locally there should exist many φ-submanifolds N in M, so the calibrated geometry of φ on (M, g) is nontrivial.

This gives us a general method for finding interesting calibrations on manifolds with reduced holonomy. Here are the most important examples of this.

- Let $G = \mathrm{U}(m) \subset \mathrm{O}(2m)$. Then G preserves a 2-form ω_0 on $\mathbb{R}^{2m}$. If g is a metric on M with holonomy $\mathrm{U}(m)$ then g is *Kähler* with complex structure J, and the 2-form ω on M associated to ω_0 is the *Kähler form* of g.

  One can show that ω is a calibration on (M, g), and the calibrated submanifolds are exactly the *holomorphic curves* in (M, J). More generally $\omega^k/k!$ is a calibration on M for $1 \leqslant k \leqslant m$, and the corresponding calibrated submanifolds are the complex k-dimensional submanifolds of (M, J).

- Let $G = \mathrm{SU}(m) \subset \mathrm{O}(2m)$. Compact manifolds (M, g) with holonomy $\mathrm{SU}(m)$ extend to *Calabi–Yau m-folds* (M, J, g, Ω), as in §4.5. The real part $\mathrm{Re}\,\Omega$ is a calibration on M, and the corresponding calibrated submanifolds are called *special Lagrangian submanifolds.*

- The group $G_2 \subset \mathrm{O}(7)$ preserves a 3-form φ_0 and a 4-form $*\varphi_0$ on $\mathbb{R}^7$. Thus a Riemannian 7-manifold (M, g) with holonomy G_2 comes with a 3-form φ and 4-form $*\varphi$, which are both calibrations. The corresponding calibrated submanifolds are called *associative 3-folds* and *coassociative 4-folds.*

- The group $\mathrm{Spin}(7) \subset \mathrm{O}(8)$ preserves a 4-form Ω_0 on $\mathbb{R}^8$. Thus a Riemannian 8-manifold (M, g) with holonomy $\mathrm{Spin}(7)$ has a 4-form Ω, which is a calibration. We call Ω-submanifolds *Cayley 4-folds*.

It is an important general principle that to each calibration φ on an n-manifold (M, g) with special holonomy we construct in this way, there corresponds a constant calibration φ_0 on $\mathbb{R}^n$. Locally, φ-submanifolds in M will look very like φ_0-submanifolds in $\mathbb{R}^n$, and have many of the same properties. Thus, to understand the calibrated submanifolds in a manifold with special holonomy, it is often a good idea to start by studying the corresponding calibrated submanifolds of $\mathbb{R}^n$.

In particular, singularities of φ-submanifolds in M will be locally modelled on singularities of φ_0-submanifolds in $\mathbb{R}^n$. (Formally, the *tangent cone* at a singular point of a φ-submanifold in M is a conical φ_0-submanifold in $\mathbb{R}^n$.) So by studying singular φ_0-submanifolds in $\mathbb{R}^n$, we may understand the singular behaviour of φ-submanifolds in M.

6.5 Exercises

6.1 The metric g and Kähler form ω on $\mathbb{C}^m$ are given by

$$g = |\mathrm{d}z_1|^2 + \cdots + |\mathrm{d}z_m|^2 \quad \text{and} \quad \omega = \frac{i}{2}(\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \cdots + \mathrm{d}z_m \wedge \mathrm{d}\bar{z}_m).$$

Show that a tangent 2-plane in $\mathbb{C}^m$ is calibrated w.r.t. ω if and only if it is a complex line in $\mathbb{C}^m$. (Harder) generalize to tangent $2k$-planes and $\frac{1}{k!}\,\omega^k$.

7 Special Lagrangian submanifolds in $\mathbb{C}^m$

We now discuss special Lagrangian submanifolds in $\mathbb{C}^m$. A reference for this section is Harvey and Lawson [13, §III.1–§III.2].

Definition 7.1 Let $\mathbb{C}^m \cong \mathbb{R}^{2m}$ have complex coordinates $(z_1, \ldots, z_m)$ and complex structure I, and define a metric g, Kähler form ω and complex volume form Ω on $\mathbb{C}^m$ by

$$\begin{gathered} g = |\mathrm{d}z_1|^2 + \cdots + |\mathrm{d}z_m|^2, \quad \omega = \frac{i}{2}(\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \cdots + \mathrm{d}z_m \wedge \mathrm{d}\bar{z}_m), \\ \text{and} \quad \Omega = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_m. \end{gathered} \tag{13}$$

Then $\mathrm{Re}\,\Omega$ and $\mathrm{Im}\,\Omega$ are real m-forms on $\mathbb{C}^m$. Let L be an oriented real submanifold of $\mathbb{C}^m$ of real dimension m. We call L a *special Lagrangian submanifold* of $\mathbb{C}^m$, or *SL m-fold* for short, if L is calibrated with respect to $\mathrm{Re}\,\Omega$, in the sense of §6.2.

In fact there is a more general definition involving a *phase* $\mathrm{e}^{i\theta}$: if $\theta \in [0, 2\pi)$, we say that L is *special Lagrangian with phase* $\mathrm{e}^{i\theta}$ if it is calibrated with respect to $\cos\theta\ \mathrm{Re}\,\Omega + \sin\theta\ \mathrm{Im}\,\Omega$. But we will not use this.

We shall identify the family $\mathcal{F}$ of tangent m-planes in $\mathbb{C}^m$ calibrated with respect to $\operatorname{Re}\Omega$. The subgroup of $\mathrm{GL}(2m,\mathbb{R})$ preserving g,ω and Ω is the Lie group $\mathrm{SU}(m)$ of complex unitary matrices with determinant 1. Define a real vector subspace U in $\mathbb{C}^m$ to be

$$U=\big\{(x_1,\ldots,x_m):x_j\in\mathbb{R}\big\}\subset\mathbb{C}^m, \tag{14}$$

and let U have the usual orientation. Then U is calibrated w.r.t. $\operatorname{Re}\Omega$.

Furthermore, any oriented real vector subspace V in $\mathbb{C}^m$ calibrated w.r.t. $\operatorname{Re}\Omega$ is of the form $V=\gamma\cdot U$ for some $\gamma\in\mathrm{SU}(m)$. Therefore $\mathrm{SU}(m)$ acts transitively on $\mathcal{F}$. The stabilizer subgroup of U in $\mathrm{SU}(m)$ is the subset of matrices in $\mathrm{SU}(m)$ with real entries, which is $\mathrm{SO}(m)$. Thus $\mathcal{F}\cong\mathrm{SU}(m)/\mathrm{SO}(m)$, and we prove:

Proposition 7.2 *The family $\mathcal{F}$ of oriented real m-dimensional vector subspaces V in $\mathbb{C}^m$ with $\operatorname{Re}\Omega|_V=\mathrm{vol}_V$ is isomorphic to $\mathrm{SU}(m)/\mathrm{SO}(m)$, and has dimension $\frac{1}{2}(m^2+m-2)$.*

The dimension follows because $\dim\mathrm{SU}(m)=m^2-1$ and $\dim\mathrm{SO}(m)=\frac{1}{2}m(m-1)$. It is easy to see that $\omega|_U=\operatorname{Im}\Omega|_U=0$. As $\mathrm{SU}(m)$ preserves ω and $\operatorname{Im}\Omega$ and acts transitively on $\mathcal{F}$, it follows that $\omega|_V=\operatorname{Im}\Omega|_V=0$ for any $V\in\mathcal{F}$. Conversely, if V is a real m-dimensional vector subspace of $\mathbb{C}^m$ and $\omega|_V=\operatorname{Im}\Omega|_V=0$, then V lies in $\mathcal{F}$, with some orientation. This implies an alternative characterization of special Lagrangian submanifolds, [13, Cor. III.1.11]:

Proposition 7.3 *Let L be a real m-dimensional submanifold of $\mathbb{C}^m$. Then L admits an orientation making it into a special Lagrangian submanifold of $\mathbb{C}^m$ if and only if $\omega|_L\equiv 0$ and $\operatorname{Im}\Omega|_L\equiv 0$.*

Note that an m-dimensional submanifold L in $\mathbb{C}^m$ is called *Lagrangian* if $\omega|_L\equiv 0$. (This is a term from symplectic geometry, and ω is a symplectic structure.) Thus special Lagrangian submanifolds are Lagrangian submanifolds satisfying the extra condition that $\operatorname{Im}\Omega|_L\equiv 0$, which is how they get their name.

7.1 Special Lagrangian 2-folds in $\mathbb{C}^2$ and the quaternions

The smallest interesting dimension, $m=2$, is a special case. Let $\mathbb{C}^2$ have complex coordinates (z_1,z_2), complex structure I, and metric g, Kähler form ω and holomorphic 2-form Ω defined in (13). Define real coordinates (x_0,x_1,x_2,x_3) on $\mathbb{C}^2\cong\mathbb{R}^4$ by $z_0=x_0+ix_1$, $z_1=x_2+ix_3$. Then

$$\begin{aligned} g&=\mathrm{d}x_0^2+\cdots+\mathrm{d}x_3^2, & \omega&=\mathrm{d}x_0\wedge\mathrm{d}x_1+\mathrm{d}x_2\wedge\mathrm{d}x_3,\\ \operatorname{Re}\Omega&=\mathrm{d}x_0\wedge\mathrm{d}x_2-\mathrm{d}x_1\wedge\mathrm{d}x_3 \quad\text{and} & \operatorname{Im}\Omega&=\mathrm{d}x_0\wedge\mathrm{d}x_3+\mathrm{d}x_1\wedge\mathrm{d}x_2.\end{aligned}$$

Now define a *different* set of complex coordinates (w_1,w_2) on $\mathbb{C}^2=\mathbb{R}^4$ by $w_1=x_0+ix_2$ and $w_2=x_1-ix_3$. Then $\omega-i\operatorname{Im}\Omega=\mathrm{d}w_1\wedge\mathrm{d}w_2$.

But by Proposition 7.3, a real 2-submanifold $L \subset \mathbb{R}^4$ is special Lagrangian if and only if $\omega|_L \equiv \operatorname{Im}\Omega|_L \equiv 0$. Thus, L is special Lagrangian if and only if $(\mathrm{d}w_1 \wedge \mathrm{d}w_2)|_L \equiv 0$. But this holds if and only if L is a *holomorphic curve* with respect to the complex coordinates (w_1, w_2).

Here is another way to say this. There are *two different* complex structures I and J involved in this problem, associated to the two different complex coordinate systems (z_1, z_2) and (w_1, w_2) on $\mathbb{R}^4$. In the coordinates $(x_0, \ldots, x_3)$, I and J are given by

$$\begin{aligned} &I\big(\tfrac{\partial}{\partial x_0}\big) = \tfrac{\partial}{\partial x_1}, \quad I\big(\tfrac{\partial}{\partial x_1}\big) = -\tfrac{\partial}{\partial x_0}, \quad I\big(\tfrac{\partial}{\partial x_2}\big) = \tfrac{\partial}{\partial x_3}, \quad I\big(\tfrac{\partial}{\partial x_3}\big) = -\tfrac{\partial}{\partial x_2}, \\ &J\big(\tfrac{\partial}{\partial x_0}\big) = \tfrac{\partial}{\partial x_2}, \quad J\big(\tfrac{\partial}{\partial x_1}\big) = -\tfrac{\partial}{\partial x_3}, \quad J\big(\tfrac{\partial}{\partial x_2}\big) = -\tfrac{\partial}{\partial x_0}, \quad J\big(\tfrac{\partial}{\partial x_3}\big) = \tfrac{\partial}{\partial x_1}. \end{aligned}$$

The usual complex structure on $\mathbb{C}^2$ is I, but a 2-fold L in $\mathbb{C}^2$ is special Lagrangian if and only if it is holomorphic w.r.t. the alternative complex structure J. This means that special Lagrangian 2-folds are already very well understood, so we generally focus our attention on dimensions $m \geqslant 3$.

We can express all this in terms of the *quaternions* $\mathbb{H}$. The complex structures I, J anticommute, so that $IJ = -JI$, and $K = IJ$ is also a complex structure on $\mathbb{R}^4$, and $\langle 1, I, J, K\rangle$ is an algebra of automorphisms of $\mathbb{R}^4$ isomorphic to $\mathbb{H}$.

7.2 Special Lagrangian submanifolds in $\mathbb{C}^m$ as graphs

In symplectic geometry, there is a well-known way of manufacturing *Lagrangian* submanifolds of $\mathbb{R}^{2m} \cong \mathbb{C}^m$, which works as follows. Let $f : \mathbb{R}^m \to \mathbb{R}$ be a smooth function, and define

$$\Gamma_f = \big\{\big(x_1 + i\tfrac{\partial f}{\partial x_1}(x_1, \ldots, x_m), \ldots, x_m + i\tfrac{\partial f}{\partial x_m}(x_1, \ldots, x_m)\big) : x_1, \ldots, x_m \in \mathbb{R}\big\}.$$

Then Γ_f is a smooth real m-dimensional submanifold of $\mathbb{C}^m$, with $\omega|_{\Gamma_f} \equiv 0$. Identifying $\mathbb{C}^m \cong \mathbb{R}^{2m} \cong \mathbb{R}^m \times (\mathbb{R}^m)^*$, we may regard Γ_f as the graph of the 1-form $\mathrm{d}f$ on $\mathbb{R}^m$, so that Γ_f is the *graph of a closed* 1*-form*. Locally, but not globally, every Lagrangian submanifold arises from this construction.

Now by Proposition 7.3, a special Lagrangian m-fold in $\mathbb{C}^m$ is a Lagrangian m-fold L satisfying the additional condition that $\operatorname{Im}\Omega|_L \equiv 0$. We shall find the condition for Γ_f to be a special Lagrangian m-fold. Define the *Hessian* $\operatorname{Hess} f$ of f to be the $m \times m$ matrix $\big(\frac{\partial^2 f}{\partial x_i \partial x_j}\big)_{i,j=1}^m$ of real functions on $\mathbb{R}^m$. Then it is easy to show that $\operatorname{Im}\Omega|_{\Gamma_f} \equiv 0$ if and only if

$$\operatorname{Im} \det{}_{\mathbb{C}}\big(I + i \operatorname{Hess} f\big) \equiv 0 \quad \text{on } \mathbb{C}^m. \tag{15}$$

This is a *nonlinear second-order elliptic partial differential equation* upon the function $f : \mathbb{R}^m \to \mathbb{R}$.

7.3 Local discussion of special Lagrangian deformations

Suppose L_0 is a special Lagrangian submanifold in $\mathbb{C}^m$ (or, more generally, in some Calabi–Yau m-fold). What can we say about the family of *special Lagrangian deformations* of L_0, that is, the set of special Lagrangian m-folds L that are 'close to L_0' in a suitable sense? Essentially, deformation theory is one way of thinking about the question 'how many special Lagrangian submanifolds are there in $\mathbb{C}^m$'?

Locally (that is, in small enough open sets), every special Lagrangian m-fold looks quite like $\mathbb{R}^m$ in $\mathbb{C}^m$. Therefore deformations of special Lagrangian m-folds should look like special Lagrangian deformations of $\mathbb{R}^m$ in $\mathbb{C}^m$. So, we would like to know what special Lagrangian m-folds L in $\mathbb{C}^m$ close to $\mathbb{R}^m$ look like.

Now $\mathbb{R}^m$ is the graph Γ_f associated to the function $f \equiv 0$. Thus, a graph Γ_f will be close to $\mathbb{R}^m$ if the function f and its derivatives are small. But then $\mathrm{Hess}\, f$ is small, so we can approximate equation (15) by its *linearization*. For

$$\operatorname{Im} \det_{\mathbb{C}}\bigl(I + i \,\mathrm{Hess}\, f\bigr) = \mathrm{Tr}\,\mathrm{Hess}\, f + \text{higher order terms}.$$

Thus, when the second derivatives of f are small, equation (15) reduces approximately to $\mathrm{Tr}\,\mathrm{Hess}\, f \equiv 0$. But $\mathrm{Tr}\,\mathrm{Hess}\, f = \frac{\partial^2 f}{(\partial x_1)^2} + \cdots + \frac{\partial^2 f}{(\partial x_m)^2} = \Delta f$, where Δ is the *Laplacian* on $\mathbb{R}^m$.

Hence, the small special Lagrangian deformations of $\mathbb{R}^m$ in $\mathbb{C}^m$ are approximately parametrized by small *harmonic functions* on $\mathbb{R}^m$. Actually, because adding a constant to f has no effect on Γ_f, this parametrization is degenerate. We can get round this by parametrizing instead by $\mathrm{d}f$, which is a closed and coclosed 1-form. This justifies the following:

Principle. *Small special Lagrangian deformations of a special Lagrangian m-fold L are approximately parametrized by closed and coclosed* 1*-forms α on L.*

This is the idea behind McLean's Theorem, Theorem 9.4 below.

We have seen using (15) that the deformation problem for special Lagrangian m-folds can be written as an *elliptic equation*. In particular, there are the same number of equations as functions, so the problem is neither overdetermined nor underdetermined. Therefore we do not expect special Lagrangian m-folds to be very few and very rigid (as would be the case if (15) were overdetermined), nor to be very abundant and very flabby (as would be the case if (15) were underdetermined).

If we think about Proposition 7.2 for a while, this may seem surprising. For the set $\mathcal{F}$ of special Lagrangian m-planes in $\mathbb{C}^m$ has dimension $\frac{1}{2}(m^2+m-2)$, but the set of all real m-planes in $\mathbb{C}^m$ has dimension m^2. So the special Lagrangian m-planes have codimension $\frac{1}{2}(m^2 - m + 2)$ in the set of all m-planes.

This means that the condition for a real m-submanifold L in $\mathbb{C}^m$ to be special Lagrangian is $\frac{1}{2}(m^2-m+2)$ real equations on each tangent space of L. However, the freedom to vary L is the sections of its normal bundle in $\mathbb{C}^m$, which is m

real functions. When $m \geqslant 3$, there are more equations than functions, so we would expect the deformation problem to be *overdetermined*.

The explanation is that because ω is a *closed* 2-form, submanifolds L with $\omega|_L \equiv 0$ are much more abundant than would otherwise be the case. So the closure of ω is a kind of integrability condition necessary for the existence of many special Lagrangian submanifolds, just as the integrability of an almost complex structure is a necessary condition for the existence of many complex submanifolds of dimension greater than 1 in a complex manifold.

7.4 Exercises

7.1 Find your own proofs of Propositions 7.2 and 7.3.

8 Constructions of SL m-folds in $\mathbb{C}^m$

We now describe five methods of constructing special Lagrangian m-folds in $\mathbb{C}^m$, drawn from papers by the author [18, 19, 20, 21, 22, 24, 25, 26], Bryant [3], Castro and Urbano [4], Goldstein [6, 7], Harvey [12, p. 139–143], Harvey and Lawson [13, §III], Haskins [14], Lawlor [32], Ma and Ma [34], McIntosh [35] and Sharipov [41]. These yield many examples of singular SL m-folds, and so hopefully will help in understanding what general singularities of SL m-folds in Calabi–Yau m-folds are like.

8.1 SL m-folds with large symmetry groups

Here is a method used in [18] (and also by Harvey and Lawson [13, §III.3], Haskins [14] and Goldstein [6, 7]) to construct examples of SL m-folds in $\mathbb{C}^m$. The group $\mathrm{SU}(m) \ltimes \mathbb{C}^m$ acts on $\mathbb{C}^m$ preserving all the structure g, ω, Ω, so that it takes SL m-folds to SL m-folds in $\mathbb{C}^m$. Let G be a Lie subgroup of $\mathrm{SU}(m) \ltimes \mathbb{C}^m$ with Lie algebra $\mathfrak{g}$, and N a connected G-invariant SL m-fold in $\mathbb{C}^m$.

Since G preserves the symplectic form ω on $\mathbb{C}^m$, one can show that it has a *moment map* $\mu : \mathbb{C}^m \to \mathfrak{g}^*$. As N is Lagrangian, one can show that μ is constant on N, that is, $\mu \equiv c$ on N for some $c \in Z(\mathfrak{g}^*)$, the *centre* of $\mathfrak{g}^*$.

If the orbits of G in N are of codimension 1 (that is, dimension $m-1$), then N is a 1-parameter family of G-orbits $\mathcal{O}_t$ for $t \in \mathbb{R}$. After reparametrizing the variable t, it can be shown that the special Lagrangian condition is equivalent to an o.d.e. in t upon the orbits $\mathcal{O}_t$.

Thus, we can construct examples of cohomogeneity one SL m-folds in $\mathbb{C}^m$ by solving an o.d.e. in the family of $(m-1)$-dimensional G-orbits $\mathcal{O}$ in $\mathbb{C}^m$ with $\mu|_{\mathcal{O}} \equiv c$, for fixed $c \in Z(\mathfrak{g}^*)$. This o.d.e. usually turns out to be *integrable*.

Now suppose N is a *special Lagrangian cone* in $\mathbb{C}^m$, invariant under a subgroup $G \subset \mathrm{SU}(m)$ which has orbits of dimension $m-2$ in N. In effect the symmetry group of N is $G \times \mathbb{R}_+$, where $\mathbb{R}_+$ acts by *dilations*, as N is a cone. Thus, in this situation too the symmetry group of N acts with cohomogeneity one, and we again expect the problem to reduce to an o.d.e.

One can show that $N \cap \mathcal{S}^{2m-1}$ is a 1-parameter family of G-orbits $\mathcal{O}_t$ in $\mathcal{S}^{2m-1} \cap \mu^{-1}(0)$ satisfying an o.d.e. By solving this o.d.e. we construct SL cones in $\mathbb{C}^m$. When $G = \mathrm{U}(1)^{m-2}$, the o.d.e. has many *periodic solutions* which give large families of distinct SL cones on T^{m-1}. In particular, we can find many examples of SL T^2-cones in $\mathbb{C}^3$.

8.2 Evolution equations for SL m-folds

The following method was used in [19] and [20] to construct many examples of SL m-folds in $\mathbb{C}^m$. A related but less general method was used by Lawlor [32], and completed by Harvey [12, p. 139–143].

Let P be a real analytic $(m-1)$-dimensional manifold, and χ a nonvanishing real analytic section of $\Lambda^{m-1}TP$. Let $\{\phi_t : t \in \mathbb{R}\}$ be a 1-parameter family of real analytic maps $\phi_t : P \to \mathbb{C}^m$. Consider the o.d.e.

$$\left(\frac{\mathrm{d}\phi_t}{\mathrm{d}t}\right)^b = (\phi_t)_*(\chi)^{a_1 \ldots a_{m-1}} (\operatorname{Re}\Omega)_{a_1 \ldots a_{m-1} a_m} g^{a_m b}, \tag{16}$$

using the index notation for (real) tensors on $\mathbb{C}^m$, where g^{ab} is the inverse of the Euclidean metric g_{ab} on $\mathbb{C}^m$.

It is shown in [19, §3] that if the ϕ_t satisfy (16) and $\phi_0^*(\omega) \equiv 0$, then $\phi_t^*(\omega) \equiv 0$ for all t, and $N = \bigl\{\phi_t(p) : p \in P,\, t \in \mathbb{R}\bigr\}$ is an SL m-fold in $\mathbb{C}^m$ wherever it is nonsingular. We think of (16) as an *evolution equation*, and N as the result of evolving a 1-parameter family of $(m-1)$-submanifolds $\phi_t(P)$ in $\mathbb{C}^m$.

Here is one way to understand this result. Suppose we are given $\phi_t : P \to \mathbb{C}^m$ for some t, and we want to find an SL m-fold N in $\mathbb{C}^m$ containing the $(m-1)$-submanifold $\phi_t(P)$. As N is Lagrangian, a necessary condition for this is that $\omega|_{\phi_t(P)} \equiv 0$, and hence $\phi_t^*(\omega) \equiv 0$ on P.

The effect of equation (16) is to flow $\phi_t(P)$ in the direction in which $\operatorname{Re}\Omega$ is 'largest'. The result is that $\operatorname{Re}\Omega$ is 'maximized' on N, given the initial conditions. But $\operatorname{Re}\Omega$ is maximal on N exactly when N is calibrated w.r.t. $\operatorname{Re}\Omega$, that is, when N is special Lagrangian. The same technique also works for other calibrations, such as the associative and coassociative calibrations on $\mathbb{R}^7$, and the Cayley calibration on $\mathbb{R}^8$.

Now (16) evolves amongst the infinite-dimensional family of real analytic maps $\phi : P \to \mathbb{C}^m$ with $\phi^*(\omega) \equiv 0$, so it is an *infinite-dimensional* problem, and thus difficult to solve explicitly. However, there are *finite-dimensional* families $\mathcal{C}$ of maps $\phi : P \to \mathbb{C}^m$ such that evolution stays in $\mathcal{C}$. This gives a *finite-dimensional* o.d.e., which can hopefully be solved fairly explicitly. For example, if we take G to be a Lie subgroup of $\mathrm{SU}(m) \ltimes \mathbb{C}^m$, P to be an $(m–1)$-dimensional homogeneous space G/H, and $\phi : P \to \mathbb{C}^m$ to be G-equivariant, we recover the construction of §8.1.

But there are also other possibilities for $\mathcal{C}$ which do not involve a symmetry assumption. Suppose P is a submanifold of $\mathbb{R}^n$, and χ the restriction to P of a linear or affine map $\mathbb{R}^n \to \Lambda^{m-1}\mathbb{R}^n$. (This is a strong condition on P and χ.) Then we can take $\mathcal{C}$ to be the set of restrictions to P of linear or affine maps $\mathbb{R}^n \to \mathbb{C}^m$.

For instance, set $m = n$ and let P be a quadric in $\mathbb{R}^m$. Then one can construct SL m-folds in $\mathbb{C}^m$ with few symmetries by evolving quadrics in Lagrangian planes $\mathbb{R}^m$ in $\mathbb{C}^m$. When P is a quadric cone in $\mathbb{R}^m$ this gives many SL cones on products of spheres $\mathcal{S}^a \times \mathcal{S}^b \times \mathcal{S}^1$.

8.3 Ruled special Lagrangian 3-folds

A 3-submanifold N in $\mathbb{C}^3$ is called *ruled* if it is fibred by a 2-dimensional family $\mathcal{F}$ of real lines in $\mathbb{C}^3$. A *cone* N_0 in $\mathbb{C}^3$ is called *two-sided* if $N_0 = -N_0$. Two-sided cones are automatically ruled. If N is a ruled 3-fold in $\mathbb{C}^3$, we define the *asymptotic cone* N_0 of N to be the two-sided cone fibred by the lines passing through 0 and parallel to those in $\mathcal{F}$.

Ruled SL 3-folds are studied in [21], and also by Harvey and Lawson [13, §III.3.C, §III.4.B] and Bryant [3, §3]. Each (oriented) real line in $\mathbb{C}^3$ is determined by its *direction* in $\mathcal{S}^5$ together with an orthogonal *translation* from the origin. Thus a ruled 3-fold N is determined by a 2-dimensional family of directions and translations.

The condition for N to be special Lagrangian turns out [21, §5] to reduce to two equations, the first involving only the direction components, and the second *linear* in the translation components. Hence, if a ruled 3-fold N in $\mathbb{C}^3$ is special Lagrangian, then so is its asymptotic cone N_0. Conversely, the ruled SL 3-folds N asymptotic to a given two-sided SL cone N_0 come from solutions of a linear equation, and so form a *vector space*.

Let N_0 be a two-sided SL cone, and let $\Sigma = N_0 \cap \mathcal{S}^5$. Then Σ is a *Riemann surface*. Holomorphic vector fields on Σ give solutions to the linear equation (though not all solutions) [21, §6], and so yield new ruled SL 3-folds. In particular, each SL T^2-cone gives a 2-dimensional family of ruled SL 3-folds, which are generically diffeomorphic to $T^2 \times \mathbb{R}$ as immersed 3-submanifolds.

8.4 Integrable systems

Let N_0 be a special Lagrangian cone in $\mathbb{C}^3$, and set $\Sigma = N_0 \cap \mathcal{S}^5$. As N_0 is calibrated, it is minimal in $\mathbb{C}^3$, and so Σ is minimal in $\mathcal{S}^5$. That is, Σ is a *minimal Legendrian surface* in $\mathcal{S}^5$. Let $\pi : \mathcal{S}^5 \to \mathbb{CP}^2$ be the Hopf projection. One can also show that $\pi(\Sigma)$ is a *minimal Lagrangian surface* in $\mathbb{CP}^2$.

Regard Σ as a *Riemann surface*. Then the inclusions $\iota : \Sigma \to \mathcal{S}^5$ and $\pi \circ \iota : \Sigma \to \mathbb{CP}^2$ are *conformal harmonic maps*. Now harmonic maps from Riemann surfaces into $\mathcal{S}^n$ and $\mathbb{CP}^m$ are an *integrable system*. There is a complicated theory for classifying them in terms of algebro-geometric 'spectral data', and finding 'explicit' solutions. In principle, this gives all harmonic maps from T^2 into $\mathcal{S}^n$ and $\mathbb{CP}^m$. So, the field of integrable systems offers the hope of a *classification* of all SL T^2-cones in $\mathbb{C}^3$.

For a good general introduction to this field, see Fordy and Wood [5]. Sharipov [41] and Ma and Ma [34] apply this integrable systems machinery to describe minimal Legendrian tori in $\mathcal{S}^5$, and minimal Lagrangian tori in $\mathbb{CP}^2$, respectively, giving explicit formulae in terms of Prym theta functions. McIntosh [35] provides a more recent, readable, and complete discussion of special Lagrangian cones in $\mathbb{C}^3$ from the integrable systems perspective.

The families of SL T^2-cones constructed by U(1)-invariance in §8.1, and by evolving quadrics in §8.2, turn out to come from a more general, very explicit, 'integrable systems' family of conformal harmonic maps $\mathbb{R}^2 \to \mathcal{S}^5$ with Legendrian image, involving two commuting, integrable o.d.e.s., described in [22]. So, we can fit some of our examples into the integrable systems framework.

However, we know a good number of other constructions of SL m-folds in $\mathbb{C}^m$ which have the classic hallmarks of integrable systems — elliptic functions, commuting o.d.e.s, and so on — but which are not yet understood from the point of view of integrable systems. I would like to ask the integrable systems community: do SL m-folds in $\mathbb{C}^m$ for $m \geqslant 3$, or at least some classes of such submanifolds, constitute some kind of higher-dimensional integrable system?

8.5 Analysis and U(1)-invariant SL 3-folds in $\mathbb{C}^3$

Next we summarize the author's three papers [24, 25, 26], which study SL 3-folds N in $\mathbb{C}^3$ invariant under the U(1)-action

$$\mathrm{e}^{i\theta} : (z_1, z_2, z_3) \mapsto (\mathrm{e}^{i\theta} z_1, \mathrm{e}^{-i\theta} z_2, z_3) \quad \text{for } \mathrm{e}^{i\theta} \in \mathrm{U}(1). \tag{17}$$

These papers are briefly surveyed in [27]. Locally we can write N in the form

$$\begin{aligned} N = \bigl\{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 z_2 = v(x, y) + iy, \quad z_3 = x + iu(x, y),& \\ |z_1|^2 - |z_2|^2 = 2a, \quad (x, y) \in S\bigr\},& \end{aligned} \tag{18}$$

where S is a domain in $\mathbb{R}^2$, $a \in \mathbb{R}$ and $u, v : S \to \mathbb{R}$ are continuous.

Here we may take $|z_1|^2 - |z_2|^2 = 2a$ to be one of the equations defining N as $|z_1|^2 - |z_2|^2$ is the *moment map* of the U(1)-action (17), and so $|z_1|^2 - |z_2|^2$ is constant on any U(1)-invariant Lagrangian 3-fold in $\mathbb{C}^3$. Effectively (18) just means that we are choosing $x = \mathrm{Re}(z_3)$ and $y = \mathrm{Im}(z_1 z_2)$ as local coordinates on the 2-manifold $N/\,\mathrm{U}(1)$. Then we find [24, Prop. 4.1]:

Proposition 8.1 *Let S, a, u, v and N be as above. Then*

(a) *If $a = 0$, then N is a (possibly singular) SL 3-fold in $\mathbb{C}^3$ if u, v are differentiable and satisfy*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \textit{and} \quad \frac{\partial v}{\partial x} = -2\bigl(v^2 + y^2\bigr)^{1/2} \frac{\partial u}{\partial y}, \tag{19}$$

except at points $(x, 0)$ in S with $v(x, 0) = 0$, where u, v need not be differentiable. The singular points of N are those of the form $(0, 0, z_3)$, where $z_3 = x + iu(x, 0)$ for $(x, 0) \in S$ with $v(x, 0) = 0$.

(b) *If $a \neq 0$, then N is a nonsingular SL 3-fold in $\mathbb{C}^3$ if and only if u, v are differentiable in S and satisfy*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \textit{and} \quad \frac{\partial v}{\partial x} = -2\bigl(v^2 + y^2 + a^2\bigr)^{1/2} \frac{\partial u}{\partial y}. \tag{20}$$

Now (19) and (20) are *nonlinear Cauchy–Riemann equations*. Thus, we may treat $u + iv$ as like a holomorphic function of $x + iy$. Many of the results in [24, 25, 26] are analogues of well-known results in elementary complex analysis.

In [24, Prop. 7.1] we show that solutions $u, v \in C^1(S)$ of (20) come from a potential $f \in C^2(S)$ satisfying a second-order quasilinear elliptic equation.

Proposition 8.2 *Let S be a domain in $\mathbb{R}^2$ and $u, v \in C^1(S)$ satisfy* (20) *for $a \neq 0$. Then there exists $f \in C^2(S)$ with $\frac{\partial f}{\partial y} = u$, $\frac{\partial f}{\partial x} = v$ and*

$$P(f) = \left(\left(\frac{\partial f}{\partial x}\right)^2 + y^2 + a^2\right)^{-1/2} \frac{\partial^2 f}{\partial x^2} + 2\,\frac{\partial^2 f}{\partial y^2} = 0. \tag{21}$$

This f is unique up to addition of a constant, $f \mapsto f + c$. Conversely, all solutions of (21) *yield solutions of* (20).

In the following result, a condensation of [24, Th. 7.6] and [25, Th.s 9.20 & 9.21], we prove existence and uniqueness for the *Dirichlet problem* for (21).

Theorem 8.3 *Suppose S is a strictly convex domain in $\mathbb{R}^2$ invariant under $(x, y) \mapsto (x, -y)$, and $\alpha \in (0, 1)$. Let $a \in \mathbb{R}$ and $\phi \in C^{3,\alpha}(\partial S)$. Then if $a \neq 0$ there exists a unique solution f of* (21) *in $C^{3,\alpha}(S)$ with $f|_{\partial S} = \phi$. If $a = 0$ there exists a unique $f \in C^1(S)$ with $f|_{\partial S} = \phi$, which is twice weakly differentiable and satisfies* (21) *with weak derivatives. Furthermore, the map $C^{3,\alpha}(\partial S) \times \mathbb{R} \to C^1(S)$ taking $(\phi, a) \mapsto f$ is continuous.*

Here a domain S in $\mathbb{R}^2$ is *strictly convex* if it is convex and the curvature of ∂S is nonzero at each point. Also domains are by definition compact, with smooth boundary, and $C^{3,\alpha}(\partial S)$ and $C^{3,\alpha}(S)$ are *Hölder spaces* of functions on ∂S and S. For more details see [24, 25].

Combining Propositions 8.1 and 8.2 and Theorem 8.3 gives existence and uniqueness for a large class of U(1)-invariant SL 3-folds in $\mathbb{C}^3$, with boundary conditions, and including *singular* SL 3-folds. It is interesting that this existence and uniqueness is *entirely unaffected* by singularities appearing in S°.

Here are some other areas covered in [24, 25, 26]. Examples of solutions u, v of (19) and (20) are given in [24, §5]. In [25] we give more precise statements on the regularity of singular solutions of (19) and (21). In [24, §6] and [26, §7] we consider the zeroes of $(u_1, v_1) - (u_2, v_2)$, where (u_j, v_j) are (possibly singular) solutions of (19) and (20).

We show that if $(u_1, v_1) \not\equiv (u_2, v_2)$ then the zeroes of $(u_1, v_1) - (u_2, v_2)$ in S° are *isolated*, with a positive integer *multiplicity*, and that the zeroes of $(u_1, v_1) - (u_2, v_2)$ in S° can be counted with multiplicity in terms of boundary data on ∂S. In particular, under some boundary conditions we can show $(u_1, v_1) - (u_2, v_2)$ has no zeroes in S°, so that the corresponding SL 3-folds do not intersect. This will be important in constructing U(1)-invariant SL fibrations in §11.5.

In [26, §9–§10] we study singularities of solutions u, v of (19). We show that either $u(x, -y) \equiv u(x, y)$ and $v(x, -y) \equiv -v(x, y)$, so that u, v are singular all along the x-axis, or else the singular points of u, v in S° are all *isolated*, with a positive integer *multiplicity*, and one of two *types*. We also show that singularities exist with every multiplicity and type, and multiplicity n singularities occur in codimension n in the family of all U(1)-invariant SL 3-folds.

8.6 Examples of singular special Lagrangian 3-folds in $\mathbb{C}^3$

We shall now describe four families of SL 3-folds in $\mathbb{C}^3$, as examples of the material of §8.1–§8.4. They have been chosen to illustrate different kinds of singular behaviour of SL 3-folds, and also to show how nonsingular SL 3-folds can converge to a singular SL 3-fold, to serve as a preparation for our discussion of singularities of SL m-folds in §10.

Our first example derives from Harvey and Lawson [13, §III.3.A], and is discussed in detail in [16, §3] and [28, §4].

Example 8.4 Define a subset L_0 in $\mathbb{C}^3$ by

$$L_0 = \bigl\{(r\mathrm{e}^{i\theta_1}, r\mathrm{e}^{i\theta_2}, r\mathrm{e}^{i\theta_3}) : r \geqslant 0, \quad \theta_1, \theta_2, \theta_3 \in \mathbb{R}, \quad \theta_1 + \theta_2 + \theta_3 = 0\bigr\}.$$

Then L_0 is a *special Lagrangian cone* on T^2. An alternative definition is

$$L_0 = \bigl\{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1| = |z_2| = |z_3|, \ \mathrm{Im}(z_1z_2z_3) = 0, \ \mathrm{Re}(z_1z_2z_3) \geqslant 0\bigr\}.$$

Let $t > 0$, write $\mathcal{S}^1 = \bigl\{\mathrm{e}^{i\theta} : \theta \in \mathbb{R}\bigr\}$, and define a map $\phi_t : \mathcal{S}^1 \times \mathbb{C} \to \mathbb{C}^3$ by

$$\phi_t : (e^{i\theta}, z) \mapsto \bigl((|z|^2 + t^2)^{1/2}\mathrm{e}^{i\theta}, z, e^{-i\theta}\bar{z}\bigr).$$

Then ϕ_t is an *embedding*. Define $L_t = \mathrm{Image}\,\phi_t$. Then L_t is a nonsingular special Lagrangian 3-fold in $\mathbb{C}^3$ diffeomorphic to $\mathcal{S}^1 \times \mathbb{R}^2$. An equivalent definition is

$$\begin{aligned} L_t = \bigl\{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 - t^2 = |z_2|^2 = |z_3|^2,& \\ \mathrm{Im}(z_1z_2z_3) = 0, \quad \mathrm{Re}(z_1z_2z_3) \geqslant 0\bigr\}&. \end{aligned}$$

As $t \to 0_+$, the nonsingular SL 3-fold L_t converges to the singular SL cone L_0. Note that L_t is *asymptotic* to L_0 at infinity, and that $L_t = t\,L_1$ for $t > 0$, so that the L_t for $t > 0$ are all homothetic to each other. Also, each L_t for $t \geqslant 0$ is invariant under the T^2 subgroup of SU(3) acting by

$$(z_1, z_2, z_3) \mapsto (\mathrm{e}^{i\theta_1}z_1, \mathrm{e}^{i\theta_2}z_2, \mathrm{e}^{i\theta_3}z_3) \ \text{ for } \theta_1, \theta_2, \theta_3 \in \mathbb{R} \text{ with } \theta_1 + \theta_2 + \theta_3 = 0,$$

and so fits into the framework of §8.1. By [24, Th. 5.1] the L_a may also be written in the form (18) for continuous $u, v : \mathbb{R}^2 \to \mathbb{R}$, as in §8.5.

Our second example is adapted from Harvey and Lawson [13, §III.3.B].

Example 8.5 For each $t > 0$, define

$$\begin{aligned} L_t = \bigl\{(\mathrm{e}^{i\theta}x_1, \mathrm{e}^{i\theta}x_2, \mathrm{e}^{i\theta}x_3) : x_j \in \mathbb{R}, \quad \theta \in (0, \pi/3),& \\ x_1^2 + x_2^2 + x_3^2 = t^2(\sin 3\theta)^{-2/3}\bigr\}&. \end{aligned}$$

Then L_t is a nonsingular embedded SL 3-fold in $\mathbb{C}^3$ diffeomorphic to $\mathcal{S}^2 \times \mathbb{R}$. As $t \to 0_+$ it converges to the singular union L_0 of the two SL 3-planes

$$\Pi_1 = \bigl\{(x_1, x_2, x_3) : x_j \in \mathbb{R}\bigr\} \text{ and } \Pi_2 = \bigl\{(\mathrm{e}^{i\pi/3}x_1, \mathrm{e}^{i\pi/3}x_2, \mathrm{e}^{i\pi/3}x_3) : x_j \in \mathbb{R}\bigr\},$$

which intersect at 0. Note that L_t is invariant under the action of the Lie subgroup SO(3) of SU(3), acting on $\mathbb{C}^3$ in the obvious way, so again this comes from the method of §8.1. Also L_t is asymptotic to L_0 at infinity.

Our third example is taken from [18, Ex. 9.4 & Ex. 9.5].

Example 8.6 Let a_1, a_2 be positive, coprime integers, and set $a_3 = -a_1 - a_2$. Let $c \in \mathbb{R}$, and define

$$L_c^{a_1,a_2} = \bigl\{(\mathrm{e}^{ia_1\theta}x_1, \mathrm{e}^{ia_2\theta}x_2, i\mathrm{e}^{ia_3\theta}x_3) : \theta \in \mathbb{R},\ x_j \in \mathbb{R},\ a_1x_1^2 + a_2x_2^2 + a_3x_3^2 = c\bigr\}.$$

Then $L_c^{a_1,a_2}$ is a special Lagrangian 3-fold, which comes from the 'evolving quadrics' construction of §8.2. It is also symmetric under the U(1)-action

$$(z_1, z_2, z_3) \mapsto (\mathrm{e}^{ia_1\theta}z_1, \mathrm{e}^{ia_2\theta}z_2, i\mathrm{e}^{ia_3\theta}z_3) \quad \text{for } \theta \in \mathbb{R},$$

but this is not a necessary feature of the construction; these are just the easiest examples to write down.

When $c = 0$ and a_3 is odd, $L_0^{a_1,a_2}$ is an embedded special Lagrangian cone on T^2, with one singular point at 0. When $c = 0$ and a_3 is even, $L_0^{a_1,a_2}$ is two opposite embedded SL T^2-cones with one singular point at 0.

When $c > 0$ and a_3 is odd, $L_c^{a_1,a_2}$ is an embedded 3-fold diffeomorphic to a nontrivial real line bundle over the Klein bottle. When $c > 0$ and a_3 is even, $L_c^{a_1,a_2}$ is an embedded 3-fold diffeomorphic to $T^2 \times \mathbb{R}$. In both cases, $L_c^{a_1,a_2}$ is a *ruled* SL 3-fold, as in §8.3, since it is fibred by hyperboloids of one sheet in $\mathbb{R}^3$, which are ruled in two different ways.

When $c < 0$ and a_3 is odd, $L_c^{a_1,a_2}$ an immersed copy of $\mathcal{S}^1 \times \mathbb{R}^2$. When $c < 0$ and a_3 is even, $L_c^{a_1,a_2}$ two immersed copies of $\mathcal{S}^1 \times \mathbb{R}^2$.

All the singular SL 3-folds we have seen so far have been *cones* in $\mathbb{C}^3$. Our final example, taken from [20], has more complicated singularities which are not cones. They are difficult to describe in a simple way, so we will not say much about them. For more details, see [20].

Example 8.7 In [20, §5] the author constructed a family of maps $\Phi : \mathbb{R}^3 \to \mathbb{C}^3$ with special Lagrangian image $N = \mathrm{Image}\,\Phi$. It is shown in [20, §6] that generic Φ in this family are immersions, so that N is nonsingular as an immersed SL 3-fold, but in codimension 1 in the family they develop isolated singularities.

Here is a rough description of these singularities, taken from [20, §6]. Taking the singular point to be at $\Phi(0,0,0) = 0$, one can write Φ as

$$\begin{aligned}\Phi(x,y,t) = &\bigl(x + \tfrac{1}{4}g(\mathbf{u},\mathbf{v})t^2\bigr)\,\mathbf{u} + \bigl(y^2 - \tfrac{1}{4}|\mathbf{u}|^2t^2\bigr)\,\mathbf{v} \\ &+ 2yt\,\mathbf{u}\times\mathbf{v} + O\bigl(x^2 + |xy| + |xt| + |y|^3 + |t|^3\bigr),\end{aligned} \tag{22}$$

where $\mathbf{u}, \mathbf{v}$ are linearly independent vectors in $\mathbb{C}^3$ with $\omega(\mathbf{u}, \mathbf{v}) = 0$, and $\times : \mathbb{C}^3 \times \mathbb{C}^3 \to \mathbb{C}^3$ is defined by

$$(r_1, r_2, r_3) \times (s_1, s_2, s_3) = \tfrac{1}{2}(\bar{r}_2\bar{s}_3 - \bar{r}_3\bar{s}_2, \bar{r}_3\bar{s}_1 - \bar{r}_1\bar{s}_3, \bar{r}_1\bar{s}_2 - \bar{r}_2\bar{s}_1).$$

The next few terms in the expansion (22) can also be given very explicitly, but we will not write them down as they are rather complex, and involve further choices of vectors $\mathbf{w}, \mathbf{x}, \ldots$.

What is going on here is that the lowest order terms in Φ are a *double cover* of the special Lagrangian plane $\langle \mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v} \rangle_{\mathbb{R}}$ in $\mathbb{C}^3$, *branched* along the real line $\langle \mathbf{u} \rangle_{\mathbb{R}}$. The branching occurs when $y = t = 0$. Higher order terms deviate from the 3-plane $\langle \mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v} \rangle_{\mathbb{R}}$, and make the singularity isolated.

8.7 Exercises

8.1 The group of automorphisms of $\mathbb{C}^m$ preserving g, ω and Ω is $\mathrm{SU}(m) \ltimes \mathbb{C}^m$, where $\mathbb{C}^m$ acts by translations. Let G be a Lie subgroup of $\mathrm{SU}(m) \ltimes \mathbb{C}^m$, let $\mathfrak{g}$ be its Lie algebra, and let $\phi : \mathfrak{g} \to \mathrm{Vect}(\mathbb{C}^m)$ be the natural map associating an element of $\mathfrak{g}$ to the corresponding vector field on $\mathbb{C}^m$.
A *moment map* for the action of G on $\mathbb{C}^m$ is a smooth map $\mu : \mathbb{C}^m \to \mathfrak{g}^*$, such that $\phi(x) \cdot \omega = x \cdot \mathrm{d}\mu$ for all $x \in \mathfrak{g}$, and $\mu : \mathbb{C}^m \to \mathfrak{g}^*$ is equivariant with respect to the G-action on $\mathbb{C}^m$ and the coadjoint G-action on $\mathfrak{g}^*$. Moment maps always exist if G is compact or semisimple, and are unique up to the addition of a constant in the centre $Z(\mathfrak{g}^*)$ of $\mathfrak{g}^*$, that is, the G-invariant subspace of $\mathfrak{g}^*$.
Suppose L is a (special) Lagrangian m-fold in $\mathbb{C}^m$ invariant under a Lie subgroup G in $\mathrm{SU}(m) \ltimes \mathbb{C}^m$, with moment map μ. Show that $\mu \equiv c$ on L for some $c \in Z(\mathfrak{g}^*)$.

8.2 Define a smooth map $f : \mathbb{C}^3 \to \mathbb{R}^3$ by

$$f(z_1, z_2, z_3) = \bigl(|z_1|^2 - |z_3|^2, |z_2|^2 - |z_3|^2, \mathrm{Im}(z_1 z_2 z_3)\bigr).$$

For each $a, b, c \in \mathbb{R}^3$, define $N_{a,b,c} = f^{-1}(a, b, c)$. Then $N_{a,b,c}$ is a real 3-dimensional submanifold of $\mathbb{C}^3$, which may be singular.

(i) At $\mathbf{z} = (z_1, z_2, z_3) \in \mathbb{C}^3$, determine $\mathrm{d}f|_{\mathbf{z}} : \mathbb{C}^3 \to \mathbb{R}^3$. Find the conditions on $\mathbf{z}$ for $\mathrm{d}f|_{\mathbf{z}}$ to be surjective.
Now $N_{a,b,c}$ is nonsingular at $\mathbf{z} \in N_{a,b,c}$ if and only if $\mathrm{d}f|_{\mathbf{z}}$ is surjective. Hence determine which of the $N_{a,b,c}$ are singular, and find their singular points.

(ii) If $\mathbf{z}$ is a nonsingular point of $N_{a,b,c}$, then $T_{\mathbf{z}} N_{a,b,c} = \mathrm{Ker}\, \mathrm{d}f|_{\mathbf{z}}$. Determine $\mathrm{Ker}\, \mathrm{d}f|_{\mathbf{z}}$ in this case, and show that it is a special Lagrangian 3-plane in $\mathbb{C}^3$.
Hence prove that $N_{a,b,c}$ is a special Lagrangian 3-fold wherever it is nonsingular, and that $f : \mathbb{C}^3 \to \mathbb{R}^3$ is a *special Lagrangian fibration.*

(iii) Observe that $N_{a,b,c}$ is invariant under the Lie group $G = \mathrm{U}(1)^2$, acting by

$$(\mathrm{e}^{i\theta_1}, \mathrm{e}^{i\theta_2}) : (z_1, z_2, z_3) \mapsto (\mathrm{e}^{i\theta_1} z_1, \mathrm{e}^{i\theta_2} z_2, \mathrm{e}^{-i\theta_1 - i\theta_2} z_3).$$

How is the form of f related to the ideas of question 8.1? How might G-invariance have been used to construct the fibration f?

(iv) Describe the topology of $N_{a,b,c}$, distinguishing different cases according to the singularities.

9 Compact SL m-folds in Calabi–Yau m-folds

In this section we shall discuss *compact* special Lagrangian submanifolds in Calabi–Yau manifolds. Here are three important questions which motivate work in this area.

1. Let N be a compact special Lagrangian m-fold in a fixed Calabi–Yau m-fold (M, J, g, Ω). Let $\mathcal{M}_N$ be the moduli space of *special Lagrangian deformations* of N, that is, the connected component of the set of special Lagrangian m-folds containing N. What can we say about $\mathcal{M}_N$? For instance, is it a smooth manifold, and of what dimension?

2. Let $\big\{(M, J_t, g_t, \Omega_t) : t \in (-\epsilon, \epsilon)\big\}$ be a smooth 1-parameter family of Calabi–Yau m-folds. Suppose N_0 is an SL m-fold in (M, J_0, g_0, Ω_0). Under what conditions can we extend N_0 to a smooth family of special Lagrangian m-folds N_t in (M, J_t, g_t, Ω_t) for $t \in (-\epsilon, \epsilon)$?

3. In general the moduli space $\mathcal{M}_N$ in Question 1 will be noncompact. Can we enlarge $\mathcal{M}_N$ to a compact space $\overline{\mathcal{M}}_N$ by adding a 'boundary' consisting of *singular* special Lagrangian m-folds? If so, what is the nature of the singularities that develop?

Briefly, these questions concern the *deformations* of special Lagrangian m-folds, *obstructions* to their existence, and their *singularities* respectively. The local answers to Questions 1 and 2 are well understood, and we shall discuss them in this section. Question 3 is the subject of §10–§11.

9.1 SL m-folds in Calabi–Yau m-folds

Here is the definition.

Definition 9.1 Let (M, J, g, Ω) be a Calabi–Yau m-fold. Then $\operatorname{Re}\Omega$ is a *calibration* on the Riemannian manifold (M, g). An oriented real m-dimensional submanifold N in M is called a *special Lagrangian submanifold (SL m-fold)* if it is calibrated with respect to $\operatorname{Re}\Omega$.

From Proposition 7.3 we deduce an *alternative definition* of SL m-folds. It is often more useful than Definition 9.1.

Proposition 9.2 *Let (M, J, g, Ω) be a Calabi–Yau m-fold, with Kähler form ω, and L a real m-dimensional submanifold in M. Then N admits an orientation making it into an SL m-fold in M if and only if $\omega|_N \equiv 0$ and $\operatorname{Im}\Omega|_N \equiv 0$.*

Regard N as an immersed submanifold, with immersion $\iota : N \to M$. Then $[\omega|_N]$ and $[\operatorname{Im}\Omega|_N]$ are unchanged under continuous variations of the immersion ι. Thus, $[\omega|_N] = [\operatorname{Im}\Omega|_N] = 0$ is a necessary condition not just for N to be special Lagrangian, but also for any isotopic submanifold N' in M to be special Lagrangian. This proves:

Corollary 9.3 *Let (M, J, g, Ω) be a Calabi–Yau m-fold, and N a compact real m-submanifold in M. Then a necessary condition for N to be isotopic to a special Lagrangian submanifold N' in M is that $[\omega|_N] = 0$ in $H^2(N, \mathbb{R})$ and $[\operatorname{Im}\Omega|_N] = 0$ in $H^m(N, \mathbb{R})$.*

This gives a simple, necessary topological condition for an isotopy class of m-submanifolds in a Calabi–Yau m-fold to contain a special Lagrangian submanifold.

9.2 Deformations of compact special Lagrangian m-folds

The deformation theory of compact special Lagrangian manifolds was studied by McLean [36], who proved the following result.

Theorem 9.4 *Let (M, J, g, Ω) be a Calabi–Yau m-fold, and N a compact special Lagrangian m-fold in M. Then the moduli space $\mathcal{M}_N$ of special Lagrangian deformations of N is a smooth manifold of dimension $b^1(N)$, the first Betti number of N.*

Sketch proof. Suppose for simplicity that N is an embedded submanifold. There is a natural orthogonal decomposition $TM|_N = TN \oplus \nu$, where $\nu \to N$ is the *normal bundle* of N in M. As N is Lagrangian, the complex structure $J : TM \to TM$ gives an isomorphism $J : \nu \to TN$. But the metric g gives an isomorphism $TN \cong T^*N$. Composing these two gives an isomorphism $\nu \cong T^*N$.

Let T be a small *tubular neighbourhood* of N in M. Then we can identify T with a neighbourhood of the zero section in ν. Using the isomorphism $\nu \cong T^*N$, we have an identification between T and a neighbourhood of the zero section in T^*N. This can be chosen to identify the Kähler form ω on T with the natural symplectic structure on T^*N. Let $\pi : T \to N$ be the obvious projection.

Under this identification, submanifolds N' in $T \subset M$ which are C^1 close to N are identified with the graphs of small smooth sections α of T^*N. That is, submanifolds N' of M close to N are identified with 1-*forms* α on N. We need to know: which 1-forms α are identified with *special Lagrangian* submanifolds N'?

Well, N' is special Lagrangian if $\omega|_{N'} \equiv \operatorname{Im}\Omega|_{N'} \equiv 0$. Now $\pi|_{N'} : N' \to N$ is a diffeomorphism, so we can push $\omega|_{N'}$ and $\operatorname{Im}\Omega|_{N'}$ down to N, and regard them as functions of α. Calculation shows that

$$\pi_*\big(\omega|_{N'}\big) = \mathrm{d}\alpha \quad \text{and} \quad \pi_*\big(\operatorname{Im}\Omega|_{N'}\big) = F(\alpha, \nabla\alpha),$$

where F is a nonlinear function of its arguments. Thus, the moduli space $\mathcal{M}_N$ is locally isomorphic to the set of small 1-forms α on N such that $\mathrm{d}\alpha \equiv 0$ and $F(\alpha, \nabla\alpha) \equiv 0$.

Now it turns out that F satisfies $F(\alpha, \nabla\alpha) \approx \mathrm{d}(*\alpha)$ when α is small. Therefore $\mathcal{M}_N$ is locally approximately isomorphic to the vector space of 1-forms α with $\mathrm{d}\alpha = \mathrm{d}(*\alpha) = 0$. But by Hodge theory, this is isomorphic to the de Rham cohomology group $H^1(N, \mathbb{R})$, and is a manifold with dimension $b^1(N)$.

To carry out this last step rigorously requires some technical machinery: one must work with certain *Banach spaces* of sections of T^*N, $\Lambda^2 T^*N$ and $\Lambda^m T^*N$, use *elliptic regularity results* to prove that the map $\alpha \mapsto \big(\mathrm{d}\alpha, F(\alpha, \nabla\alpha)\big)$ has *closed image* in these Banach spaces, and then use the *Implicit Function Theorem for Banach spaces* to show that the kernel of the map is what we expect. □

9.3 Obstructions to the existence of compact SL m-folds

Next we address Question 2 above. Let $\big\{(M, J_t, g_t, \Omega_t) : t \in (-\epsilon, \epsilon)\big\}$ be a smooth 1-parameter family of Calabi–Yau m-folds. Suppose N_0 is a special Lagrangian m-fold of (M, J_0, g_0, Ω_0). When can we extend N_0 to a smooth family of special Lagrangian m-folds N_t in (M, J_t, g_t, Ω_t) for $t \in (-\epsilon, \epsilon)$?

By Corollary 9.3, a necessary condition is that $[\omega_t|_{N_0}] = [\mathrm{Im}\,\Omega_t|_{N_0}] = 0$ for all t. Our next result shows that locally, this is also a *sufficient* condition.

Theorem 9.5 *Let $\big\{(M, J_t, g_t, \Omega_t) : t \in (-\epsilon, \epsilon)\big\}$ be a smooth 1-parameter family of Calabi–Yau m-folds, with Kähler forms ω_t. Let N_0 be a compact SL m-fold in (M, J_0, g_0, Ω_0), and suppose that $[\omega_t|_{N_0}] = 0$ in $H^2(N_0, \mathbb{R})$ and $[\mathrm{Im}\,\Omega_t|_{N_0}] = 0$ in $H^m(N_0, \mathbb{R})$ for all $t \in (-\epsilon, \epsilon)$. Then N_0 extends to a smooth 1-parameter family $\big\{N_t : t \in (-\delta, \delta)\big\}$, where $0 < \delta \leqslant \epsilon$ and N_t is a compact SL m-fold in (M, J_t, g_t, Ω_t).*

This can be proved using similar techniques to Theorem 9.4, though McLean did not prove it. Note that the condition $[\mathrm{Im}\,\Omega_t|_{N_0}] = 0$ for all t can be satisfied by choosing the phases of the Ω_t appropriately, and if the image of $H_2(N, \mathbb{Z})$ in $H_2(M, \mathbb{R})$ is zero, then the condition $[\omega|_N] = 0$ holds automatically.

Thus, the obstructions $[\omega_t|_{N_0}] = [\mathrm{Im}\,\Omega_t|_{N_0}] = 0$ in Theorem 9.5 are actually fairly mild restrictions, and special Lagrangian m-folds should be thought of as pretty stable under small deformations of the Calabi–Yau structure.

Remark. The deformation and obstruction theory of compact special Lagrangian m-folds are *extremely well-behaved* compared to many other moduli space problems in differential geometry. In other geometric problems (such as the deformations of complex structures on a complex manifold, or pseudo-holomorphic curves in an almost complex manifold, or instantons on a Riemannian 4-manifold, and so on), the deformation theory often has the following general structure.

There are vector bundles E, F over a compact manifold M, and an elliptic operator $P : C^\infty(E) \to C^\infty(F)$, usually first-order. The kernel $\mathrm{Ker}\,P$ is the set of *infinitesimal deformations*, and the cokernel $\mathrm{Coker}\, P$ the set of *obstructions*. The actual moduli space $\mathcal{M}$ is locally the zeros of a nonlinear map $\Psi : \mathrm{Ker}\, P \to \mathrm{Coker}\, P$.

In a *generic* case, $\mathrm{Coker}\, P = 0$, and then the moduli space $\mathcal{M}$ is locally isomorphic to $\mathrm{Ker}\, P$, and so is locally a manifold with dimension $\mathrm{ind}(P)$. However, in nongeneric situations $\mathrm{Coker}\, P$ may be nonzero, and then the moduli space $\mathcal{M}$ may be nonsingular, or have an unexpected dimension.

However, special Lagrangian submanifolds do not follow this pattern. Instead, the obstructions are *topologically determined*, and the moduli space is *always* smooth, with dimension given by a topological formula. This should be regarded as a minor mathematical miracle.

9.4 Natural coordinates on the moduli space $\mathcal{M}_N$

Let N be a compact SL m-fold in a Calabi–Yau m-fold (M, J, g, Ω). Theorem 9.4 shows that the moduli space $\mathcal{M}_N$ has dimension $b^1(N)$. By Poincaré duality $b^1(N) = b^{m-1}(N)$. Thus $\mathcal{M}_N$ has the same dimension as the de Rham cohomology groups $H^1(M, \mathbb{R})$ and $H^{m-1}(M, \mathbb{R})$.

We shall construct natural local diffeomorphisms Φ from $\mathcal{M}_N$ to $H^1(N, \mathbb{R})$, and Ψ from $\mathcal{M}_N$ to $H^{m-1}(N, \mathbb{R})$. These induce two natural *affine structures* on $\mathcal{M}_N$, and can be thought of as two *natural coordinate systems* on $\mathcal{M}_N$. The material of this section can be found in Hitchin [15, §4].

Here is how to define Φ and Ψ. Let U be a connected and simply-connected open neighbourhood of N in $\mathcal{M}_N$. We will construct smooth maps $\Phi : U \to H^1(N, \mathbb{R})$ and $\Psi : U \to H^{m-1}(N, \mathbb{R})$ with $\Phi(N) = \Psi(N) = 0$, which are local diffeomorphisms.

Let $N' \in U$. Then as U is connected, there exists a smooth path $\gamma : [0, 1] \to U$ with $\gamma(0) = N$ and $\gamma(1) = N'$, and as U is simply-connected, γ is unique up to isotopy. Now γ parametrizes a family of submanifolds of M diffeomorphic to N, which we can lift to a smooth map $\Gamma : N \times [0, 1] \to M$ with $\Gamma(N \times \{t\}) = \gamma(t)$.

Consider the 2-form $\Gamma^*(\omega)$ on $N \times [0, 1]$. As each fibre $\gamma(t)$ is Lagrangian, we have $\Gamma^*(\omega)|_{N\times\{t\}} \equiv 0$ for each $t \in [0, 1]$. Therefore we may write $\Gamma^*(\omega) = \alpha_t \wedge \mathrm{d}t$, where α_t is a closed 1-form on N for $t \in [0, 1]$. Define $\Phi(N') = \bigl[\int_0^1 \alpha_t \,\mathrm{d}t\bigr] \in H^1(N, \mathbb{R})$. That is, we integrate the 1-forms α_t with respect to t to get a closed 1-form $\int_0^1 \alpha_t \,\mathrm{d}t$, and then take its cohomology class.

Similarly, write $\Gamma^*(\mathrm{Im}\,\Omega) = \beta_t \wedge \mathrm{d}t$, where β_t is a closed $(m-1)$-form on N for $t \in [0, 1]$, and define $\Psi(N') = \bigl[\int_0^1 \beta_t \,\mathrm{d}t\bigr] \in H^{m-1}(N, \mathbb{R})$. Then Φ and Ψ are independent of choices made in the construction (exercise). We need to restrict to a simply-connected subset U of $\mathcal{M}_N$ so that γ is unique up to isotopy. Alternatively, one can define Φ and Ψ on the universal cover $\widetilde{\mathcal{M}}_N$ of $\mathcal{M}_N$.

9.5 SL m-folds in almost Calabi–Yau manifolds

Next we explain a generalization of special Lagrangian geometry to the class of *almost Calabi–Yau manifolds*.

Definition 9.6 Let $m \geqslant 2$. An *almost Calabi–Yau m-fold*, or *ACY m-fold* for short, is a quadruple (M, J, g, Ω) such that (M, J, g) is a compact m-dimensional Kähler manifold, and Ω is a non-vanishing holomorphic $(m, 0)$-form on M.

The difference between this and Definition 4.3 is that we do not require Ω and the Kähler form ω of g and Ω to satisfy equation (11), and hence g need not be Ricci-flat, nor have holonomy $\mathrm{SU}(m)$. Here is the appropriate definition of special Lagrangian m-folds in ACY m-folds.

Definition 9.7 Let (M, J, g, Ω) be an almost Calabi–Yau m-fold with Kähler form ω, and N a real m-dimensional submanifold of M. We call N a *special Lagrangian submanifold*, or *SL m-fold* for short, if $\omega|_N \equiv \operatorname{Im}\Omega|_N \equiv 0$. It easily follows that $\operatorname{Re}\Omega|_N$ is a nonvanishing m-form on N. Thus N is orientable, with a unique orientation in which $\operatorname{Re}\Omega|_N$ is positive.

By Proposition 9.2, if (M, J, g, Ω) is Calabi–Yau rather than almost Calabi–Yau, then N is special Lagrangian in the sense of Definition 4.3. Thus, this is a genuine extension of the idea of special Lagrangian submanifold. Many of the good properties of special Lagrangian submanifolds in Calabi–Yau manifolds also apply in almost Calabi–Yau manifolds. In particular:

Theorem 9.8 *Corollary 9.3 and Theorems 9.4 and 9.5 also hold in almost Calabi–Yau manifolds rather than Calabi–Yau manifolds.*

This is because the proofs of these results only really depend on the conditions $\omega|_N \equiv \operatorname{Im}\Omega|_N \equiv 0$, and the pointwise connection (11) between ω and Ω is not important.

Let (M, J, g, Ω) be an ACY m-fold, with metric g. In general, SL m-folds in M are neither calibrated nor minimal with respect to g. However, let $f : M \to (0, \infty)$ be the unique smooth function such that $f^{2m}\omega^m/m! = (-1)^{m(m-1)/2}(i/2)^m \Omega \wedge \bar{\Omega}$, and define $\tilde{g}$ to be the conformally equivalent metric $f^2 g$ on M. Then $\operatorname{Re}\Omega$ is a calibration on the Riemannian manifold $(M, \tilde{g})$, and that SL m-folds N in (M, J, g, Ω) are calibrated with respect to it, so that they are minimal with respect to $\tilde{g}$.

The idea of extending special Lagrangian geometry to almost Calabi–Yau manifolds appears in the work of Goldstein [6, §3.1], Bryant [3, §1], who uses the term 'special Kähler' instead of 'almost Calabi–Yau', and the author [28].

One important reason for considering SL m-folds in almost Calabi–Yau rather than Calabi–Yau m-folds is that they have much stronger *genericness properties*. There are many situations in geometry in which one uses a genericity assumption to control singular behaviour.

For instance, pseudo-holomorphic curves in an arbitrary almost complex manifold may have bad singularities, but the possible singularities in a generic almost complex manifold are much simpler. In the same way, it is reasonable to hope that in a *generic* Calabi–Yau m-fold, compact SL m-folds may have better singular behaviour than in an arbitrary Calabi–Yau m-fold.

But because Calabi–Yau manifolds come in only finite-dimensional families, choosing a generic Calabi–Yau structure is a fairly weak assumption, and probably will not help very much. However, almost Calabi–Yau manifolds come in *infinite-dimensional* families, so choosing a generic almost Calabi–Yau structure is a much more powerful thing to do, and will probably simplify the singular behaviour of compact SL m-folds considerably. We will return to this idea in §10.

9.6 Exercises

9.1 Show that the maps Φ, Ψ between special Lagrangian moduli space $\mathcal{M}_N$ and $H^1(N, \mathbb{R})$, $H^{m-1}(N, \mathbb{R})$ defined in §9.4 are well-defined and independent of choices.
Prove also that Φ and Ψ are *local diffeomorphisms*, that is, that $\mathrm{d}\Phi|_{N'}$ and $\mathrm{d}\Psi|_{N'}$ are isomorphisms between $T_{N'}\mathcal{M}_N$ and $H^1(N, \mathbb{R})$, $H^{m-1}(N, \mathbb{R})$ for each $N' \in U$.

9.2 Putting together the maps Φ, Ψ of Question 9.1 gives a map $\Phi \times \Psi : U \to H^1(N, \mathbb{R}) \times H^{m-1}(N, \mathbb{R})$. Now $H^1(N, \mathbb{R})$ and $H^{m-1}(N, \mathbb{R})$ are dual by Poincaré duality, so $H^1(N, \mathbb{R}) \times H^{m-1}(N, \mathbb{R})$ has a natural *symplectic structure*. Show that the image of U is a *Lagrangian submanifold* in $H^1(N, \mathbb{R}) \times H^{m-1}(N, \mathbb{R})$.

Hint: From the proof of McLean's theorem in §9.2, the tangent space $T_N\mathcal{M}_N$ is isomorphic to the vector space of 1-forms α with $\mathrm{d}\alpha = \mathrm{d}(*\alpha) = 0$. Then $\mathrm{d}\Phi|_N : T_N\mathcal{M} \to H^1(M, \mathbb{R})$ takes $\alpha \mapsto [\alpha]$, and $\mathrm{d}\Psi|_N : T_N\mathcal{M} \to H^{m-1}(M, \mathbb{R})$ takes $\alpha \mapsto [*\alpha]$. Use the fact that for 1-forms α, β on an oriented Riemannian manifold we have $\alpha \wedge (*\beta) = \beta \wedge (*\alpha)$.

10 Singularities of special Lagrangian m-folds

Now we move on to Question 3 of §9, and discuss the *singularities* of special Lagrangian m-folds. We can divide it into two sub-questions:

3(a) What kinds of singularities are possible in singular special Lagrangian m-folds, and what do they look like?

3(b) How can singular SL m-folds arise as limits of nonsingular SL m-folds, and what does the limiting behaviour look like near the singularities?

The basic premise of the author's approach to special Lagrangian singularities is that singularities of SL m-folds in Calabi–Yau m-folds should look locally like singularities of SL m-folds in $\mathbb{C}^m$, to the first few orders of approximation. That is, if M is a Calabi–Yau m-fold and N an SL m-fold in M with a singular point at $x \in M$, then near x, M resembles $\mathbb{C}^m = T_xM$, and N resembles an SL m-fold L in $\mathbb{C}^m$ with a singular point at 0. We call L a *local model* for N near x.

Therefore, to understand singularities of SL m-folds in Calabi–Yau manifolds, we begin by studying singularities of SL m-folds in $\mathbb{C}^m$, first by constructing as many examples as we can, and then by aiming for some kind of rough classification of the most common kinds of special Lagrangian singularities, at least in low dimensions such as $m = 3$.

10.1 Cones, and asymptotically conical SL m-folds

In Examples 8.4–8.6, the singular SL 3-folds we constructed were cones. Here a closed SL m-fold C in $\mathbb{C}^m$ is called a *cone* if $C = tC$ for all $t > 0$, where $tC = \{tx : x \in C\}$. Note that 0 is always a singular point of C, unless C is a special Lagrangian plane $\mathbb{R}^m$ in $\mathbb{C}^m$. The simplest kind of SL cones (from the point of view of singular behaviour) are those in which 0 is the only singular point. Then $\Sigma = C \cap \mathcal{S}^{2m-1}$ is a nonsingular, compact, minimal, Legendrian $(m-1)$-submanifold in the unit sphere $\mathcal{S}^{2m-1}$ in $\mathbb{C}^m$.

In one sense, *all* singularities of SL m-folds are modelled on special Lagrangian cones, to highest order. It follows from [13, §II.5] that if M is a Calabi–Yau m-fold and N an SL m-fold in M with a singular point at x, then N has a *tangent cone* at x in the sense of Geometric Measure Theory, which is a special Lagrangian cone C in $\mathbb{C}^m = T_xM$.

If C has multiplicity one and 0 is its only singular point, then N really does look like C near x. However, things become more complicated if the singularities of C are not isolated (for instance, if C is the union of two SL 3-planes $\mathbb{R}^3$ in $\mathbb{C}^3$ intersecting in a line) or if the multiplicity of C is greater than 1 (this happens in Example 8.7, as the tangent cone is a double $\mathbb{R}^3$). In these cases, the tangent cone captures only the simplest part of the singular behaviour, and we have to work harder to find out what is going on.

Now suppose for simplicity that we are interested in SL m-folds with singularities modelled on a multiplicity 1 SL cone C in $\mathbb{C}^m$ with an isolated singular point. To answer question 3(b) above, and understand how such singularities arise as limits of nonsingular SL m-folds, we need to study *asymptotically conical (AC)* SL m-folds L in $\mathbb{C}^m$ asymptotic to C.

We shall be interested in two classes of asymptotically conical SL m-folds L, *weakly AC*, which converge to C like $o(r)$, and *strongly AC*, which converge to C like $O(r^{-1})$. Here is a more precise definition.

Definition 10.1 Let C be a special Lagrangian cone in $\mathbb{C}^m$ with isolated singularity at 0, and let $\Sigma = C \cap \mathcal{S}^{2m-1}$, so that Σ is a compact, nonsingular $(m-1)$-manifold. Define the *number of ends of C at infinity* to be the number of connected components of Σ. Let h be the metric on Σ induced by the metric g on $\mathbb{C}^m$, and r the radius function on $\mathbb{C}^m$. Define $\iota : \Sigma \times (0,\infty) \to \mathbb{C}^m$ by $\iota(\sigma, r) = r\sigma$. Then the image of ι is $C \setminus \{0\}$, and $\iota^*(g) = r^2 h + \mathrm{d}r^2$ is the cone metric on $C \setminus \{0\}$.

Let L be a closed, nonsingular SL m-fold in $\mathbb{C}^m$. We call L *weakly asymptotically conical (weakly AC)* with cone C if there exists a compact subset $K \subset L$ and a diffeomorphism $\phi : \Sigma \times (R,\infty) \to L \setminus K$ for some $R > 0$, such that $|\phi - \iota| = o(r)$ and $\left|\nabla^k(\phi - \iota)\right| = o(r^{1-k})$ as $r \to \infty$ for $k = 1, 2, \ldots$, where ∇ is the Levi-Civita connection of the cone metric $\iota^*(g)$, and $|\,.\,|$ is computed using $\iota^*(g)$.

Similarly, we call L *strongly asymptotically conical (strongly AC)* with cone C if $|\phi - \iota| = O(r^{-1})$ and $\left|\nabla^k(\phi - \iota)\right| = O(r^{-1-k})$ as $r \to \infty$ for $k = 1, 2, \ldots$, using the same notation.

These two asymptotic conditions are useful for different purposes. If L is a weakly AC SL m-fold then tL converges to C as $t \to 0_+$. Thus, weakly AC SL m-folds provide models for how singularities modelled on cones C can arise as limits of nonsingular SL m-folds. The weakly AC condition is in practice the weakest asymptotic condition which ensures this; if the $o(r)$ condition were made any weaker then the asymptotic cone C at infinity might not be unique or well-defined.

On the other hand, explicit constructions tend to produce strongly AC SL m-folds, and they appear to be the easiest class to prove results about. For example, one can show:

Proposition 10.2 *Suppose C is an SL cone in $\mathbb{C}^m$ with an isolated singular point at* 0*, invariant under a connected Lie subgroup G of* $\mathrm{SU}(m)$*. Then any strongly AC SL m-fold L in $\mathbb{C}^m$ with cone C is also G-invariant.*

This should be a help in classifying strongly AC SL m-folds with cones with a lot of symmetry. For instance, using $\mathrm{U}(1)^2$ symmetry one can show that the only strongly AC SL 3-folds in $\mathbb{C}^3$ with cone L_0 from Example 8.4 are the SL 3-folds L_t from Example 8.4 for $t > 0$, and two other families obtained from the L_t by cyclic permutations of coordinates (z_1, z_2, z_3).

10.2 Moduli spaces of AC SL m-folds

Next we discuss moduli space problems for AC SL m-folds. I shall state our problems as conjectures, because the proofs are not yet complete. One should be able to prove analogues of Theorem 9.4 for AC SL m-folds. Here is the appropriate result for strongly AC SL m-folds.

Conjecture 10.3 *Let L be a strongly AC SL m-fold in $\mathbb{C}^m$, with cone C, and let k be the number of ends of C at infinity. Then the moduli space $\mathcal{M}_L^s$ of strongly AC SL m-folds in $\mathbb{C}^m$ with cone C is near L a smooth manifold of dimension $b^1(L) + k - 1$.*

Before generalizing this to the weak case, here is a definition.

Definition 10.4 Let C be a special Lagrangian cone in $\mathbb{C}^m$ with an isolated singularity at 0, and let $\Sigma = C \cap \mathcal{S}^{2m-1}$. Regard Σ as a compact Riemannian manifold, with metric induced from the round metric on $\mathcal{S}^{2m-1}$. Let $\Delta = \mathrm{d}^*\mathrm{d}$ be the Laplacian on functions on Σ. Define the *Legendrian index* $\text{l-ind}(C)$ to be the number of eigenvalues of Δ in $(0, 2m)$, counted with multiplicity.

We call this the *Legendrian index* since it is the index of the area functional at Σ under Legendrian variations of the submanifold Σ in $\mathcal{S}^{2m-1}$. It is not difficult to show that the restriction of any real linear function on $\mathbb{C}^m$ to Σ is an eigenfunction of Δ with eigenvalue $m-1$. These contribute m to l-ind(C) for each connected component of Σ which is a round unit sphere $\mathcal{S}^{m-1}$, and $2m$ to l-ind(C) for each other connected component of Σ. This gives a useful lower bound for l-ind(C). In particular, l-ind$(C) \geqslant 2m$.

Conjecture 10.5 *Let L be a weakly AC SL m-fold in $\mathbb{C}^m$, with cone C, and let k be the number of ends of C at infinity. Then the moduli space $\mathcal{M}_L^w$ of weakly AC SL m-folds in $\mathbb{C}^m$ with cone C is near L a smooth manifold of dimension $b^1(L)+k-1+\text{l-ind}(C)$.*

Here are some remarks on these conjectures:

- The author's student, Stephen Marshall, is working on proofs of Conjectures 10.3 and 10.5, which we hope to be able to publish soon. Some related results have recently been proved by Tommaso Pacini, [38, §3].
- If L is a weakly AC SL m-fold in $\mathbb{C}^m$, then any translation of L is also weakly AC, with the same cone. Since C has an isolated singularity by assumption, it cannot have translation symmetries. Hence L also has no translation symmetries, so the translations of L are all distinct, and $\mathcal{M}_L^w$ has dimension at least $2m$. The inequality l-ind$(C) \geqslant 2m$ above ensures this.
- The dimension of $\mathcal{M}_L^s$ in Conjecture 10.3 is purely topological, as in Theorem 9.4, which is another indication that strongly AC is in many ways the nicest asymptotic condition to work with. But the dimension of $\mathcal{M}_L^w$ in Conjecture 10.5 has an analytic component, the eigenvalue count in l-ind(C).
- It is an interesting question whether moduli spaces of weakly AC SL m-folds always contain a strongly AC SL m-fold.

10.3 SL singularities in generic almost Calabi–Yau m-folds

We move on to discuss the singular behaviour of compact SL m-folds in Calabi–Yau m-folds. For simplicity we shall restrict our attention to a class of SL cones with no nontrivial deformations.

Definition 10.6 Let C be a special Lagrangian cone in $\mathbb{C}^m$ with an isolated singularity at 0 and k ends at infinity, and set $\Sigma = C \cap \mathcal{S}^{2m-1}$. Let $\Sigma_1, \ldots, \Sigma_k$ be the connected components of Σ. Regard each Σ_j as a compact Riemannian manifold, and let $\Delta_j = \mathrm{d}^*\mathrm{d}$ be the Laplacian on functions on Σ_j. Let G_j be the Lie subgroup of $\mathrm{SU}(m)$ preserving Σ_j, and V_j the eigenspace of Δ_j with eigenvalue $2m$. We call the SL cone C *rigid* if $\dim V_j = \dim \mathrm{SU}(m) - \dim G_j$ for each $j = 1, \ldots, k$.

Here is how to understand this definition. We can regard C as the union of one-ended SL cones $C_1, \ldots, C_k$ intersecting at 0, where $C_j \setminus \{0\}$ is naturally identified with $\Sigma_j \times (0, \infty)$. The cone metric on C_j is $g_j = r^2 h_j + \mathrm{d}r^2$, where h_j

is the metric on Σ_j. Suppose f_j is an eigenfunction of Δ_j on Σ_j with eigenvalue $2m$. Then $r^2 f_j$ is *harmonic* on C_j.

Hence, $\mathrm{d}(r^2 f_j)$ is a closed, coclosed 1-form on C_j which is linear in r. By the Principle in §7.3, the basis of the proof of Theorem 9.4, such 1-forms correspond to small deformations of C_j as an SL cone in $\mathbb{C}^m$. Therefore, we can interpret V_j as the space of *infinitesimal deformations* of C_j as a special Lagrangian cone.

Clearly, one way to deform C_j as a special Lagrangian cone is to apply elements of $\mathrm{SU}(m)$. This gives a family $\mathrm{SU}(m)/G_j$ of deformations of C_j, with dimension $\dim \mathrm{SU}(m) - \dim G_j$, so that $\dim V_j \geqslant \dim \mathrm{SU}(m) - \dim G_j$. (The corresponding functions in V_j are moment maps of $\mathfrak{su}(m)$ vector fields.) We call C *rigid* if equality holds for all j, that is, if all infinitesimal deformations of C come from applying motions in $\mathfrak{su}(m)$ to the component cones $C_1, \ldots, C_k$.

Not all SL cones in $\mathbb{C}^m$ are rigid. One can show using integrable systems that there exist families of SL T^2-cones C in $\mathbb{C}^3$ up to $\mathrm{SU}(3)$ equivalence, of arbitrarily high dimension. If the dimension of the family is greater than $\dim \mathrm{SU}(3) - \dim G_1$, where G_1 is the Lie subgroup of $\mathrm{SU}(3)$ preserving C, then C is not rigid.

Now we can give a first approximation to the kinds of results the author expects to hold for singular SL m-folds in (almost) Calabi–Yau m-folds.

Conjecture 10.7 *Let C be a rigid special Lagrangian cone in $\mathbb{C}^m$ with an isolated singularity at 0 and k ends at infinity, and L a weakly AC SL m-fold in $\mathbb{C}^m$ with cone C. Let (M, J, g, Ω) be a generic almost Calabi–Yau m-fold, and $\mathcal{M}$ a connected moduli space of compact nonsingular SL m-folds N in M.*

Suppose that at the boundary of $\mathcal{M}$ there is a moduli space $\mathcal{M}_C$ of compact, singular SL m-folds with one isolated singular point modelled on the cone C, which arise as limits of SL m-folds in $\mathcal{M}$ by collapsing weakly AC SL m-folds with the topology of L. Then

$$\dim \mathcal{M} = \dim \mathcal{M}_C + b^1(L) + k - 1 + \text{l-ind}(C) - 2m. \tag{23}$$

Here are some remarks on the conjecture:

- I have an outline proof of this conjecture which works when $m < 6$. The analytic difficulties increase with dimension; I am not sure whether the conjecture holds in high dimensions.

- Similar results should hold for non-rigid singularities, but the dimension formulae will be more complicated.

- Closely associated to this result is an analogue of Theorem 9.4 for SL m-folds with isolated conical singularities of a given kind, under an appropriate genericity assumption.

- Here is one way to arrive at equation (23). Assuming Conjecture 10.5, the moduli space $\mathcal{M}_L^w$ of weakly AC SL m-folds containing L has dimension

$b^1(L)+k-1+\text{l-ind}(C)$. Now translations in $\mathbb{C}^m$ act freely on $\mathcal{M}_L^w$, so the family of weakly AC SL m-folds up to translations has dimension $b^1(L)+k-1+\text{l-ind}(C)-2m$.

The idea is that each singular SL m-fold N_0 in $\mathcal{M}_C$ can be 'resolved' to give a nonsingular SL m-fold N by gluing in any 'sufficiently small' weakly AC SL m-fold L', up to translation. Thus, desingularizing should add $b^1(L)+k-1+\text{l-ind}(C)-2m$ degrees of freedom, which is how we get equation (23).

- However, (23) may not give the right answer in every case. One can imagine situations in which there are *cohomological obstructions* to gluing in AC SL m-folds L'. For instance, it might be necessary that the symplectic area of a disc in $\mathbb{C}^m$ with boundary in L' be zero to make $N = L' \# N_0$ Lagrangian. These could reduce the number of degrees of freedom in desingularizing N_0, and then (23) would require correction. An example of this is considered in [16, §4.4].

Now we can introduce the final important idea in this section. Suppose we have a suitably generic (almost) Calabi–Yau m-fold M and a compact, singular SL m-fold N_0 in M, which is the limit of a family of compact nonsingular SL m-folds N in M.

We (loosely) define the *index* of the singularities of N_0 to be the codimension of the family of singular SL m-folds with singularities like those of N_0 in the family of nonsingular SL m-folds N. Thus, in the situation of Conjecture 10.7, the index of the singularities is $b^1(L)+k-1+\text{l-ind}(C)-2m$.

More generally, one can work not just with a fixed generic almost Calabi–Yau m-fold, but with a *generic family* of almost Calabi–Yau m-folds. So, for instance, if we have a generic k-dimensional family of almost Calabi–Yau m-folds M, and in each M we have an l-dimensional family of SL m-folds, then in the total $(k+l)$-dimensional family of SL m-folds we are guaranteed to meet singularities of index at most $k+l$.

Now singularities with *small index* are the most commonly occurring, and so arguably the most interesting kinds of singularity. Also, for various problems it will only be necessary to know about singularities with index up to a certain value.

For example, in [16] the author proposed to define an invariant of almost Calabi–Yau 3-folds by counting special Lagrangian homology 3-spheres (which occur in 0-dimensional moduli spaces) in a given homology class, with a certain topological weight. This invariant will only be interesting if it is essentially conserved under deformations of the underlying almost Calabi–Yau 3-fold. During such a deformation, nonsingular SL 3-folds can develop singularities and disappear, or new ones appear, which might change the invariant.

To prove the invariant is conserved, we need to show that it is unchanged along generic 1-parameter families of Calabi–Yau 3-folds. The only kinds of singularities of SL homology 3-spheres that arise in such families will have index

1. Thus, to resolve the conjectures in [16], we only have to know about index 1 singularities of SL 3-folds in almost Calabi–Yau 3-folds.

Another problem in which the index of singularities will be important is the *SYZ Conjecture*, to be discussed in §11. This has to do with dual 3-dimensional families $\mathcal{M}_X$, $\mathcal{M}_{\hat{X}}$ of SL 3-tori in (almost) Calabi–Yau 3-folds $X, \hat{X}$. If $X, \hat{X}$ are generic then the only kinds of singularities that can occur at the boundaries of $\mathcal{M}_X, \mathcal{M}_{\hat{X}}$ are of index 1, 2 or 3. So, to study the SYZ Conjecture in the generic case, we only have to know about singularities of SL 3-folds with index 1, 2, 3 (and possibly 4).

It would be an interesting and useful project to find examples of, and eventually to classify, special Lagrangian singularities with small index, at least in dimension 3. For instance, consider rigid SL cones C in $\mathbb{C}^3$ as in Conjecture 10.7, of index 1. Then $b^1(L) + k - 1 + \text{l-ind}(C) - 6 = 1$, and $\text{l-ind}(C) \geqslant 6$, so $b^1(L) + k \leqslant 2$. But $k \geqslant 1$ and $b^1(L) \geqslant \frac{1}{2}b^1(\Sigma)$, so $b^1(\Sigma) \leqslant 2$. As Σ is oriented, one can show that either $k = 1$, $\text{l-ind}(C) = 6$ and Σ is a torus T^2, or $k = 2$, $\text{l-ind}(C) = 6$, Σ is 2 copies of $\mathcal{S}^2$, and C is the union of two SL 3-planes in $\mathbb{C}^3$ intersecting only at 0.

The eigenvalue count $\text{l-ind}(C)$ implies an upper bound for the area of Σ. Hopefully, one can then use integrable systems results as in §8.4 to pin down the possibilities for C. The author guesses that the T^2-cone L_0 of Example 8.4, and perhaps also the T^2-cone $L_0^{1,2}$ of Example 8.6, are the only examples of index 1 SL T^2-cones in $\mathbb{C}^3$ up to SU(3) isomorphisms.

10.4 Exercises

10.1 Prove Proposition 10.2.

11 The SYZ Conjecture, and SL fibrations

Mirror Symmetry is a mysterious relationship between pairs of Calabi–Yau 3-folds $X, \hat{X}$, arising from a branch of physics known as *String Theory*, and leading to some very strange and exciting conjectures about Calabi–Yau 3-folds, many of which have been proved in special cases.

The *SYZ Conjecture* is an attempt to explain Mirror Symmetry in terms of dual 'fibrations' $f : X \to B$ and $\hat{f} : \hat{X} \to B$ of $X, \hat{X}$ by special Lagrangian 3-folds, including singular fibres. We give brief introductions to String Theory, Mirror Symmetry, and the SYZ Conjecture, and then a short survey of the state of mathematical research into the SYZ Conjecture, biased in favour of the author's own interests.

11.1 String Theory and Mirror Symmetry

String Theory is a branch of high-energy theoretical physics in which particles are modelled not as points but as 1-dimensional objects – 'strings' – propagating in some background space-time M. String theorists aim to construct a *quantum theory* of the string's motion. The process of quantization is extremely complicated, and fraught with mathematical difficulties that are as yet still poorly understood.

The most popular version of String Theory requires the universe to be 10-dimensional for this quantization process to work. Therefore, String Theorists suppose that the space we live in looks locally like $M = \mathbb{R}^4 \times X$, where $\mathbb{R}^4$ is Minkowski space, and X is a compact Riemannian 6-manifold with radius of order 10^{-33}cm, the Planck length. Since the Planck length is so small, space then appears to macroscopic observers to be 4-dimensional.

Because of supersymmetry, X has to be a *Calabi–Yau 3-fold*. Therefore String Theorists are very interested in Calabi–Yau 3-folds. They believe that each Calabi–Yau 3-fold X has a quantization, which is a *Super Conformal Field Theory* (SCFT), a complicated mathematical object. Invariants of X such as the Dolbeault groups $H^{p,q}(X)$ and the number of holomorphic curves in X translate to properties of the SCFT.

However, two entirely different Calabi–Yau 3-folds X and $\hat{X}$ may have the *same* SCFT. In this case, there are powerful relationships between the invariants of X and of $\hat{X}$ that translate to properties of the SCFT. This is the idea behind *Mirror Symmetry* of Calabi–Yau 3-folds.

It turns out that there is a very simple automorphism of the structure of a SCFT — changing the sign of a U(1)-action — which does *not* correspond to a classical automorphism of Calabi–Yau 3-folds. We say that X and $\hat{X}$ are *mirror* Calabi–Yau 3-folds if their SCFT's are related by this automorphism. Then one can argue using String Theory that

$$H^{1,1}(X) \cong H^{2,1}(\hat{X}) \quad \text{and} \quad H^{2,1}(X) \cong H^{1,1}(\hat{X}).$$

Effectively, the mirror transform exchanges even- and odd-dimensional cohomology. This is a very surprising result!

More involved String Theory arguments show that, in effect, the Mirror Transform exchanges things related to the complex structure of X with things related to the symplectic structure of $\hat{X}$, and vice versa. Also, a generating function for the number of holomorphic rational curves in X is exchanged with a simple invariant to do with variation of complex structure on $\hat{X}$, and so on.

Because the quantization process is poorly understood and not at all rigorous — it involves non-convergent path-integrals over horrible infinite-dimensional spaces — String Theory generates only conjectures about Mirror Symmetry, not proofs. However, many of these conjectures have been verified in particular cases.

11.2 Mathematical interpretations of Mirror Symmetry

In the beginning (the 1980's), Mirror Symmetry seemed mathematically completely mysterious. But there are now two complementary conjectural theories, due to Kontsevich and Strominger–Yau–Zaslow, which explain Mirror Symmetry in a fairly mathematical way. Probably both are true, at some level.

The first proposal was due to Kontsevich [31] in 1994. This says that for mirror Calabi–Yau 3-folds X and $\hat{X}$, the derived category $D^b(X)$ of coherent sheaves on X is equivalent to the derived category $D^b(\mathrm{Fuk}(\hat{X}))$ of the Fukaya category of $\hat{X}$, and vice versa. Basically, $D^b(X)$ has to do with X as a complex manifold, and $D^b(\mathrm{Fuk}(\hat{X}))$ with $\hat{X}$ as a symplectic manifold, and its Lagrangian submanifolds. We shall not discuss this here.

The second proposal, due to Strominger, Yau and Zaslow [42] in 1996, is known as the *SYZ Conjecture*. Here is an attempt to state it.

The SYZ Conjecture *Suppose X and $\hat{X}$ are mirror Calabi–Yau 3-folds. Then (under some additional conditions) there should exist a compact topological 3-manifold B and surjective, continuous maps $f : X \to B$ and $\hat{f} : \hat{X} \to B$, such that*

(i) *There exists a dense open set $B_0 \subset B$, such that for each $b \in B_0$, the fibres $f^{-1}(b)$ and $\hat{f}^{-1}(b)$ are nonsingular special Lagrangian 3-tori T^3 in X and $\hat{X}$. Furthermore, $f^{-1}(b)$ and $\hat{f}^{-1}(b)$ are in some sense dual to one another.*

(ii) *For each $b \in \Delta = B \setminus B_0$, the fibres $f^{-1}(b)$ and $\hat{f}^{-1}(b)$ are expected to be singular special Lagrangian 3-folds in X and $\hat{X}$.*

We call f and $\hat{f}$ *special Lagrangian fibrations*, and the set of singular fibres Δ is called the *discriminant*. In part (i), the nonsingular fibres of f and $\hat{f}$ are supposed to be *dual tori*. What does this mean?

On the topological level, we can define duality between two tori $T, \hat{T}$ to be a choice of isomorphism $H^1(T, \mathbb{Z}) \cong H_1(\hat{T}, \mathbb{Z})$. We can also define duality between tori equipped with flat Riemannian metrics. Write $T = V/\Lambda$, where V is a Euclidean vector space and Λ a *lattice* in V. Then the dual torus $\hat{T}$ is defined to be V^*/Λ^*, where V^* is the dual vector space and Λ^* the dual lattice. However, there is no notion of duality between non-flat metrics on dual tori.

Strominger, Yau and Zaslow argue only that their conjecture holds when $X, \hat{X}$ are close to the 'large complex structure limit'. In this case, the diameters of the fibres $f^{-1}(b), \hat{f}^{-1}(b)$ are expected to be small compared to the diameter of the base space B, and away from singularities of $f, \hat{f}$, the metrics on the nonsingular fibres are expected to be approximately flat.

So, part (i) of the SYZ Conjecture says that for $b \in B \setminus B_0$, $f^{-1}(b)$ is approximately a flat Riemannian 3-torus, and $\hat{f}^{-1}(b)$ is approximately the dual flat Riemannian torus. Really, the SYZ Conjecture makes most sense as a statement about the limiting behaviour of *families* of mirror Calabi–Yau 3-folds X_t, $\hat{X}_t$ which approach the 'large complex structure limit' as $t \to 0$.

11.3 The symplectic topological approach to SYZ

The most successful approach to the SYZ Conjecture so far could be described as *symplectic topological*. In this approach, we mostly forget about complex structures, and treat $X, \hat{X}$ just as *symplectic manifolds*. We mostly forget about the 'special' condition, and treat $f, \hat{f}$ just as *Lagrangian fibrations*. We also impose the condition that B is a *smooth* 3-manifold and $f : X \to B$ and $\hat{f} : \hat{X} \to B$ are *smooth maps*. (It is not clear that $f, \hat{f}$ can in fact be smooth at every point, though).

Under these simplifying assumptions, Gross [8, 9, 10, 11], Ruan [39, 40], and others have built up a beautiful, detailed picture of how dual SYZ fibrations work at the global topological level, in particular for examples such as the quintic and its mirror, and for Calabi–Yau 3-folds constructed as hypersurfaces in toric 4-folds, using combinatorial data.

11.4 Local geometric approach, and SL singularities

There is also another approach to the SYZ Conjecture, begun by the author in [26, 28], and making use of the ideas and philosophy set out in §10. We could describe it as a *local geometric* approach.

In it we try to take the special Lagrangian condition seriously from the outset, and our focus is on the local behaviour of special Lagrangian submanifolds, and especially their singularities, rather than on global topological questions. Also, we are interested in what fibrations of *generic* (almost) Calabi–Yau 3-folds might look like.

One of the first-fruits of this approach has been the understanding that for *generic* (almost) Calabi–Yau 3-folds X, special Lagrangian fibrations $f : X \to B$ will not be smooth maps, but only piecewise smooth. Furthermore, their behaviour at the singular set is rather different to the smooth Lagrangian fibrations discussed in §11.3.

For smooth special Lagrangian fibrations $f : X \to B$, the discriminant Δ is of codimension 2 in B, and the typical singular fibre is singular along an $\mathcal{S}^1$. But in a generic special Lagrangian fibration $f : X \to B$ the discriminant Δ is of codimension 1 in B, and the typical singular fibre is singular at finitely many points.

One can also show that if $X, \hat{X}$ are a mirror pair of generic (almost) Calabi–Yau 3-folds and $f : X \to B$ and $\hat{f} : \hat{X} \to B$ are dual special Lagrangian fibrations, then in general the discriminants Δ of f and $\hat{\Delta}$ of $\hat{f}$ cannot coincide in B, because they have different topological properties in the neighbourhood of a certain kind of codimension 3 singular fibre.

This contradicts part (ii) of the SYZ Conjecture, as we have stated it in §11.2. In the author's view, these calculations support the idea that the SYZ Conjecture in its present form should be viewed primarily as a limiting statement, about what happens at the 'large complex structure limit', rather than as simply being about pairs of Calabi–Yau 3-folds. A similar conclusion is reached by Mark Gross in [11, §5].

11.5 U(1)-invariant SL fibrations in $\mathbb{C}^3$

We finish by describing work of the author in [26, §8] and [28], which aims to describe what the singularities of SL fibrations of *generic* (almost) Calabi–

Yau 3-folds look like, providing they exist. This proceeds by first studying SL fibrations of subsets of $\mathbb{C}^3$ invariant under the U(1)-action (17), using the ideas of §8.5. For a brief survey of the main results, see [27].

Then we argue (without a complete proof, as yet) that the kinds of singularities we see in codimension 1 and 2 in generic U(1)-invariant SL fibrations in $\mathbb{C}^3$, also occur in codimension 1 and 2 in SL fibrations of generic (almost) Calabi–Yau 3-folds, without any assumption of U(1)-invariance.

Following [26, Def. 8.1], we use the results of §8.5 to construct a family of SL 3-folds $N_{\boldsymbol{\alpha}}$ in $\mathbb{C}^3$, depending on boundary data $\Phi(\boldsymbol{\alpha})$.

Definition 11.1 Let S be a strictly convex domain in $\mathbb{R}^2$ invariant under $(x, y) \mapsto (x, -y)$, let U be an open set in $\mathbb{R}^3$, and $\alpha \in (0, 1)$. Suppose $\Phi : U \to C^{3,\alpha}(\partial S)$ is a continuous map such that if $(a, b, c) \neq (a, b', c')$ in U then $\Phi(a, b, c) - \Phi(a, b', c')$ has exactly one local maximum and one local minimum in ∂S.

For $\boldsymbol{\alpha} = (a, b, c) \in U$, let $f_{\boldsymbol{\alpha}} \in C^{3,\alpha}(S)$ or $C^1(S)$ be the unique (weak) solution of (21) with $f_{\boldsymbol{\alpha}}|_{\partial S} = \Phi(\boldsymbol{\alpha})$, which exists by Theorem 8.3. Define $u_{\boldsymbol{\alpha}} = \frac{\partial f_{\boldsymbol{\alpha}}}{\partial y}$ and $v_{\boldsymbol{\alpha}} = \frac{\partial f_{\boldsymbol{\alpha}}}{\partial x}$. Then $(u_{\boldsymbol{\alpha}}, v_{\boldsymbol{\alpha}})$ is a solution of (20) in $C^{2,\alpha}(S)$ if $a \neq 0$, and a weak solution of (19) in $C^0(S)$ if $a = 0$. Also $u_{\boldsymbol{\alpha}}, v_{\boldsymbol{\alpha}}$ depend continuously on $\boldsymbol{\alpha} \in U$ in $C^0(S)$, by Theorem 8.3.

For each $\boldsymbol{\alpha} = (a, b, c)$ in U, define $N_{\boldsymbol{\alpha}}$ in $\mathbb{C}^3$ by

$$\begin{aligned} N_{\boldsymbol{\alpha}} = \bigl\{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 z_2 = v_{\boldsymbol{\alpha}}(x, y) + iy, \quad z_3 = x + iu_{\boldsymbol{\alpha}}(x, y), \\ |z_1|^2 - |z_2|^2 = 2a, \quad (x, y) \in S^\circ\bigr\}. \end{aligned} \tag{24}$$

Then $N_{\boldsymbol{\alpha}}$ is a noncompact SL 3-fold without boundary in $\mathbb{C}^3$, which is nonsingular if $a \neq 0$, by Proposition 8.1.

In [26, Th. 8.2] we show that the $N_{\boldsymbol{\alpha}}$ are the fibres of an *SL fibration.*

Theorem 11.2 *In the situation of Definition 11.1, if $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}'$ in U then $N_{\boldsymbol{\alpha}} \cap N_{\boldsymbol{\alpha}'} = \emptyset$. There exists an open set $V \subset \mathbb{C}^3$ and a continuous, surjective map $F : V \to U$ such that $F^{-1}(\boldsymbol{\alpha}) = N_{\boldsymbol{\alpha}}$ for all $\boldsymbol{\alpha} \in U$. Thus, F is a special Lagrangian fibration of $V \subset \mathbb{C}^3$, which may include singular fibres.*

It is easy to produce families Φ satisfying Definition 11.1. For example [26, Ex. 8.3], given any $\phi \in C^{3,\alpha}(\partial S)$ we may define $U = \mathbb{R}^3$ and $\Phi : \mathbb{R}^3 \to C^{3,\alpha}(\partial S)$ by $\Phi(a, b, c) = \phi + bx + cy$. So this construction produces very large families of U(1)-invariant SL fibrations, including singular fibres, which can have any multiplicity and type.

Here is a simple, explicit example. Define $F : \mathbb{C}^3 \to \mathbb{R} \times \mathbb{C}$ by

$$\begin{aligned} &F(z_1, z_2, z_3) = (a, b), \quad \text{where} \quad 2a = |z_1|^2 - |z_2|^2 \\ &\text{and} \quad b = \begin{cases} z_3, & a = z_1 = z_2 = 0, \\ z_3 + \bar{z}_1 \bar{z}_2 / |z_1|, & a \geqslant 0, \ z_1 \neq 0, \\ z_3 + \bar{z}_1 \bar{z}_2 / |z_2|, & a < 0. \end{cases} \end{aligned} \tag{25}$$

This is a piecewise-smooth SL fibration of $\mathbb{C}^3$. It is not smooth on $|z_1| = |z_2|$.

The fibres $F^{-1}(a, b)$ are T^2-cones singular at $(0, 0, b)$ when $a = 0$, and nonsingular $\mathcal{S}^1 \times \mathbb{R}^2$ when $a \neq 0$. They are isomorphic to the SL 3-folds of Example 8.4 under transformations of $\mathbb{C}^3$, but they are assembled to make a fibration in a novel way.

As a goes from positive to negative the fibres undergo a surgery, a Dehn twist on $\mathcal{S}^1$. The reason why the fibration is only piecewise-smooth, rather than smooth, is really this topological transition, rather than the singularities themselves. The fibration is not differentiable at every point of a singular fibre, rather than just at singular points, and this is because we are jumping from one moduli space of SL 3-folds to another at the singular fibres.

I conjecture that F is the local model for codimension one singularities of SL fibrations of generic almost Calabi–Yau 3-folds. The justification for this is that the T^2-cone singularities have 'index one' in the sense of §10.3, and so should occur in codimension one in families of SL 3-folds in generic almost Calabi–Yau 3-folds. Since they occur in codimension one in this family, the singular behaviour should be stable under small perturbations of the underlying almost Calabi–Yau structure.

I also have a U(1)-invariant model for codimension two singularities, described in [28], in which two of the codimension one T^2-cones come together and cancel out. I conjecture that it too is a typical codimension two singular behaviour in SL fibrations of generic almost Calabi–Yau 3-folds. I do not expect codimension three singularities in generic SL fibrations to be locally U(1)-invariant, and so this approach will not help.

11.6 Exercises

11.1 Let $f : \mathbb{C}^3 \to \mathbb{R} \times \mathbb{C}$ be as in equation (25).

- **(a)** Show that f is continuous, surjective, and piecewise smooth.
- **(b)** Show that $f^{-1}(a, b)$ is a (possibly singular) special Lagrangian 3-fold for all $(a, b) \in \mathbb{R} \times \mathbb{C}$.
- **(c)** Identify the singular fibres and describe their singularities. Describe the topology of the singular and the nonsingular fibres.

The idea of a 'special Lagrangian fibration' $f : X \to B$ is in some ways a rather unnatural one. One of the problems is that the map f doesn't satisfy a particularly nice equation, locally; the level sets of f do, but the 'coordinates' on B are determined globally rather than locally. To understand the problems with special Lagrangian fibrations, try the following (rather difficult) exercise.

11.2 Let X be a Calabi–Yau 3-fold, N a compact SL 3-fold in X diffeomorphic to T^3, $\mathcal{M}_N$ the family of special Lagrangian deformations of N, and $\overline{\mathcal{M}}_N$ be $\mathcal{M}_N$ together with the singular SL 3-folds occurring as limits of elements of $\mathcal{M}_N$.

In good cases, SYZ hope that $\overline{\mathcal{M}}_N$ is the family of level sets of an SL fibration $f : X \to B$, where B is homeomorphic to $\overline{\mathcal{M}}_N$. How many different ways can you think of for this not to happen? (There are at least two mechanisms not involving singular fibres, and others which do).

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