The exceptional holonomy groups and calibrated geometry

Dominic Joyce
Lincoln College, Oxford, OX1 3DR

1 Introduction

In the theory of Riemannian holonomy groups, perhaps the most mysterious are the two exceptional cases, the holonomy group G_2 in 7 dimensions and the holonomy group Spin(7) in 8 dimensions. This is a survey paper on the exceptional holonomy groups, in two parts. Part I collects together useful facts about G_2 and Spin(7) in §2, and explains constructions of compact 7-manifolds with holonomy G_2 in §3, and of compact 8-manifolds with holonomy Spin(7) in §4.

Part II discusses the *calibrated submanifolds* of manifolds of exceptional holonomy, namely *associative* 3-*folds* and *coassociative* 4-*folds* in G_2-manifolds, and *Cayley* 4-*folds* in Spin(7)-manifolds. We introduce calibrations in §5, defining the three geometries and giving examples. Finally, §6 explains their *deformation theory*.

Sections 3 and 4 describe my own work, for which the main reference is my book [18]. Part II describes work by other people, principally the very important papers by Harvey and Lawson [12] and McLean [28], but also more recent developments.

This paper was written to accompany lectures at the 11^{th} Gökova Geometry and Topology Conference in May 2004, sponsored by TUBITAK. In keeping with the theme of the conference, I have focussed mostly on G_2, at the expense of Spin(7). The paper is based in part on the books [18] and [11, Part I], and the survey paper [21].

Acknowledgements. I would like to thank the conference organizers Turgut Onder and Selman Akbulut for their hospitality. Many people have helped me develop my ideas on exceptional holonomy and calibrated geometry; I would particularly like to thank Simon Salamon and Robert Bryant.

Part I. Exceptional Holonomy

2 Introduction to G_2 and Spin(7)

We introduce the notion of *Riemannian holonomy groups*, and their classification by Berger. Then we give short descriptions of the holonomy groups G_2, Spin(7) and SU(m), and the relations between them. All the results below can be found in my book [18].

2.1 Riemannian holonomy groups

Let M be a connected n-dimensional manifold, g a Riemannian metric on M, and ∇ the Levi-Civita connection of g. Let x, y be points in M joined by a smooth path γ. Then *parallel transport* along γ using ∇ defines an isometry between the tangent spaces T_xM, T_yM at x and y.

Definition 2.1. The *holonomy group* $\mathrm{Hol}(g)$ of g is the group of isometries of T_xM generated by parallel transport around piecewise-smooth closed loops based at x in M. We consider $\mathrm{Hol}(g)$ to be a subgroup of $\mathrm{O}(n)$, defined up to conjugation by elements of $\mathrm{O}(n)$. Then $\mathrm{Hol}(g)$ is independent of the base point x in M.

Let ∇ be the *Levi-Civita connection* of g. A tensor S on M is *constant* if $\nabla S = 0$. An important property of $\mathrm{Hol}(g)$ is that it *determines the constant tensors on* M.

Theorem 2.2. *Let (M, g) be a Riemannian manifold, and ∇ the Levi-Civita connection of g. Fix a base point $x \in M$, so that $\mathrm{Hol}(g)$ acts on T_xM, and so on the tensor powers $\bigotimes^k T_xM \otimes \bigotimes^l T_x^*M$. Suppose $S \in C^\infty\big(\bigotimes^k TM \otimes \bigotimes^l T^*M\big)$ is a constant tensor. Then $S|_x$ is fixed by the action of $\mathrm{Hol}(g)$. Conversely, if $S|_x \in \bigotimes^k T_xM \otimes \bigotimes^l T_x^*M$ is fixed by $\mathrm{Hol}(g)$, it extends to a unique constant tensor $S \in C^\infty\big(\bigotimes^k TM \otimes \bigotimes^l T^*M\big)$.*

The main idea in the proof is that if S is a constant tensor and $\gamma : [0, 1] \to M$ is a path from x to y, then $P_\gamma(S|_x) = S|_y$, where P_γ is the *parallel transport map* along γ. Thus, constant tensors are invariant under parallel transport. In particular, they are invariant under parallel transport around closed loops based at x, that is, under elements of $\mathrm{Hol}(g)$.

The classification of holonomy groups was achieved by Berger [1] in 1955.

Theorem 2.3. *Let M be a simply-connected, n-dimensional manifold, and g an irreducible, nonsymmetric Riemannian metric on M. Then either*

(i) $\mathrm{Hol}(g) = \mathrm{SO}(n)$,

(ii) $n = 2m$ *and* $\mathrm{Hol}(g) = \mathrm{SU}(m)$ *or* $\mathrm{U}(m)$,

(iii) $n = 4m$ *and* $\mathrm{Hol}(g) = \mathrm{Sp}(m)$ *or* $\mathrm{Sp}(m)\,\mathrm{Sp}(1)$,

(iv) $n = 7$ *and* $\mathrm{Hol}(g) = G_2$*, or*

(v) $n = 8$ *and* $\mathrm{Hol}(g) = \mathrm{Spin}(7)$*.*

Here are some brief remarks about each group on Berger's list.

(i) $\mathrm{SO}(n)$ is the holonomy group of generic Riemannian metrics.

(ii) Riemannian metrics g with $\mathrm{Hol}(g) \subseteq \mathrm{U}(m)$ are called *Kähler metrics*. Kähler metrics are a natural class of metrics on complex manifolds, and generic Kähler metrics on a given complex manifold have holonomy $\mathrm{U}(m)$.

Metrics g with $\mathrm{Hol}(g) = \mathrm{SU}(m)$ are called *Calabi–Yau metrics*. Since $\mathrm{SU}(m)$ is a subgroup of $\mathrm{U}(m)$, all Calabi–Yau metrics are Kähler. If g is Kähler and M is simply-connected, then $\mathrm{Hol}(g) \subseteq \mathrm{SU}(m)$ if and only if g is Ricci-flat. Thus Calabi–Yau metrics are locally more or less the same as Ricci-flat Kähler metrics.

(iii) metrics g with $\mathrm{Hol}(g) = \mathrm{Sp}(m)$ are called *hyperkähler*. As $\mathrm{Sp}(m) \subseteq \mathrm{SU}(2m) \subset \mathrm{U}(2m)$, hyperkähler metrics are Ricci-flat and Kähler.

Metrics g with holonomy group $\mathrm{Sp}(m)\,\mathrm{Sp}(1)$ for $m \geqslant 2$ are called *quaternionic Kähler*. (Note that quaternionic Kähler metrics are not in fact Kähler.) They are Einstein, but not Ricci-flat.

(iv),(v) G_2 and $\mathrm{Spin}(7)$ are the exceptional cases, so they are called the *exceptional holonomy groups*. Metrics with these holonomy groups are Ricci-flat.

The groups can be understood in terms of the four *division algebras*: the *real numbers* $\mathbb{R}$, the *complex numbers* $\mathbb{C}$, the *quaternions* $\mathbb{H}$, and the *octonions* or *Cayley numbers* $\mathbb{O}$.

- $\mathrm{SO}(n)$ is a group of automorphisms of $\mathbb{R}^n$.
- $\mathrm{U}(m)$ and $\mathrm{SU}(m)$ are groups of automorphisms of $\mathbb{C}^m$
- $\mathrm{Sp}(m)$ and $\mathrm{Sp}(m)\,\mathrm{Sp}(1)$ are automorphism groups of $\mathbb{H}^m$.
- G_2 is the automorphism group of $\mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$. $\mathrm{Spin}(7)$ is a group of automorphisms of $\mathbb{O} \cong \mathbb{R}^8$, preserving part of the structure on $\mathbb{O}$.

For some time after Berger's classification, the exceptional holonomy groups remained a mystery. In 1987, Bryant [6] used the theory of exterior differential systems to show that locally there exist many metrics with these holonomy groups, and gave some explicit, incomplete examples. Then in 1989, Bryant and Salamon [8] found explicit, *complete* metrics with holonomy G_2 and $\mathrm{Spin}(7)$ on noncompact manifolds.

In 1994-5 the author constructed the first examples of metrics with holonomy G_2 and $\mathrm{Spin}(7)$ on *compact* manifolds [14, 15, 16]. These, and the more complicated constructions developed later by the author [17, 18] and by Kovalev [22], are the subject of Part I.

2.2 The holonomy group G_2

Let $(x_1, \dots, x_7)$ be coordinates on $\mathbb{R}^7$. Write $\mathrm{d}\mathbf{x}_{ij\dots l}$ for the exterior form $\mathrm{d}x_i \wedge \mathrm{d}x_j \wedge \cdots \wedge \mathrm{d}x_l$ on $\mathbb{R}^7$. Define a metric g_0, a 3-form φ_0 and a 4-form $*\varphi_0$ on $\mathbb{R}^7$ by $g_0 = \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_7^2$,

$$\begin{aligned} \varphi_0 &= \mathrm{d}\mathbf{x}_{123} + \mathrm{d}\mathbf{x}_{145} + \mathrm{d}\mathbf{x}_{167} + \mathrm{d}\mathbf{x}_{246} - \mathrm{d}\mathbf{x}_{257} - \mathrm{d}\mathbf{x}_{347} - \mathrm{d}\mathbf{x}_{356} \ \text{ and} \\ *\varphi_0 &= \mathrm{d}\mathbf{x}_{4567} + \mathrm{d}\mathbf{x}_{2367} + \mathrm{d}\mathbf{x}_{2345} + \mathrm{d}\mathbf{x}_{1357} - \mathrm{d}\mathbf{x}_{1346} - \mathrm{d}\mathbf{x}_{1256} - \mathrm{d}\mathbf{x}_{1247}. \end{aligned} \tag{1}$$

The subgroup of $GL(7, \mathbb{R})$ preserving φ_0 is the *exceptional Lie group* G_2. It also preserves $g_0, *\varphi_0$ and the orientation on $\mathbb{R}^7$. It is a compact, semisimple, 14-dimensional Lie group, a subgroup of SO(7).

A *G_2-structure* on a 7-manifold M is a principal subbundle of the frame bundle of M, with structure group G_2. Each G_2-structure gives rise to a 3-form φ and a metric g on M, such that every tangent space of M admits an isomorphism with $\mathbb{R}^7$ identifying φ and g with φ_0 and g_0 respectively. By an abuse of notation, we will refer to (φ, g) as a G_2-structure.

Proposition 2.4. *Let M be a 7-manifold and (φ, g) a G_2-structure on M. Then the following are equivalent:*

(i) $\mathrm{Hol}(g) \subseteq G_2$*, and φ is the induced* 3*-form,*

(ii) $\nabla\varphi = 0$ *on M, where ∇ is the Levi-Civita connection of g, and*

(iii) $\mathrm{d}\varphi = \mathrm{d}^*\varphi = 0$ *on M.*

Note that $\mathrm{Hol}(g) \subseteq G_2$ if and only if $\nabla\varphi = 0$ follows from Theorem 2.2. We call $\nabla\varphi$ the *torsion* of the G_2-structure (φ, g), and when $\nabla\varphi = 0$ the G_2-structure is *torsion-free*. A triple (M, φ, g) is called a *G_2-manifold* if M is a 7-manifold and (φ, g) a torsion-free G_2-structure on M. If g has holonomy $\mathrm{Hol}(g) \subseteq G_2$, then g is Ricci-flat.

Theorem 2.5. *Let M be a compact* 7*-manifold, and suppose that (φ, g) is a torsion-free G_2-structure on M. Then* $\mathrm{Hol}(g) = G_2$ *if and only if $\pi_1(M)$ is finite. In this case the moduli space of metrics with holonomy G_2 on M, up to diffeomorphisms isotopic to the identity, is a smooth manifold of dimension $b^3(M)$.*

2.3 The holonomy group Spin(7)

Let $\mathbb{R}^8$ have coordinates $(x_1, \dots, x_8)$. Define a 4-form Ω_0 on $\mathbb{R}^8$ by

$$\begin{aligned} \Omega_0 = {} & \mathrm{d}\mathbf{x}_{1234} + \mathrm{d}\mathbf{x}_{1256} + \mathrm{d}\mathbf{x}_{1278} + \mathrm{d}\mathbf{x}_{1357} - \mathrm{d}\mathbf{x}_{1368} - \mathrm{d}\mathbf{x}_{1458} - \mathrm{d}\mathbf{x}_{1467} \\ & - \mathrm{d}\mathbf{x}_{2358} - \mathrm{d}\mathbf{x}_{2367} - \mathrm{d}\mathbf{x}_{2457} + \mathrm{d}\mathbf{x}_{2468} + \mathrm{d}\mathbf{x}_{3456} + \mathrm{d}\mathbf{x}_{3478} + \mathrm{d}\mathbf{x}_{5678}. \end{aligned} \tag{2}$$

The subgroup of $\mathrm{GL}(8, \mathbb{R})$ preserving Ω_0 is the holonomy group Spin(7). It also preserves the orientation on $\mathbb{R}^8$ and the Euclidean metric $g_0 = \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_8^2$. It is a compact, semisimple, 21-dimensional Lie group, a subgroup of SO(8).

A Spin(7)-structure on an 8-manifold M gives rise to a 4-form Ω and a metric g on M, such that each tangent space of M admits an isomorphism with $\mathbb{R}^8$ identifying Ω and g with Ω_0 and g_0 respectively. By an abuse of notation we will refer to the pair (Ω, g) as a Spin(7)-structure.

Proposition 2.6. *Let M be an 8-manifold and (Ω, g) a* Spin(7)*-structure on M. Then the following are equivalent:*

(i) $\mathrm{Hol}(g) \subseteq \mathrm{Spin}(7)$*, and Ω is the induced 4-form,*

(ii) $\nabla\Omega = 0$ *on M, where ∇ is the Levi-Civita connection of g, and*

(iii) $\mathrm{d}\Omega = 0$ *on M.*

We call $\nabla\Omega$ the *torsion* of the Spin(7)-structure (Ω, g), and (Ω, g) *torsion-free* if $\nabla\Omega = 0$. A triple (M, Ω, g) is called a Spin(7)-*manifold* if M is an 8-manifold and (Ω, g) a torsion-free Spin(7)-structure on M. If g has holonomy $\mathrm{Hol}(g) \subseteq \mathrm{Spin}(7)$, then g is Ricci-flat.

Here is a result on *compact* 8-manifolds with holonomy Spin(7).

Theorem 2.7. *Let (M, Ω, g) be a compact* Spin(7)*-manifold. Then* $\mathrm{Hol}(g) = \mathrm{Spin}(7)$ *if and only if M is simply-connected, and $b^3(M) + b^4_+(M) = b^2(M) + 2b^4_-(M) + 25$. In this case the moduli space of metrics with holonomy* Spin(7) *on M, up to diffeomorphisms isotopic to the identity, is a smooth manifold of dimension $1 + b^4_-(M)$.*

2.4 The holonomy groups SU(m)

Let $\mathbb{C}^m \cong \mathbb{R}^{2m}$ have complex coordinates $(z_1, \ldots, z_m)$, and define the metric g_0, Kähler form ω_0 and complex volume form θ_0 on $\mathbb{C}^m$ by

$$\begin{gathered} g_0 = |\mathrm{d}z_1|^2 + \cdots + |\mathrm{d}z_m|^2, \quad \omega_0 = \frac{i}{2}(\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \cdots + \mathrm{d}z_m \wedge \mathrm{d}\bar{z}_m), \\ \text{and} \quad \theta_0 = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_m. \end{gathered} \tag{3}$$

The subgroup of $\mathrm{GL}(2m, \mathbb{R})$ preserving g_0, ω_0 and θ_0 is the special unitary group $\mathrm{SU}(m)$. Manifolds with holonomy $\mathrm{SU}(m)$ are called *Calabi–Yau manifolds.*

Calabi–Yau manifolds are automatically Ricci-flat and Kähler, with trivial canonical bundle. Conversely, any Ricci-flat Kähler manifold (M, J, g) with trivial canonical bundle has $\mathrm{Hol}(g) \subseteq \mathrm{SU}(m)$. By Yau's proof of the Calabi Conjecture [31], we have:

Theorem 2.8. *Let (M, J) be a compact complex m-manifold admitting Kähler metrics, with trivial canonical bundle. Then there is a unique Ricci-flat Kähler metric g in each Kähler class on M, and* $\mathrm{Hol}(g) \subseteq \mathrm{SU}(m)$.

Using this and complex algebraic geometry one can construct many examples of compact Calabi–Yau manifolds. The theorem also applies in the orbifold category, yielding examples of *Calabi–Yau orbifolds.*

2.5 Relations between G_2, Spin(7) and SU(m)

Here are the inclusions between the holonomy groups $\mathrm{SU}(m), G_2$ and Spin(7):

$$\begin{array}{ccccc} \mathrm{SU}(2) & \longrightarrow & \mathrm{SU}(3) & \longrightarrow & G_2 \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{SU}(2)\times\mathrm{SU}(2) & \longrightarrow & \mathrm{SU}(4) & \longrightarrow & \mathrm{Spin}(7). \end{array}$$

We shall illustrate what we mean by this using the inclusion $\mathrm{SU}(3) \hookrightarrow G_2$. As SU(3) acts on $\mathbb{C}^3$, it also acts on $\mathbb{R}\oplus\mathbb{C}^3 \cong \mathbb{R}^7$, taking the SU(3)-action on $\mathbb{R}$ to be trivial. Thus we embed SU(3) as a subgroup of $\mathrm{GL}(7,\mathbb{R})$. It turns out that SU(3) is a subgroup of the subgroup G_2 of $\mathrm{GL}(7,\mathbb{R})$ defined in §2.2.

Here is a way to see this in terms of differential forms. Identify $\mathbb{R}\oplus\mathbb{C}^3$ with $\mathbb{R}^7$ in the obvious way in coordinates, so that $\bigl(x_1,(x_2+ix_3,x_4+ix_5,x_6+ix_7)\bigr)$ in $\mathbb{R}\oplus\mathbb{C}^3$ is identified with $(x_1,\ldots,x_7)$ in $\mathbb{R}^7$. Then $\varphi_0 = \mathrm{d}x_1\wedge\omega_0 + \mathrm{Re}\,\theta_0$, where φ_0 is defined in (1) and ω_0,θ_0 in (3). Since SU(3) preserves ω_0 and θ_0, the action of SU(3) on $\mathbb{R}^7$ preserves φ_0, and so $\mathrm{SU}(3)\subset G_2$.

It follows that if (M,J,h) is Calabi–Yau 3-fold, then $\mathbb{R}\times M$ and $\mathcal{S}^1\times M$ have torsion-free G_2-structures, that is, are G_2-manifolds.

Proposition 2.9. *Let (M,J,h) be a Calabi–Yau 3-fold, with Kähler form ω and complex volume form θ. Let x be a coordinate on $\mathbb{R}$ or $\mathcal{S}^1$. Define a metric $g=\mathrm{d}x^2+h$ and a 3-form $\varphi=\mathrm{d}x\wedge\omega+\mathrm{Re}\,\theta$ on $\mathbb{R}\times M$ or $\mathcal{S}^1\times M$. Then (φ,g) is a torsion-free G_2-structure on $\mathbb{R}\times M$ or $\mathcal{S}^1\times M$, and $*\varphi=\frac{1}{2}\omega\wedge\omega-\mathrm{d}x\wedge\mathrm{Im}\,\theta$.*

Similarly, the inclusions $\mathrm{SU}(2)\hookrightarrow G_2$ and $\mathrm{SU}(4)\hookrightarrow \mathrm{Spin}(7)$ give:

Proposition 2.10. *Let (M,J,h) be a Calabi–Yau 2-fold, with Kähler form ω and complex volume form θ. Let (x_1,x_2,x_3) be coordinates on $\mathbb{R}^3$ or T^3. Define a metric $g=\mathrm{d}x_1^2+\mathrm{d}x_2^2+\mathrm{d}x_3^2+h$ and a 3-form φ on $\mathbb{R}^3\times M$ or $T^3\times M$ by*

$$\varphi=\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\mathrm{d}x_3+\mathrm{d}x_1\wedge\omega+\mathrm{d}x_2\wedge\mathrm{Re}\,\theta-\mathrm{d}x_3\wedge\mathrm{Im}\,\theta. \tag{4}$$

Then (φ,g) is a torsion-free G_2-structure on $\mathbb{R}^3\times M$ or $T^3\times M$, and

$$*\varphi=\tfrac{1}{2}\omega\wedge\omega+\mathrm{d}x_2\wedge\mathrm{d}x_3\wedge\omega-\mathrm{d}x_1\wedge\mathrm{d}x_3\wedge\mathrm{Re}\,\theta-\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\mathrm{Im}\,\theta. \tag{5}$$

Proposition 2.11. *Let (M,J,g) be a Calabi–Yau 4-fold, with Kähler form ω and complex volume form θ. Define a 4-form Ω on M by $\Omega=\frac{1}{2}\omega\wedge\omega+\mathrm{Re}\,\theta$. Then (Ω,g) is a torsion-free* Spin(7)*-structure on M.*

3 Constructing G_2-manifolds from orbifolds T^7/Γ

We now explain the method used in [14, 15] and [18, §11–§12] to construct examples of compact 7-manifolds with holonomy G_2. It is based on the *Kummer construction* for Calabi–Yau metrics on the $K3$ surface, and may be divided into four steps.

Step 1. Let T^7 be the 7-torus and (φ_0, g_0) a flat G_2-structure on T^7. Choose a finite group Γ of isometries of T^7 preserving (φ_0, g_0). Then the quotient T^7/Γ is a singular, compact 7-manifold, an *orbifold*.

Step 2. For certain special groups Γ there is a method to resolve the singularities of T^7/Γ in a natural way, using complex geometry. We get a nonsingular, compact 7-manifold M, together with a map $\pi : M \to T^7/\Gamma$, the resolving map.

Step 3. On M, we explicitly write down a 1-parameter family of G_2-structures (φ_t, g_t) depending on $t \in (0, \epsilon)$. They are not torsion-free, but have small torsion when t is small. As $t \to 0$, the G_2-structure (φ_t, g_t) converges to the singular G_2-structure $\pi^*(\varphi_0, g_0)$.

Step 4. We prove using analysis that for sufficiently small t, the G_2-structure (φ_t, g_t) on M, with small torsion, can be deformed to a G_2-structure $(\tilde{\varphi}_t, \tilde{g}_t)$, with zero torsion. Finally, we show that $\tilde{g}_t$ is a metric with holonomy G_2 on the compact 7-manifold M.

We will now explain each step in greater detail.

3.1 Step 1: Choosing an orbifold

Let (φ_0, g_0) be the Euclidean G_2-structure on $\mathbb{R}^7$ defined in §2.2. Suppose Λ is a *lattice* in $\mathbb{R}^7$, that is, a discrete additive subgroup isomorphic to $\mathbb{Z}^7$. Then $\mathbb{R}^7/\Lambda$ is the torus T^7, and (φ_0, g_0) pushes down to a torsion-free G_2-structure on T^7. We must choose a finite group Γ acting on T^7 preserving (φ_0, g_0). That is, the elements of Γ are the push-forwards to T^7/Λ of affine transformations of $\mathbb{R}^7$ which fix (φ_0, g_0), and take Λ to itself under conjugation.

Here is an example of a suitable group Γ, taken from [18, §12.2].

Example 3.1. Let $(x_1, \ldots, x_7)$ be coordinates on $T^7 = \mathbb{R}^7/\mathbb{Z}^7$, where $x_i \in \mathbb{R}/\mathbb{Z}$. Let (φ_0, g_0) be the flat G_2-structure on T^7 defined by (1). Let α, β and γ be the involutions of T^7 defined by

$$\alpha : (x_1, \ldots, x_7) \mapsto (x_1, x_2, x_3, -x_4, -x_5, -x_6, -x_7), \tag{6}$$

$$\beta : (x_1, \ldots, x_7) \mapsto (x_1, -x_2, -x_3, x_4, x_5, \tfrac{1}{2} - x_6, -x_7), \tag{7}$$

$$\gamma : (x_1, \ldots, x_7) \mapsto \big(-x_1, x_2, -x_3, x_4, \tfrac{1}{2} - x_5, x_6, \tfrac{1}{2} - x_7). \tag{8}$$

By inspection, α, β and γ preserve (φ_0, g_0), because of the careful choice of exactly which signs to change. Also, $\alpha^2 = \beta^2 = \gamma^2 = 1$, and α, β and γ commute. Thus they generate a group $\Gamma = \langle \alpha, \beta, \gamma \rangle \cong \mathbb{Z}_2^3$ of isometries of T^7 preserving the flat G_2-structure (φ_0, g_0).

Having chosen a lattice Λ and finite group Γ, the quotient T^7/Γ is an *orbifold*, a singular manifold with only quotient singularities. The singularities of T^7/Γ come from the fixed points of non-identity elements of Γ. We now describe the singularities in our example.

Lemma 3.2. *In Example 3.1, $\beta\gamma, \gamma\alpha, \alpha\beta$ and $\alpha\beta\gamma$ have no fixed points on T^7. The fixed points of α, β, γ are each 16 copies of T^3. The singular set S of T^7/Γ is a disjoint union of 12 copies of T^3, 4 copies from each of α, β, γ. Each component of S is a singularity modelled on that of $T^3 \times \mathbb{C}^2/\{\pm 1\}$.*

The most important consideration in choosing Γ is that we should be able to resolve the singularities of T^7/Γ within holonomy G_2. We will explain how to do this next.

3.2 Step 2: Resolving the singularities

Our goal is to resolve the singular set S of T^7/Γ to get a compact 7-manifold M with holonomy G_2. How can we do this? In general we cannot, because we have no idea of how to resolve general orbifold singularities with holonomy G_2. However, suppose we can arrange that every connected component of S is locally isomorphic to either

(a) $T^3 \times \mathbb{C}^2/G$, for G a finite subgroup of SU(2), or

(b) $\mathcal{S}^1 \times \mathbb{C}^3/G$, for G a finite subgroup of SU(3) acting freely on $\mathbb{C}^3 \setminus \{0\}$.

One can use complex algebraic geometry to find a *crepant resolution* X of $\mathbb{C}^2/G$ or Y of $\mathbb{C}^3/G$. Then $T^3 \times X$ or $\mathcal{S}^1 \times Y$ gives a local model for how to resolve the corresponding component of S in T^7/Γ. Thus we construct a nonsingular, compact 7-manifold M by using the patches $T^3 \times X$ or $\mathcal{S}^1 \times Y$ to repair the singularities of T^7/Γ. In the case of Example 3.1, this means gluing 12 copies of $T^3 \times X$ into T^7/Γ, where X is the blow-up of $\mathbb{C}^2/\{\pm 1\}$ at its singular point.

Now the point of using crepant resolutions is this. In both case (a) and (b), there exists a Calabi–Yau metric on X or Y which is asymptotic to the flat Euclidean metric on $\mathbb{C}^2/G$ or $\mathbb{C}^3/G$. Such metrics are called *Asymptotically Locally Euclidean (ALE)*. In case (a), the ALE Calabi–Yau metrics were classified by Kronheimer [23, 24], and exist for all finite $G \subset$ SU(2). In case (b), crepant resolutions of $\mathbb{C}^3/G$ exist for all finite $G \subset$ SU(3) by Roan [29], and the author [19], [18, §8] proved that they carry ALE Calabi–Yau metrics, using a noncompact version of the Calabi Conjecture.

By Propositions 2.9 and 2.10, we can use the Calabi–Yau metrics on X or Y to construct a torsion-free G_2-structure on $T^3 \times X$ or $\mathcal{S}^1 \times Y$. This gives a local model for how to resolve the singularity $T^3 \times \mathbb{C}^2/G$ or $\mathcal{S}^1 \times \mathbb{C}^3/G$ with holonomy G_2. So, this method gives not only a way to smooth out the singularities of T^7/Γ as a manifold, but also a family of torsion-free G_2-structures on the resolution which show how to smooth out the singularities of the G_2-structure.

The requirement above that S be divided into connected components of the form (a) and (b) is in fact unnecessarily restrictive. There is a more complicated and powerful method, described in [18, §11–§12], for resolving singularities of a more general kind. We require only that the singularities should *locally* be of the form $\mathbb{R}^3 \times \mathbb{C}^2/G$ or $\mathbb{R} \times \mathbb{C}^3/G$, for G a finite subgroup of SU(2) or SU(3), and when $G \subset$ SU(3) we do *not* require that G act freely on $\mathbb{C}^3 \setminus \{0\}$.

If X is a crepant resolution of $\mathbb{C}^3/G$, where G does not act freely on $\mathbb{C}^3\setminus\{0\}$, then the author shows [18, §9], [20] that X carries a family of Calabi–Yau metrics satisfying a complicated asymptotic condition at infinity, called *Quasi-ALE* metrics. These yield the local models necessary to resolve singularities locally of the form $\mathbb{R}\times\mathbb{C}^3/G$ with holonomy G_2. Using this method we can resolve many orbifolds T^7/Γ, and prove the existence of large numbers of compact 7-manifolds with holonomy G_2.

3.3 Step 3: Finding G_2-structures with small torsion

For each resolution X of $\mathbb{C}^2/G$ in case (a), and Y of $\mathbb{C}^3/G$ in case (b) above, we can find a 1-parameter family $\{h_t : t>0\}$ of metrics with the properties

(a) h_t is a Kähler metric on X with $\mathrm{Hol}(h_t)=\mathrm{SU}(2)$. Its injectivity radius satisfies $\delta(h_t)=O(t)$, its Riemann curvature satisfies $\|R(h_t)\|_{C^0}=O(t^{-2})$, and $h_t=h+O(t^4r^{-4})$ for large r, where h is the Euclidean metric on $\mathbb{C}^2/G$, and r the distance from the origin.

(b) h_t is Kähler on Y with $\mathrm{Hol}(h_t)=\mathrm{SU}(3)$, where $\delta(h_t)=O(t)$, $\|R(h_t)\|_{C^0}=O(t^{-2})$, and $h_t=h+O(t^6r^{-6})$ for large r.

In fact we can choose h_t to be isometric to t^2h_1, and then (a), (b) are easy to prove.

Suppose one of the components of the singular set S of T^7/Γ is locally modelled on $T^3\times\mathbb{C}^2/G$. Then T^3 has a natural flat metric h_{T^3}. Let X be the crepant resolution of $\mathbb{C}^2/G$ and let $\{h_t : t>0\}$ satisfy property (a). Then Proposition 2.10 gives a 1-parameter family of torsion-free G_2-structures $(\hat\varphi_t,\hat g_t)$ on $T^3\times X$ with $\hat g_t=h_{T^3}+h_t$. Similarly, if a component of S is modelled on $\mathcal{S}^1\times\mathbb{C}^3/G$, using Proposition 2.9 we get a family of torsion-free G_2-structures $(\hat\varphi_t,\hat g_t)$ on $\mathcal{S}^1\times Y$.

The idea is to make a G_2-structure (φ_t,g_t) on M by gluing together the torsion-free G_2-structures $(\hat\varphi_t,\hat g_t)$ on the patches $T^3\times X$ and $\mathcal{S}^1\times Y$, and (φ_0,g_0) on T^7/Γ. The gluing is done using a partition of unity. Naturally, the first derivative of the partition of unity introduces 'errors', so that (φ_t,g_t) is not torsion-free. The size of the torsion $\nabla\varphi_t$ depends on the difference $\hat\varphi_t-\varphi_0$ in the region where the partition of unity changes. On the patches $T^3\times X$, since $h_t-h=O(t^4r^{-4})$ and the partition of unity has nonzero derivative when $r=O(1)$, we find that $\nabla\varphi_t=O(t^4)$. Similarly $\nabla\varphi_t=O(t^6)$ on the patches $\mathcal{S}^1\times Y$, and so $\nabla\varphi_t=O(t^4)$ on M.

For small t, the dominant contributions to the injectivity radius $\delta(g_t)$ and Riemann curvature $R(g_t)$ are made by those of the metrics h_t on X and Y, so we expect $\delta(g_t)=O(t)$ and $\|R(g_t)\|_{C^0}=O(t^{-2})$ by properties (a) and (b) above. In this way we prove the following result [18, Th. 11.5.7], which gives the estimates on (φ_t,g_t) that we need.

Theorem 3.3. *On the compact* 7*-manifold* M *described above, and on many other* 7*-manifolds constructed in a similar fashion, one can write down the following data explicitly in coordinates:*

- *Positive constants A_1, A_2, A_3 and ϵ,*
- *A G_2-structure (φ_t, g_t) on M with $\mathrm{d}\varphi_t = 0$ for each $t \in (0, \epsilon)$, and*
- *A 3-form ψ_t on M with $\mathrm{d}^*\psi_t = \mathrm{d}^*\varphi_t$ for each $t \in (0, \epsilon)$.*

These satisfy three conditions:

(i) $\|\psi_t\|_{L^2} \leqslant A_1 t^4$, $\|\psi_t\|_{C^0} \leqslant A_1 t^3$ *and* $\|\mathrm{d}^*\psi_t\|_{L^{14}} \leqslant A_1 t^{16/7}$,

(ii) *the injectivity radius* $\delta(g_t)$ *satisfies* $\delta(g_t) \geqslant A_2 t$,

(iii) *the Riemann curvature* $R(g_t)$ *of* g_t *satisfies* $\big\|R(g_t)\big\|_{C^0} \leqslant A_3 t^{-2}$.

Here the operator d^* *and the norms* $\|\,.\,\|_{L^2}$, $\|\,.\,\|_{L^{14}}$ *and* $\|\,.\,\|_{C^0}$ *depend on* g_t.

Here one should regard ψ_t as a *first integral* of the torsion $\nabla\varphi_t$ of (φ_t, g_t). Thus the norms $\|\psi_t\|_{L^2}$, $\|\psi_t\|_{C^0}$ and $\|\mathrm{d}^*\psi_t\|_{L^{14}}$ are measures of $\nabla\varphi_t$. So parts (i)–(iii) say that $\nabla\varphi_t$ is small compared to the injectivity radius and Riemann curvature of (M, g_t).

3.4 Step 4: Deforming to a torsion-free G_2-structure

We prove the following analysis result.

Theorem 3.4. *Let A_1, A_2, A_3 be positive constants. Then there exist positive constants κ, K such that whenever $0 < t \leqslant \kappa$, the following is true.*

Let M be a compact 7-manifold, and (φ, g) a G_2-structure on M with $\mathrm{d}\varphi = 0$. Suppose ψ is a smooth 3-form on M with $\mathrm{d}^*\psi = \mathrm{d}^*\varphi$, and

(i) $\|\psi\|_{L^2} \leqslant A_1 t^4$, $\|\psi\|_{C^0} \leqslant A_1 t^{1/2}$ *and* $\|\mathrm{d}^*\psi\|_{L^{14}} \leqslant A_1$,

(ii) *the injectivity radius* $\delta(g)$ *satisfies* $\delta(g) \geqslant A_2 t$, *and*

(iii) *the Riemann curvature* $R(g)$ *satisfies* $\big\|R(g)\big\|_{C^0} \leqslant A_3 t^{-2}$.

Then there exists a smooth, torsion-free G_2-structure $(\tilde\varphi, \tilde g)$ on M with $\|\tilde\varphi - \varphi\|_{C^0} \leqslant K t^{1/2}$.

Basically, this result says that if (φ, g) is a G_2-structure on M, and the torsion $\nabla\varphi$ is sufficiently small, then we can deform to a nearby G_2-structure $(\tilde\varphi, \tilde g)$ that is torsion-free. Here is a sketch of the proof of Theorem 3.4, ignoring several technical points. The proof is that given in [18, §11.6–§11.8], which is an improved version of the proof in [14].

We have a 3-form φ with $\mathrm{d}\varphi = 0$ and $\mathrm{d}^*\varphi = \mathrm{d}^*\psi$ for small ψ, and we wish to construct a nearby 3-form $\tilde\varphi$ with $\mathrm{d}\tilde\varphi = 0$ and $\tilde{\mathrm{d}}^*\tilde\varphi = 0$. Set $\tilde\varphi = \varphi + \mathrm{d}\eta$, where η is a small 2-form. Then η must satisfy a nonlinear p.d.e., which we write as

$$\mathrm{d}^*\mathrm{d}\eta = -\mathrm{d}^*\psi + \mathrm{d}^*F(\mathrm{d}\eta), \tag{9}$$

where F is nonlinear, satisfying $F(\mathrm{d}\eta) = O\big(|\mathrm{d}\eta|^2\big)$.

We solve (9) by iteration, introducing a sequence $\{\eta_j\}_{j=0}^{\infty}$ with $\eta_0 = 0$, satisfying the inductive equations

$$\mathrm{d}^*\mathrm{d}\eta_{j+1} = -\mathrm{d}^*\psi + \mathrm{d}^*F(\mathrm{d}\eta_j), \qquad \mathrm{d}^*\eta_{j+1} = 0. \tag{10}$$

If such a sequence exists and converges to η, then taking the limit in (10) shows that η satisfies (9), giving us the solution we want.

The key to proving this is an *inductive estimate* on the sequence $\{\eta_j\}_{j=0}^{\infty}$. The inductive estimate we use has three ingredients, the equations

$$\|\mathrm{d}\eta_{j+1}\|_{L^2} \leqslant \|\psi\|_{L^2} + C_1\|\mathrm{d}\eta_j\|_{L^2}\|\mathrm{d}\eta_j\|_{C^0}, \tag{11}$$

$$\|\nabla\mathrm{d}\eta_{j+1}\|_{L^{14}} \leqslant C_2\big(\|\mathrm{d}^*\psi\|_{L^{14}} + \|\nabla\mathrm{d}\eta_j\|_{L^{14}}\|\mathrm{d}\eta_j\|_{C^0} + t^{-4}\|\mathrm{d}\eta_{j+1}\|_{L^2}\big), \tag{12}$$

$$\|\mathrm{d}\eta_j\|_{C^0} \leqslant C_3\big(t^{1/2}\|\nabla\mathrm{d}\eta_j\|_{L^{14}} + t^{-7/2}\|\mathrm{d}\eta_j\|_{L^2}\big). \tag{13}$$

Here C_1, C_2, C_3 are positive constants independent of t. Equation (11) is obtained from (10) by taking the L^2-inner product with η_{j+1} and integrating by parts. Using the fact that $\mathrm{d}^*\varphi = \mathrm{d}^*\psi$ and $\|\psi\|_{L^2} = O(t^4)$, $|\psi| = O(t^{1/2})$ we get a powerful estimate of the L^2-norm of $\mathrm{d}\eta_{j+1}$.

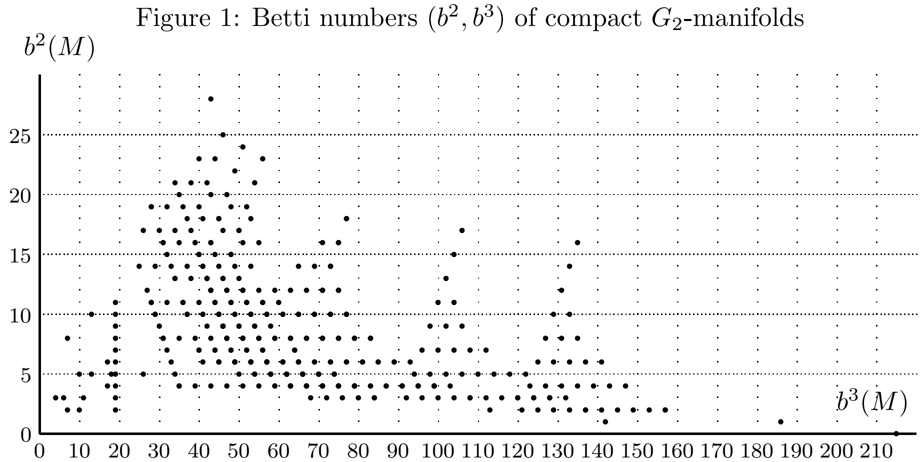
Equation (12) is derived from an *elliptic regularity estimate* for the operator $\mathrm{d}+\mathrm{d}^*$ acting on 3-forms on M. Equation (13) follows from the *Sobolev embedding theorem*, since $L^{14}_1(M) \hookrightarrow C^0(M)$. Both (12) and (13) are proved on small balls of radius $O(t)$ in M, using parts (ii) and (iii) of Theorem 3.3, and this is where the powers of t come from.

Using (11)-(13) and part (i) of Theorem 3.3 we show that if

$$\|\mathrm{d}\eta_j\|_{L^2} \leqslant C_4 t^4, \quad \|\nabla\mathrm{d}\eta_j\|_{L^{14}} \leqslant C_5, \quad \text{and } \|\mathrm{d}\eta_j\|_{C^0} \leqslant K t^{1/2}, \tag{14}$$

where C_4, C_5 and K are positive constants depending on C_1, C_2, C_3 and A_1, and if t is sufficiently small, then the same inequalities (14) apply to $\mathrm{d}\eta_{j+1}$. Since $\eta_0 = 0$, by induction (14) applies for all j and the sequence $\{\mathrm{d}\eta_j\}_{j=0}^{\infty}$ is bounded in the Banach space $L^{14}_1(\Lambda^3 T^*M)$. One can then use standard techniques in analysis to prove that this sequence converges to a smooth limit $\mathrm{d}\eta$. This concludes the proof of Theorem 3.4.

Figure 1: Betti numbers (b^2, b^3) of compact G_2-manifolds



From Theorems 3.3 and 3.4 we see that the compact 7-manifold M constructed in Step 2 admits torsion-free G_2-structures $(\tilde{\varphi}, \tilde{g})$. Theorem 2.5 then shows that $\mathrm{Hol}(\tilde{g}) = G_2$ if and only if $\pi_1(M)$ is finite. In the example above M is simply-connected, and so $\pi_1(M) = \{1\}$ and M has metrics with holonomy G_2, as we want.

By considering different groups Γ acting on T^7, and also by finding topologically distinct resolutions $M_1, \ldots, M_k$ of the same orbifold T^7/Γ, we can construct many compact Riemannian 7-manifolds with holonomy G_2. A good number of examples are given in [18, §12]. Figure 1 displays the Betti numbers of compact, simply-connected 7-manifolds with holonomy G_2 constructed there. There are 252 different sets of Betti numbers.

Examples are also known [18, §12.4] of compact 7-manifolds with holonomy G_2 with finite, nontrivial fundamental group. It seems likely to the author that the Betti numbers given in Figure 1 are only a small proportion of the Betti numbers of all compact, simply-connected 7-manifolds with holonomy G_2.

3.5 Other constructions of compact G_2-manifolds

Here are two other methods, taken from [18, §11.9], of constructing compact 7-manifolds with holonomy G_2. The first was outlined by the author in [15, §4.3].

Method 1. Let (Y, J, h) be a Calabi–Yau 3-fold, with Kähler form ω and holomorphic volume form θ. Suppose $\sigma : Y \to Y$ is an involution, satisfying $\sigma^*(h) = h$, $\sigma^*(J) = -J$ and $\sigma^*(\theta) = \bar{\theta}$. We call σ a *real structure* on Y. Let N be the fixed point set of σ in Y. Then N is a real 3-dimensional submanifold of Y, and is in fact a special Lagrangian 3-fold.

Let $\mathcal{S}^1 = \mathbb{R}/\mathbb{Z}$, and define a torsion-free G_2-structure (φ, g) on $\mathcal{S}^1 \times Y$ as in Proposition 2.9. Then $\varphi = \mathrm{d}x \wedge \omega + \mathrm{Re}\,\theta$, where $x \in \mathbb{R}/\mathbb{Z}$ is the coordinate on $\mathcal{S}^1$. Define $\hat{\sigma} : \mathcal{S}^1 \times Y \to \mathcal{S}^1 \times Y$ by $\hat{\sigma}\bigl((x, y)\bigr) = \bigl(-x, \sigma(y)\bigr)$. Then $\hat{\sigma}$ preserves (φ, g) and $\hat{\sigma}^2 = 1$. The fixed points of $\hat{\sigma}$ in $\mathcal{S}^1 \times Y$ are $\{\mathbb{Z}, \frac{1}{2} + \mathbb{Z}\} \times N$. Thus $(\mathcal{S}^1 \times Y)/\langle\hat{\sigma}\rangle$ is an orbifold. Its singular set is 2 copies of N, and each singular point is modelled on $\mathbb{R}^3 \times \mathbb{R}^4/\{\pm 1\}$.

We aim to resolve $(\mathcal{S}^1 \times Y)/\langle\hat{\sigma}\rangle$ to get a compact 7-manifold M with holonomy G_2. Locally, each singular point should be resolved like $\mathbb{R}^3 \times X$, where X is an ALE Calabi–Yau 2-fold asymptotic to $\mathbb{C}^2/\{\pm 1\}$. There is a 3-dimensional family of such X, and we need to choose one member of this family for each singular point in the singular set.

Calculations by the author indicate that the data needed to do this is a closed, coclosed 1-form α on N that is nonzero at every point of N. The existence of a suitable 1-form α depends on the metric on N, which is the restriction of the metric g on Y. But g comes from the solution of the Calabi Conjecture, so we know little about it. This may make the method difficult to apply in practice.

The second method has been successfully applied by Kovalev [22], and is based on an idea due to Simon Donaldson.

Method 2. Let X be a projective complex 3-fold with canonical bundle K_X, and s a holomorphic section of K_X^{-1} which vanishes to order 1 on a smooth divisor D in X. Then D has trivial canonical bundle, so D is T^4 or $K3$. Suppose D is a $K3$ surface. Define $Y = X \setminus D$, and suppose Y is simply-connected.

Then Y is a noncompact complex 3-fold with K_Y trivial, and one infinite end modelled on $D \times \mathcal{S}^1 \times [0, \infty)$. Using a version of the proof of the Calabi Conjecture for noncompact manifolds one constructs a complete Calabi–Yau metric h on Y, which is asymptotic to the product on $D \times \mathcal{S}^1 \times [0, \infty)$ of a Calabi–Yau metric on D, and Euclidean metrics on $\mathcal{S}^1$ and $[0, \infty)$. We call such metrics *Asymptotically Cylindrical*.

Suppose we have such a metric on Y. Define a torsion-free G_2-structure (φ, g) on $\mathcal{S}^1 \times Y$ as in Proposition 2.9. Then $\mathcal{S}^1 \times Y$ is a noncompact G_2-manifold with one end modelled on $D \times T^2 \times [0, \infty)$, whose metric is asymptotic to the product on $D \times T^2 \times [0, \infty)$ of a Calabi–Yau metric on D, and Euclidean metrics on T^2 and $[0, \infty)$.

Donaldson and Kovalev's idea is to take two such products $\mathcal{S}^1 \times Y_1$ and $\mathcal{S}^1 \times Y_2$ whose infinite ends are isomorphic in a suitable way, and glue them together to get a compact 7-manifold M with holonomy G_2. The gluing process swaps round the $\mathcal{S}^1$ factors. That is, the $\mathcal{S}^1$ factor in $\mathcal{S}^1 \times Y_1$ is identified with the asymptotic $\mathcal{S}^1$ factor in $Y_2 \sim D_2 \times \mathcal{S}^1 \times [0, \infty)$, and vice versa.

4 Compact Spin(7)-manifolds from Calabi–Yau 4-orbifolds

In a very similar way to the G_2 case, one can construct examples of compact 8-manifolds with holonomy Spin(7) by resolving the singularities of torus orbifolds T^8/Γ. This is done in [16] and [18, §13–§14]. In [18, §14], examples are constructed which realize 181 different sets of Betti numbers. Two compact 8-manifolds with holonomy Spin(7) and the same Betti numbers may be distinguished by the cup products on their cohomologies (examples of this are given in [16, §3.4]), so they probably represent rather more than 181 topologically distinct 8-manifolds.

The main differences with the G_2 case are, firstly, that the technical details of the analysis are different and harder, and secondly, that the singularities that arise are typically more complicated and more tricky to resolve. One reason for this is that in the G_2 case the singular set is made up of 1 and 3-dimensional pieces in a 7-dimensional space, so one can often arrange for the pieces to avoid each other, and resolve them independently.

But in the Spin(7) case the singular set is typically made up of 4-dimensional pieces in an 8-dimensional space, so they nearly always intersect. There are also topological constraints arising from the $\hat{A}$-genus, which do not apply in the G_2 case. The moral appears to be that when you increase the dimension, things become more difficult.

Anyway, we will not discuss this further, as the principles are very similar to the G_2 case above. Instead, we will discuss an entirely different construction of compact 8-manifolds with holonomy Spin(7) developed by the author in [17] and [18, §15], a little like Method 1 of §3.5. In this we start from a *Calabi–Yau 4-orbifold* rather than from T^8. The construction can be divided into five steps.

Step 1. Find a compact, complex 4-orbifold (Y, J) satisfying the conditions:

(a) Y has only finitely many singular points $p_1, \dots, p_k$, for $k \geqslant 1$.
(b) Y is modelled on $\mathbb{C}^4/\langle i \rangle$ near each p_j, where i acts on $\mathbb{C}^4$ by complex multiplication.
(c) There exists an antiholomorphic involution $\sigma : Y \to Y$ whose fixed point set is $\{p_1, \dots, p_k\}$.
(d) $Y \setminus \{p_1, \dots, p_k\}$ is simply-connected, and $h^{2,0}(Y) = 0$.

Step 2. Choose a σ-invariant Kähler class on Y. Then by Theorem 2.8 there exists a unique σ-invariant Ricci-flat Kähler metric g in this Kähler class. Let ω be the Kähler form of g. Let θ be a holomorphic volume form for (Y, J, g). By multiplying θ by $e^{i\phi}$ if necessary, we can arrange that $\sigma^*(\theta) = \bar{\theta}$.

Define $\Omega = \frac{1}{2}\omega\wedge\omega + \mathrm{Re}\,\theta$. Then (Ω, g) is a torsion-free Spin(7)-structure on Y, by Proposition 2.11. Also, (Ω, g) is σ-invariant, as $\sigma^*(\omega) = -\omega$ and $\sigma^*(\theta) = \bar{\theta}$. Define $Z = Y/\langle\sigma\rangle$. Then Z is a compact real 8-orbifold with isolated singular points $p_1, \dots, p_k$, and (Ω, g) pushes down to a torsion-free Spin(7)-structure (Ω, g) on Z.

Step 3. Z is modelled on $\mathbb{R}^8/G$ near each p_j, where G is a certain finite subgroup of Spin(7) with $|G| = 8$. We can write down two explicit, topologically distinct ALE Spin(7)-manifolds X_1, X_2 asymptotic to $\mathbb{R}^8/G$. Each carries a 1-parameter family of homothetic ALE metrics h_t for $t > 0$ with $\mathrm{Hol}(h_t) = \mathbb{Z}_2 \ltimes \mathrm{SU}(4) \subset \mathrm{Spin}(7)$.

For $j = 1, \dots, k$ we choose $i_j = 1$ or 2, and resolve the singularities of Z by gluing in X_{i_j} at the singular point p_j for $j = 1, \dots, k$, to get a compact, nonsingular 8-manifold M, with projection $\pi : M \to Z$.

Step 4. On M, we explicitly write down a 1-parameter family of Spin(7)-structures (Ω_t, g_t) depending on $t \in (0, \epsilon)$. They are not torsion-free, but have small torsion when t is small. As $t \to 0$, the Spin(7)-structure (Ω_t, g_t) converges to the singular Spin(7)-structure $\pi^*(\Omega_0, g_0)$.

Step 5. We prove using analysis that for sufficiently small t, the Spin(7)-structure (Ω_t, g_t) on M, with small torsion, can be deformed to a Spin(7)-structure $(\tilde{\Omega}_t, \tilde{g}_t)$, with zero torsion.

It turns out that if $i_j = 1$ for $j = 1, \dots, k$ we have $\pi_1(M) \cong \mathbb{Z}_2$ and $\mathrm{Hol}(\tilde{g}_t) = \mathbb{Z}_2 \ltimes \mathrm{SU}(4)$, and for the other $2^k - 1$ choices of $i_1, \dots, i_k$ we have $\pi_1(M) = \{1\}$ and $\mathrm{Hol}(\tilde{g}_t) = \mathrm{Spin}(7)$. So $\tilde{g}_t$ is a metric with holonomy Spin(7) on the compact 8-manifold M for $(i_1, \dots, i_k) \neq (1, \dots, 1)$.

Once we have completed Step 1, Step 2 is immediate. Steps 4 and 5 are analogous to Steps 3 and 4 of §3, and can be done using the techniques and analytic results developed by the author for the first T^8/Γ construction of compact Spin(7)-manifolds, [16], [18, §13]. So the really new material is in Steps 1 and 3, and we will discuss only these.

4.1 Step 1: An example

We do Step 1 using complex algebraic geometry. The problem is that conditions (a)–(d) above are very restrictive, so it is not that easy to find *any* Y satisfying all four conditions. All the examples Y the author has found are constructed using *weighted projective spaces*, an important class of complex orbifolds.

Definition 4.1. Let $m \geqslant 1$ be an integer, and $a_0, a_1, \ldots, a_m$ positive integers with highest common factor 1. Let $\mathbb{C}^{m+1}$ have complex coordinates on $(z_0, \ldots, z_m)$, and define an action of the complex Lie group $\mathbb{C}^*$ on $\mathbb{C}^{m+1}$ by

$$(z_0, \ldots, z_m) \xmapsto{u} (u^{a_0} z_0, \ldots, u^{a_m} z_m), \qquad \text{for } u \in \mathbb{C}^*.$$

The *weighted projective space* $\mathbb{CP}^m_{a_0,\ldots,a_m}$ is $\bigl(\mathbb{C}^{m+1} \setminus \{0\}\bigr)/\mathbb{C}^*$. The $\mathbb{C}^*$-orbit of $(z_0, \ldots, z_m)$ is written $[z_0, \ldots, z_m]$.

Here is the simplest example the author knows.

Example 4.2. Let Y be the hypersurface of degree 12 in $\mathbb{CP}^5_{1,1,1,1,4,4}$ given by

$$Y = \bigl\{[z_0, \ldots, z_5] \in \mathbb{CP}^5_{1,1,1,1,4,4} : z_0^{12} + z_1^{12} + z_2^{12} + z_3^{12} + z_4^3 + z_5^3 = 0\bigr\}.$$

Calculation shows that Y has trivial canonical bundle and three singular points $p_1 = [0,0,0,0,1,-1]$, $p_2 = [0,0,0,0,1,e^{\pi i/3}]$ and $p_3 = [0,0,0,0,1,e^{-\pi i/3}]$, modelled on $\mathbb{C}^4/\langle i\rangle$.

Now define a map $\sigma : Y \to Y$ by

$$\sigma : [z_0, \ldots, z_5] \longmapsto [\bar{z}_1, -\bar{z}_0, \bar{z}_3, -\bar{z}_2, \bar{z}_5, \bar{z}_4].$$

Note that $\sigma^2 = 1$, though this is not immediately obvious, because of the geometry of $\mathbb{CP}^5_{1,1,1,1,4,4}$. It can be shown that conditions (a)–(d) of Step 1 above hold for Y and σ.

More suitable 4-folds Y may be found by taking hypersurfaces or complete intersections in other weighted projective spaces, possibly also dividing by a finite group, and then doing a crepant resolution to get rid of any singularities that we don't want. Examples are given in [17], [18, §15].

4.2 Step 3: Resolving $\mathbb{R}^8/G$

Define $\alpha, \beta : \mathbb{R}^8 \to \mathbb{R}^8$ by

$$\alpha : (x_1, \ldots, x_8) \mapsto (-x_2, x_1, -x_4, x_3, -x_6, x_5, -x_8, x_7),$$
$$\beta : (x_1, \ldots, x_8) \mapsto (x_3, -x_4, -x_1, x_2, x_7, -x_8, -x_5, x_6).$$

Then α, β preserve Ω_0 given in (2), so they lie in Spin(7). Also $\alpha^4 = \beta^4 = 1$, $\alpha^2 = \beta^2$ and $\alpha\beta = \beta\alpha^3$. Let $G = \langle \alpha, \beta \rangle$. Then G is a finite nonabelian subgroup of Spin(7) of order 8, which acts freely on $\mathbb{R}^8 \setminus \{0\}$. One can show that if Z is the compact Spin(7)-orbifold constructed in Step 2 above, then $T_{p_j} Z$ is isomorphic to $\mathbb{R}^8/G$ for $j = 1, \ldots, k$, with an isomorphism identifying the Spin(7)-structures (Ω, g) on Z and (Ω_0, g_0) on $\mathbb{R}^8/G$, such that β corresponds to the σ-action on Y.

In the next two examples we shall construct two different ALE Spin(7)-manifolds (X_1, Ω_1, g_1) and (X_2, Ω_2, g_2) asymptotic to $\mathbb{R}^8/G$.

Example 4.3. Define complex coordinates $(z_1, \ldots, z_4)$ on $\mathbb{R}^8$ by

$$(z_1, z_2, z_3, z_4) = (x_1 + ix_2, x_3 + ix_4, x_5 + ix_6, x_7 + ix_8),$$

Then $g_0 = |\mathrm{d}z_1|^2 + \cdots + |\mathrm{d}z_4|^2$, and $\Omega_0 = \frac{1}{2}\omega_0 \wedge \omega_0 + \mathrm{Re}(\theta_0)$, where ω_0 and θ_0 are the usual Kähler form and complex volume form on $\mathbb{C}^4$. In these coordinates, α and β are given by

$$\begin{aligned} \alpha &: (z_1, \ldots, z_4) \mapsto (iz_1, iz_2, iz_3, iz_4), \\ \beta &: (z_1, \ldots, z_4) \mapsto (\bar{z}_2, -\bar{z}_1, \bar{z}_4, -\bar{z}_3). \end{aligned} \tag{15}$$

Now $\mathbb{C}^4/\langle \alpha \rangle$ is a complex singularity, as $\alpha \in \mathrm{SU}(4)$. Let (Y_1, π_1) be the blow-up of $\mathbb{C}^4/\langle \alpha \rangle$ at 0. Then Y_1 is the unique crepant resolution of $\mathbb{C}^4/\langle \alpha \rangle$. The action of β on $\mathbb{C}^4/\langle \alpha \rangle$ lifts to a *free* antiholomorphic map $\beta : Y_1 \to Y_1$ with $\beta^2 = 1$. Define $X_1 = Y_1/\langle \beta \rangle$. Then X_1 is a nonsingular 8-manifold, and the projection $\pi_1 : Y_1 \to \mathbb{C}^4/\langle \alpha \rangle$ pushes down to $\pi_1 : X_1 \to \mathbb{R}^8/G$.

There exist ALE Calabi–Yau metrics g_1 on Y_1, which were written down explicitly by Calabi [9, p. 285], and are invariant under the action of β on Y_1. Let ω_1 be the Kähler form of g_1, and $\theta_1 = \pi_1^*(\theta_0)$ the holomorphic volume form on Y_1. Define $\Omega_1 = \frac{1}{2}\omega_1 \wedge \omega_1 + \mathrm{Re}(\theta_1)$. Then (Ω_1, g_1) is a torsion-free Spin(7)-structure on Y_1, as in Proposition 2.11.

As $\beta^*(\omega_1) = -\omega_1$ and $\beta^*(\theta_1) = \bar{\theta}_1$, we see that β preserves (Ω_1, g_1). Thus (Ω_1, g_1) pushes down to a torsion-free Spin(7)-structure (Ω_1, g_1) on X_1. Then (X_1, Ω_1, g_1) is an *ALE* Spin(7)-*manifold* asymptotic to $\mathbb{R}^8/G$.

Example 4.4. Define new complex coordinates $(w_1, \ldots, w_4)$ on $\mathbb{R}^8$ by

$$(w_1, w_2, w_3, w_4) = (-x_1 + ix_3, x_2 + ix_4, -x_5 + ix_7, x_6 + ix_8).$$

Again we find that $g_0 = |\mathrm{d}w_1|^2 + \cdots + |\mathrm{d}w_4|^2$ and $\Omega_0 = \frac{1}{2}\omega_0 \wedge \omega_0 + \mathrm{Re}(\theta_0)$. In these coordinates, α and β are given by

$$\begin{aligned} \alpha &: (w_1, \ldots, w_4) \mapsto (\bar{w}_2, -\bar{w}_1, \bar{w}_4, -\bar{w}_3), \\ \beta &: (w_1, \ldots, w_4) \mapsto (iw_1, iw_2, iw_3, iw_4). \end{aligned} \tag{16}$$

Observe that (15) and (16) are the same, except that the rôles of α, β are reversed. Therefore we can use the ideas of Example 4.3 again.

Let Y_2 be the crepant resolution of $\mathbb{C}^4/\langle \beta \rangle$. The action of α on $\mathbb{C}^4/\langle \beta \rangle$ lifts to a free antiholomorphic involution of Y_2. Let $X_2 = Y_2/\langle \alpha \rangle$. Then X_2 is nonsingular, and carries a torsion-free Spin(7)-structure (Ω_2, g_2), making (X_2, Ω_2, g_2) into an ALE Spin(7)-manifold asymptotic to $\mathbb{R}^8/G$.

We can now explain the remarks on holonomy groups at the end of Step 5. The holonomy groups $\mathrm{Hol}(g_i)$ of the metrics g_1, g_2 in Examples 4.3 and 4.4 are both isomorphic to $\mathbb{Z}_2 \ltimes \mathrm{SU}(4)$, a subgroup of Spin(7). However, they are two *different* inclusions of $\mathbb{Z}_2 \ltimes \mathrm{SU}(4)$ in Spin(7), as in the first case the complex structure is α and in the second β.

The Spin(7)-structure (Ω, g) on Z also has holonomy $\mathrm{Hol}(g) = \mathbb{Z}_2 \ltimes \mathrm{SU}(4)$. Under the natural identifications we have $\mathrm{Hol}(g_1) = \mathrm{Hol}(g)$ but $\mathrm{Hol}(g_2) \neq \mathrm{Hol}(g)$ as subgroups of Spin(7). Therefore, if we choose $i_j = 1$ for all $j = 1, \ldots, k$, then Z and X_{i_j} all have the same holonomy group $\mathbb{Z}_2 \ltimes \mathrm{SU}(4)$, so they combine to give metrics $\tilde{g}_t$ on M with $\mathrm{Hol}(\tilde{g}_t) = \mathbb{Z}_2 \ltimes \mathrm{SU}(4)$.

However, if $i_j = 2$ for some j then the holonomy of g on Z and g_{i_j} on X_{i_j} are *different* $\mathbb{Z}_2 \ltimes \mathrm{SU}(4)$ subgroups of Spin(7), which together generate the whole group Spin(7). Thus they combine to give metrics $\tilde{g}_t$ on M with $\mathrm{Hol}(\tilde{g}_t) = \mathrm{Spin}(7)$.

4.3 Conclusions

The author was able in [17] and [18, Ch. 15] to construct compact 8-manifolds with holonomy Spin(7) realizing 14 distinct sets of Betti numbers, which are given in Table 1. Probably there are many other examples which can be produced by similar methods.

Table 1: Betti numbers (b^2, b^3, b^4) of compact Spin(7)-manifolds

$(4, 33, 200)$	$(3, 33, 202)$	$(2, 33, 204)$	$(1, 33, 206)$	$(0, 33, 208)$
$(1, 0, 908)$	$(0, 0, 910)$	$(1, 0, 1292)$	$(0, 0, 1294)$	$(1, 0, 2444)$
$(0, 0, 2446)$	$(0, 6, 3730)$	$(0, 0, 4750)$	$(0, 0, 11\,662)$	

Comparing these Betti numbers with those of the compact 8-manifolds constructed in [18, Ch. 14] by resolving torus orbifolds T^8/Γ, we see that these examples the middle Betti number b^4 is much bigger, as much as 11 662 in one case.

Given that the two constructions of compact 8-manifolds with holonomy Spin(7) that we know appear to produce sets of 8-manifolds with rather different 'geography', it is tempting to speculate that the set of all compact 8-manifolds with holonomy Spin(7) may be rather large, and that those constructed so far are a small sample with atypical behaviour.

Part II. Calibrated Geometry

5 Introduction to calibrated geometry

Calibrated geometry was introduced in the seminal paper of Harvey and Lawson [12]. We introduce the basic ideas in §5.1–§5.2, and then discuss the G_2 calibrations in more detail in §5.3–§5.5, and the Spin(7) calibration in §5.6.

5.1 Calibrations and calibrated submanifolds

We begin by defining *calibrations* and *calibrated submanifolds*, following Harvey and Lawson [12].

Definition 5.1. Let (M, g) be a Riemannian manifold. An *oriented tangent k-plane* V on M is a vector subspace V of some tangent space T_xM to M with $\dim V = k$, equipped with an orientation. If V is an oriented tangent k-plane on M then $g|_V$ is a Euclidean metric on V, so combining $g|_V$ with the orientation on V gives a natural *volume form* vol_V on V, which is a k-form on V.

Now let φ be a closed k-form on M. We say that φ is a *calibration* on M if for every oriented k-plane V on M we have $\varphi|_V \leqslant \mathrm{vol}_V$. Here $\varphi|_V = \alpha \cdot \mathrm{vol}_V$ for some $\alpha \in \mathbb{R}$, and $\varphi|_V \leqslant \mathrm{vol}_V$ if $\alpha \leqslant 1$. Let N be an oriented submanifold of M with dimension k. Then each tangent space T_xN for $x \in N$ is an oriented tangent k-plane. We say that N is a *calibrated submanifold* if $\varphi|_{T_xN} = \mathrm{vol}_{T_xN}$ for all $x \in N$.

It is easy to show that calibrated submanifolds are automatically *minimal submanifolds* [12, Th. II.4.2]. We prove this in the compact case, but noncompact calibrated submanifolds are locally volume-minimizing as well.

Proposition 5.2. *Let (M, g) be a Riemannian manifold, φ a calibration on M, and N a compact φ-submanifold in M. Then N is volume-minimizing in its homology class.*

Proof. Let $\dim N = k$, and let $[N] \in H_k(M, \mathbb{R})$ and $[\varphi] \in H^k(M, \mathbb{R})$ be the homology and cohomology classes of N and φ. Then

$$[\varphi] \cdot [N] = \int_{x \in N} \varphi\big|_{T_xN} = \int_{x \in N} \mathrm{vol}_{T_xN} = \mathrm{Vol}(N),$$

since $\varphi|_{T_xN} = \mathrm{vol}_{T_xN}$ for each $x \in N$, as N is a calibrated submanifold. If N' is any other compact k-submanifold of M with $[N'] = [N]$ in $H_k(M, \mathbb{R})$, then

$$[\varphi] \cdot [N] = [\varphi] \cdot [N'] = \int_{x \in N'} \varphi\big|_{T_xN'} \leqslant \int_{x \in N'} \mathrm{vol}_{T_xN'} = \mathrm{Vol}(N'),$$

since $\varphi|_{T_xN'} \leqslant \mathrm{vol}_{T_xN'}$ because φ is a calibration. The last two equations give $\mathrm{Vol}(N) \leqslant \mathrm{Vol}(N')$. Thus N is volume-minimizing in its homology class. □

Now let (M, g) be a Riemannian manifold with a calibration φ, and let $\iota : N \to M$ be an immersed submanifold. Whether N is a φ-submanifold depends upon the tangent spaces of N. That is, it depends on ι and its first derivative. So, to be calibrated with respect to φ is a *first-order* partial differential equation on ι. But if N is calibrated then N is minimal, and to be minimal is a *second-order* partial differential equation on ι.

One moral is that the calibrated equations, being first-order, are often easier to solve than the minimal submanifold equations, which are second-order. So calibrated geometry is a fertile source of examples of minimal submanifolds.

5.2 Calibrated submanifolds and special holonomy

Next we explain the connection with Riemannian holonomy. Let $G \subset \mathrm{O}(n)$ be a possible holonomy group of a Riemannian metric. In particular, we can take G to be one of the holonomy groups $\mathrm{U}(m)$, $\mathrm{SU}(m)$, $\mathrm{Sp}(m)$, G_2 or $\mathrm{Spin}(7)$ from Berger's classification. Then G acts on the k-forms $\Lambda^k(\mathbb{R}^n)^*$ on $\mathbb{R}^n$, so we can look for G-invariant k-forms on $\mathbb{R}^n$.

Suppose φ_0 is a nonzero, G-invariant k-form on $\mathbb{R}^n$. By rescaling φ_0 we can arrange that for each oriented k-plane $U \subset \mathbb{R}^n$ we have $\varphi_0|_U \leqslant \mathrm{vol}_U$, and that $\varphi_0|_U = \mathrm{vol}_U$ for at least one such U. Then $\varphi_0|_{\gamma \cdot U} = \mathrm{vol}_{\gamma \cdot U}$ by G-invariance, so $\gamma \cdot U$ is a calibrated k-plane for all $\gamma \in G$. Thus the family of φ_0-calibrated k-planes in $\mathbb{R}^n$ is reasonably large, and it is likely the calibrated submanifolds will have an interesting geometry.

Now let M be a manifold of dimension n, and g a metric on M with Levi-Civita connection ∇ and holonomy group G. Then by Theorem 2.2 there is a k-form φ on M with $\nabla\varphi = 0$, corresponding to φ_0. Hence $\mathrm{d}\varphi = 0$, and φ is closed. Also, the condition $\varphi_0|_U \leqslant \mathrm{vol}_U$ for all oriented k-planes U in $\mathbb{R}^n$ implies that $\varphi|_V \leqslant \mathrm{vol}_V$ for all oriented tangent k-planes V in M. Thus φ is a *calibration* on M.

This gives us a general method for finding interesting calibrations on manifolds with reduced holonomy. Here are the most significant examples of this.

- Let $G = \mathrm{U}(m) \subset \mathrm{O}(2m)$. Then G preserves a 2-form ω_0 on $\mathbb{R}^{2m}$. If g is a metric on M with holonomy $\mathrm{U}(m)$ then g is *Kähler* with complex structure J, and the 2-form ω on M associated to ω_0 is the *Kähler form* of g.

  One can show that ω is a calibration on (M, g), and the calibrated submanifolds are exactly the *holomorphic curves* in (M, J). More generally $\omega^k/k!$ is a calibration on M for $1 \leqslant k \leqslant m$, and the corresponding calibrated submanifolds are the complex k-dimensional submanifolds of (M, J).

- Let $G = \mathrm{SU}(m) \subset \mathrm{O}(2m)$. Then G preserves a *complex volume form* $\Omega_0 = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_m$ on $\mathbb{C}^m$. Thus a *Calabi–Yau m-fold* (M, g) with $\mathrm{Hol}(g) = \mathrm{SU}(m)$ has a *holomorphic volume form* Ω. The real part $\mathrm{Re}\,\Omega$ is a calibration on M, and the corresponding calibrated submanifolds are called *special Lagrangian submanifolds*.

- The group $G_2 \subset \mathrm{O}(7)$ preserves a 3-form φ_0 and a 4-form $*\varphi_0$ on $\mathbb{R}^7$. Thus a Riemannian 7-manifold (M,g) with holonomy G_2 comes with a 3-form φ and 4-form $*\varphi$, which are both calibrations. The corresponding calibrated submanifolds are called *associative 3-folds* and *coassociative 4-folds*.
- The group $\mathrm{Spin}(7) \subset \mathrm{O}(8)$ preserves a 4-form Ω_0 on $\mathbb{R}^8$. Thus a Riemannian 8-manifold (M,g) with holonomy $\mathrm{Spin}(7)$ has a 4-form Ω, which is a calibration. We call Ω-submanifolds *Cayley 4-folds*.

It is an important general principle that to each calibration φ on an n-manifold (M,g) with special holonomy we construct in this way, there corresponds a constant calibration φ_0 on $\mathbb{R}^n$. Locally, φ-submanifolds in M will look very like φ_0-submanifolds in $\mathbb{R}^n$, and have many of the same properties. Thus, to understand the calibrated submanifolds in a manifold with special holonomy, it is often a good idea to start by studying the corresponding calibrated submanifolds of $\mathbb{R}^n$.

In particular, singularities of φ-submanifolds in M will be locally modelled on singularities of φ_0-submanifolds in $\mathbb{R}^n$. (In the sense of Geometric Measure Theory, the *tangent cone* at a singular point of a φ-submanifold in M is a conical φ_0-submanifold in $\mathbb{R}^n$.) So by studying singular φ_0-submanifolds in $\mathbb{R}^n$, we may understand the singular behaviour of φ-submanifolds in M.

5.3 Associative and coassociative submanifolds

We now discuss the calibrated submanifolds of G_2-manifolds.

Definition 5.3. Let (M,φ,g) be a G_2-*manifold*, as in §2.2. Then the 3-form φ is a *calibration* on (M,g). We define an *associative 3-fold* in M to be a 3-submanifold of M calibrated with respect to φ. Similarly, the Hodge star $*\varphi$ of φ is a calibration 4-form on (M,g). We define a *coassociative 4-fold* in M to be a 4-submanifold of M calibrated with respect to $*\varphi$.

To understand these, it helps to begin with some calculations on $\mathbb{R}^7$. Let the metric g_0, 3-form φ_0 and 4-form $*\varphi_0$ on $\mathbb{R}^7$ be as in §2.2. Define an *associative 3-plane* to be an oriented 3-dimensional vector subspace V of $\mathbb{R}^7$ with $\varphi_0|_V = \mathrm{vol}_V$, and a *coassociative 4-plane* to be an oriented 4-dimensional vector subspace V of $\mathbb{R}^7$ with $*\varphi_0|_V = \mathrm{vol}_V$. From [12, Th. IV.1.8, Def. IV.1.15] we have:

Proposition 5.4. *The family $\mathcal{F}^3$ of associative 3-planes in $\mathbb{R}^7$ and the family $\mathcal{F}^4$ of coassociative 4-planes in $\mathbb{R}^7$ are both isomorphic to $G_2/SO(4)$, with dimension 8.*

Examples of an associative 3-plane U and a coassociative 4-plane V are

$$U = \bigl\{(x_1,x_2,x_3,0,0,0,0) : x_j \in \mathbb{R}\bigr\} \text{ and } V = \bigl\{(0,0,0,x_4,x_5,x_6,x_7) : x_j \in \mathbb{R}\bigr\}. \tag{17}$$

As G_2 acts *transitively* on the set of associative 3-planes by Proposition 5.4, every associative 3-plane is of the form $\gamma \cdot U$ for $\gamma \in G_2$. Similarly, every coassociative 4-plane is of the form $\gamma \cdot V$ for $\gamma \in G_2$.

Now $\varphi_0|_V \equiv 0$. As φ_0 is G_2-invariant, this gives $\varphi_0|_{\gamma \cdot V} \equiv 0$ for all $\gamma \in G_2$, so φ_0 restricts to zero on all coassociative 4-planes. In fact the converse is true: if W is a 4-plane in $\mathbb{R}^7$ with $\varphi_0|_W \equiv 0$, then W is coassociative with some orientation. From this we deduce an alternative characterization of coassociative 4-folds:

Proposition 5.5. *Let (M, φ, g) be a G_2-manifold, and L a 4-dimensional submanifold of M. Then L admits an orientation making it into a coassociative 4-fold if and only if $\varphi|_L \equiv 0$.*

Trivially, $\varphi|_L \equiv 0$ implies that $[\varphi|_L] = 0$ in $H^3(L, \mathbb{R})$. Regard L as an immersed 4-submanifold, with immersion $\iota : L \to M$. Then $[\varphi|_L] \in H^3(L, \mathbb{R})$ is unchanged under continuous variations of the immersion ι. Thus, $[\varphi|_L] = 0$ is a necessary condition not just for L to be coassociative, but also for any isotopic 4-fold N in M to be coassociative. This gives a *topological restriction* on coassociative 4-folds.

Corollary 5.6. *Let (φ, g) be a torsion-free G_2-structure on a 7-manifold M, and L a real 4-submanifold in M. Then a necessary condition for L to be isotopic to a coassociative 4-fold N in M is that $[\varphi|_L] = 0$ in $H^3(L, \mathbb{R})$.*

5.4 Examples of associative 3-submanifolds

Here are some sources of examples of associative 3-folds in $\mathbb{R}^7$:

- Write $\mathbb{R}^7 = \mathbb{R} \oplus \mathbb{C}^3$. Then $\mathbb{R} \times \Sigma$ is an associative 3-fold in $\mathbb{R}^7$ for any *holomorphic curve* Σ in $\mathbb{C}^3$. Also, if L is any *special Lagrangian 3-fold* in $\mathbb{C}^3$ and $x \in \mathbb{R}$ then $\{x\} \times L$ is associative 3-fold in $\mathbb{R}^7$. For examples of special Lagrangian 3-folds see [11, §9], and references therein.

- Bryant [5, §4] studies compact Riemann surfaces Σ in $\mathcal{S}^6$ pseudoholomorphic with respect to the almost complex structure J on $\mathcal{S}^6$ induced by its inclusion in $\operatorname{Im}\mathbb{O} \cong \mathbb{R}^7$. Then the cone on Σ is an *associative cone* on $\mathbb{R}^7$. He shows that any Σ has a *torsion* τ, a holomorphic analogue of the Serret–Frenet torsion of real curves in $\mathbb{R}^3$.

  The torsion τ is a section of a holomorphic line bundle on Σ, and $\tau = 0$ if $\Sigma \cong \mathbb{CP}^1$. If $\tau = 0$ then Σ is the projection to $\mathcal{S}^6 = G_2/\operatorname{SU}(3)$ of a *holomorphic curve* $\tilde{\Sigma}$ in the *projective complex manifold* $G_2/\operatorname{U}(2)$. This reduces the problem of understanding null-torsion associative cones in $\mathbb{R}^7$ to that of finding holomorphic curves $\tilde{\Sigma}$ in $G_2/\mathrm{U}(2)$ satisfing a *horizontality condition*, which is a problem in *complex algebraic geometry.* In integrable systems language, null torsion curves are called *superminimal.*

  Bryant also shows that *every* Riemann surface Σ may be embedded in $\mathcal{S}^6$ with null torsion in infinitely many ways, of arbitrarily high degree. This shows that there are *many associative cones in* $\mathbb{R}^7$, on *oriented surfaces of every genus.* These provide many local models for *singularities* of associative 3-folds.

Perhaps the simplest nontrivial example of a pseudoholomorphic curve Σ in $\mathcal{S}^6$ with null torsion is the *Borůvka sphere* [4], which is an $\mathcal{S}^2$ orbit of an SO(3) subgroup of G_2 acting irreducibly on $\mathbb{R}^7$. Other examples are given by Ejiri [10, §5–§6], who classifies pseudoholomorphic $\mathcal{S}^2$'s in $\mathcal{S}^6$ invariant under a U(1) subgroup of G_2, and Sekigawa [30].

- Bryant's paper is one of the first steps in the study of associative cones in $\mathbb{R}^7$ using the theory of *integrable systems*. Bolton et al. [2], [3, §6] use integrable systems methods to prove important results on pseudoholomorphic curves Σ in $\mathcal{S}^6$. When Σ is a torus T^2, they show it is of *finite type* [3, Cor. 6.4], and so can be classified in terms of algebro-geometric *spectral data*, and perhaps even in principle be written down explicitly.

- *Curvature properties* of pseudoholomorphic curves in $\mathcal{S}^6$ are studied by Hashimoto [13] and Sekigawa [30].

- Lotay [25] studies constructions for associative 3-folds N in $\mathbb{R}^7$. These generally involve writing N as the total space of a 1-parameter family of surfaces P_t in $\mathbb{R}^7$ of a prescribed form, and reducing the condition for N to be associative to an o.d.e. in t, which can be (partially) solved fairly explicitly.

  Lotay also considers *ruled associative 3-folds* [25, §6], which are associative 3-folds N in $\mathbb{R}^7$ fibred by a 2-parameter family of affine straight lines $\mathbb{R}$. He shows that any *associative cone* N_0 on a Riemann surface Σ in $\mathcal{S}^6$ is the limit of a 6-dimensional family of *Asymptotically Conical* ruled associative 3-folds if $\Sigma \cong \mathbb{CP}^1$, and of a 2-dimensional family if $\Sigma \cong T^2$.

  Combined with the results of Bryant [5, §4] above, this yields many examples of generically nonsingular Asymptotically Conical associative 3-folds in $\mathbb{R}^7$, diffeomorphic to $\mathcal{S}^2 \times \mathbb{R}$ or $T^2 \times \mathbb{R}$.

Examples of associative 3-folds in other explicit G_2-manifolds, such as those of Bryant and Salamon [8], may also be constructed using similar techniques. For finding associative 3-folds in *nonexplicit* G_2-manifolds, such as the compact examples of §3 which are known only through existence theorems, there is one method [18, §12.6], which we now explain.

Suppose $\gamma \in G_2$ with $\gamma^2 = 1$ but $\gamma \neq 1$. Then γ is conjugate in G_2 to

$$(x_1, \ldots, x_7) \longmapsto (x_1, x_2, x_3, -x_4, -x_5, -x_6, -x_7).$$

The fixed point set of this involution is the associative 3-plane U of (17). It follows that any $\gamma \in G_2$ with $\gamma^2 = 1$ but $\gamma \neq 1$ has fixed point set an associative 3-plane. Thus we deduce [18, Prop. 10.8.1]:

Proposition 5.7. *Let (M, φ, g) be a G_2-manifold, and $\sigma : M \to M$ be a nontrivial isometric involution with $\sigma^*(\varphi) = \varphi$. Then $N = \{p \in M : \sigma(p) = p\}$ is an associative 3-fold in M.*

Here a *nontrivial isometric involution* of (M,g) is a diffeomorphism $\sigma : M \to M$ such that $\sigma^*(g) = g$, and $\sigma \neq \mathrm{id}$ but $\sigma^2 = \mathrm{id}$, where id is the identity on M. Following [18, Ex. 12.6.1], we can use the proposition in to construct *examples* of compact associative 3-folds in the compact 7-manifolds with holonomy G_2 constructed in §3.

Example 5.8. Let $T^7 = \mathbb{R}^7/\mathbb{Z}^7$ and Γ be as in Example 3.1. Define $\sigma : T^7 \to T^7$ by

$$\sigma : (x_1, \ldots, x_7) \mapsto (x_1, x_2, x_3, \tfrac{1}{2} - x_4, -x_5, -x_6, -x_7).$$

Then σ preserves (φ_0, g_0) and commutes with Γ, and so its action pushes down to T^7/Γ. The fixed points of σ on T^7 are 16 copies of T^3, and $\sigma\delta$ has no fixed points in T^7 for all $\delta \neq 1$ in Γ. Thus the fixed points of σ in T^7/Γ are the image of the 16 T^3 fixed by σ in T^7.

But calculation shows that these 16 T^3 do not intersect the fixed points of α, β or γ, and that Γ acts freely on the set of 16 T^3 fixed by σ. So the image of the 16 T^3 in T^7 is 2 T^3 in T^7/Γ, which do not intersect the singular set of T^7/Γ, and which are *associative* 3*-folds* in T^7/Γ by Proposition 5.7.

Now the resolution of T^7/Γ to get a compact G_2-manifold $(M, \tilde\varphi, \tilde g)$ with $\mathrm{Hol}(\tilde g) = G_2$ described in §3 may be done in a σ*-equivariant way*, so that σ lifts to $\sigma : M \to M$ with $\sigma^*(\tilde\varphi) = \tilde\varphi$. The fixed points of σ in M are again 2 copies of T^3, which are *associative* 3*-folds* by Proposition 5.7.

5.5 Examples of coassociative 4-submanifolds

Here are some sources of examples of coassociative 4-folds in $\mathbb{R}^7$:

- Write $\mathbb{R}^7 = \mathbb{R} \oplus \mathbb{C}^3$. Then $\{x\} \times S$ is a coassociative 4-fold in $\mathbb{R}^7$ for any *holomorphic surface* S in $\mathbb{C}^3$ and $x \in \mathbb{R}$. Also, $\mathbb{R} \times L$ is a coassociative 4-fold in $\mathbb{R}^7$ for any *special Lagrangian* 3*-fold* L *in* $\mathbb{C}^3$ *with phase* i. For examples of special Lagrangian 3-folds see [11, §9], and references therein.

- Harvey and Lawson [12, §IV.3] give examples of coassociative 4-folds in $\mathbb{R}^7$ invariant under SU(2), acting on $\mathbb{R}^7 \cong \mathbb{R}^3 \oplus \mathbb{C}^2$ as $\mathrm{SO}(3) = \mathrm{SU}(2)/\{\pm 1\}$ on the $\mathbb{R}^3$ and SU(2) on the $\mathbb{C}^2$ factor. Such 4-folds correspond to solutions of an o.d.e., which Harvey and Lawson solve.

- Mashimo [27] classifies *coassociative cones* N in $\mathbb{R}^7$ with $N \cap \mathcal{S}^6$ homogeneous under a 3-dimensional simple subgroup H of G_2.

- Lotay [26] studies 2*-ruled coassociative* 4*-folds* in $\mathbb{R}^7$, that is, coassociative 4-folds N which are fibred by a 2-dimensional family of affine 2-planes $\mathbb{R}^2$ in $\mathbb{R}^7$, with base space a Riemann surface Σ. He shows that such 4-folds arise locally from data $\phi_1, \phi_2 : \Sigma \to \mathcal{S}^6$ and $\psi : \Sigma \to \mathbb{R}^7$ satisfying nonlinear p.d.e.s similar to the Cauchy–Riemann equations.

  For ϕ_1, ϕ_2 fixed, the remaining equations on ψ are *linear*. This means that the family of 2-ruled associative 4-folds N in $\mathbb{R}^7$ asymptotic to a fixed 2-ruled coassociative cone N_0 has the structure of a *vector space*. It can be used to generate families of examples of coassociative 4-folds in $\mathbb{R}^7$.

We can also use the fixed-point set technique of §5.4 to find examples of coassociative 4-folds in other G_2-manifolds. If $\alpha : \mathbb{R}^7 \to \mathbb{R}^7$ is linear with $\alpha^2 = 1$ and $\alpha^*(\varphi_0) = -\varphi_0$, then either $\alpha = -1$, or α is conjugate under an element of G_2 to the map

$$(x_1, \ldots, x_7) \longmapsto (-x_1, -x_2, -x_3, x_4, x_5, x_6, x_7).$$

The fixed set of this map is the coassociative 4-plane V of (17). Thus, the fixed point set of α is either $\{0\}$, or a coassociative 4-plane in $\mathbb{R}^7$. So we find [18, Prop. 10.8.5]:

Proposition 5.9. *Let (M, φ, g) be a G_2-manifold, and $\sigma : M \to M$ an isometric involution with $\sigma^*(\varphi) = -\varphi$. Then each connected component of the fixed point set $\{p \in M : \sigma(p) = p\}$ of σ is either a coassociative 4-fold or a single point.*

Bryant [7] uses this idea to construct many *local* examples of compact coassociative 4-folds in G_2-manifolds.

Theorem 5.10 (Bryant [7]). *Let (N, g) be a compact, real analytic, oriented Riemannian 4-manifold whose bundle of self-dual 2-forms is trivial. Then N may be embedded isometrically as a coassociative 4-fold in a G_2-manifold (M, φ, g), as the fixed point set of an involution σ.*

Note here that M need not be *compact*, nor (M, g) *complete*. Roughly speaking, Bryant's proof constructs (φ, g) as the sum of a power series on $\Lambda^2_+ T^*N$ converging near the zero section $N \subset \Lambda^2 T^*N$, using the theory of *exterior differential systems*. The involution σ acts as -1 on $\Lambda^2_+ T^*N$, fixing the zero section. One moral of Theorem 5.10 is that to be coassociative places no significant local restrictions on a 4-manifold, other than orientability.

Examples of *compact* coassociative 4-folds in *compact* G_2-manifolds with holonomy G_2 are constructed in [18, §12.6], using Proposition 5.9. Here [18, Ex. 12.6.4] are examples in the G_2-manifolds of §3.

Example 5.11. Let $T^7 = \mathbb{R}^7/\mathbb{Z}^7$ and Γ be as in Example 3.1. Define $\sigma : T^7 \to T^7$ by

$$\sigma : (x_1, \ldots, x_7) \mapsto (\tfrac{1}{2} - x_1, x_2, x_3, x_4, x_5, \tfrac{1}{2} - x_6, \tfrac{1}{2} - x_7).$$

Then σ commutes with Γ, preserves g_0 and takes φ_0 to $-\varphi_0$. The fixed points of σ in T^7 are 8 copies of T^4, and the fixed points of $\sigma\alpha\beta$ in T^7 are 128 points. If $\delta \in \Gamma$ then $\sigma\delta$ has no fixed points unless $\delta = 1, \alpha\beta$. Thus the fixed points of σ in T^7/Γ are the image of the fixed points of σ and $\sigma\alpha\beta$ in T^7.

Now Γ acts freely on the sets of 8 σ T^4 and 128 $\sigma\alpha\beta$ points. So the fixed point set of σ in T^7/Γ is the union of T^4 and 16 isolated points, none of which intersect the singular set of T^7/Γ. When we resolve T^7/Γ to get $(M, \tilde{\varphi}, \tilde{g})$ with $\mathrm{Hol}(\tilde{g}) = G_2$ in a σ-equivariant way, the action of σ on M has $\sigma^*(\tilde{\varphi}) = -\tilde{\varphi}$, and again fixes T^4 and 16 points. By Proposition 5.9, this T^4 is *coassociative*.

More examples of associative and coassociative submanifolds with different topologies are given in [18, §12.6].

5.6 Cayley 4-folds

The calibrated geometry of Spin(7) is similar to the G_2 case above, so we shall be brief.

Definition 5.12. Let (M,Ω,g) be a Spin(7)-*manifold*, as in §2.3. Then the 4-form Ω is a *calibration* on (M,g). We define a *Cayley* 4-*fold* in M to be a 4-submanifold of M calibrated with respect to Ω.

Let the metric g_0, and 4-form Ω_0 on $\mathbb{R}^8$ be as in §2.3. Define a *Cayley* 4-*plane* to be an oriented 4-dimensional vector subspace V of $\mathbb{R}^8$ with $\Omega_0|_V = \mathrm{vol}_V$. Then we have an analogue of Proposition 5.4:

Proposition 5.13. *The family $\mathcal{F}$ of Cayley* 4-*planes in* $\mathbb{R}^8$ *is isomorphic to* $\mathrm{Spin}(7)/K$, *where* $K \cong \bigl(\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2)\bigr)/\mathbb{Z}_2$ *is a Lie subgroup of* Spin(7), *and* $\dim\mathcal{F} = 12$.

Here are some sources of examples of Cayley 4-folds in $\mathbb{R}^8$:

- Write $\mathbb{R}^8 = \mathbb{C}^4$. Then any *holomorphic surface* S in $\mathbb{C}^4$ is Cayley in $\mathbb{R}^8$, and any *special Lagrangian* 4-*fold* N in $\mathbb{C}^4$ is Cayley in $\mathbb{R}^8$.

  Write $\mathbb{R}^8 = \mathbb{R}\times\mathbb{R}^7$. Then $\mathbb{R}\times L$ is Cayley for any *associative* 3-*fold* L in $\mathbb{R}^7$.

- Lotay [26] studies 2-*ruled Cayley* 4-*folds* in $\mathbb{R}^8$, that is, Cayley 4-folds N fibred by a 2-dimensional family Σ of affine 2-planes $\mathbb{R}^2$ in $\mathbb{R}^8$, as for the coassociative case in §5.5. He constructs explicit families of 2-ruled Cayley 4-folds in $\mathbb{R}^8$, including some depending on an arbitrary holomorphic function $w : \mathbb{C} \to \mathbb{C}$, [26, Th. 5.1].

By the method of Propositions 5.7 and 5.9 one can prove [18, Prop. 10.8.6]:

Proposition 5.14. *Let (M,Ω,g) be a* Spin(7)-*manifold, and $\sigma : M \to M$ a nontrivial isometric involution with $\sigma^*(\Omega) = \Omega$. Then each connected component of the fixed point set $\bigl\{p \in M : \sigma(p) = p\bigr\}$ is either a Cayley* 4-*fold or a single point.*

Using this, [18, §14.3] constructs examples of *compact* Cayley 4-folds in compact 8-manifolds with holonomy Spin(7).

6 Deformations of calibrated submanifolds

Finally we discuss *deformations* of associative, coassociative and Cayley submanifolds. In §6.1 we consider the local equations for such submanifolds in $\mathbb{R}^7$ and $\mathbb{R}^8$, following Harvey and Lawson [12, §IV.2]. Then §6.2 explains the deformation theory of *compact* coassociative 4-folds, following McLean [28, §4]. This has a particularly simple structure, as coassociative 4-folds are defined by the vanishing of φ. The deformation theory of compact associative 3-folds and Cayley 4-folds is more complex, and is sketched in §6.3.

6.1 Parameter counting and the local equations

We now study the local equations for 3- or 4-folds to be (co)associative or Cayley.

Associative 3-folds. The set of all 3-planes in $\mathbb{R}^7$ has dimension 12, and the set of associative 3-planes in $\mathbb{R}^7$ has dimension 8 by Proposition 5.4. Thus the associative 3-planes are of *codimension* 4 in the set of all 3-planes. Therefore the condition for a 3-fold L in $\mathbb{R}^7$ to be associative is 4 equations on each tangent space. The freedom to vary L is the sections of its normal bundle in $\mathbb{R}^7$, which is 4 real functions. Thus, the deformation problem for associative 3-folds involves 4 *equations on* 4 *functions*, so it is a *determined* problem.

To illustrate this, let $f : \mathbb{R}^3 \to \mathbb{H}$ be a smooth function, written

$$f(x_1,x_2,x_3) = f_0(x_1,x_2,x_3) + f_1(x_1,x_2,x_3)i + f_2(x_1,x_2,x_3)j + f_3(x_1,x_2,x_3)k.$$

Define a 3-submanifold L in $\mathbb{R}^7$ by

$$L = \bigl\{\bigl(x_1,x_2,x_3,f_0(x_1,x_2,x_3),\ldots,f_3(x_1,x_2,x_3)\bigr) : x_j \in \mathbb{R}\bigr\}.$$

Then Harvey and Lawson [12, §IV.2.A] calculate the conditions on f for L to be associative. With the conventions of §2.1, the equation is

$$i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2} - k\frac{\partial f}{\partial x_3} = C\Bigl(\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2},\frac{\partial f}{\partial x_3}\Bigr), \tag{18}$$

where $C : \mathbb{H}\times\mathbb{H}\times\mathbb{H} \to \mathbb{H}$ is a trilinear cross product.

Here (18) is 4 equations on 4 functions, as we claimed, and is a *first order nonlinear elliptic p.d.e.* When $f, \partial f$ are small, so that L approximates the associative 3-plane U of (17), equation (18) reduces approximately to the linear equation $i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2} - k\frac{\partial f}{\partial x_3} = 0$, which is equivalent to the *Dirac equation* on $\mathbb{R}^3$. More generally, first order deformations of an associative 3-fold L in a G_2-manifold (M,φ,g) correspond to solutions of a *twisted Dirac equation* on L.

Coassociative 4-folds. The set of all 4-planes in $\mathbb{R}^7$ has dimension 12, and the set of coassociative 4-planes in $\mathbb{R}^7$ has dimension 8 by Proposition 5.4. Thus the coassociative 4-planes are of *codimension* 4 in the set of all 4-planes. Therefore the condition for a 4-fold N in $\mathbb{R}^7$ to be coassociative is 4 equations on each tangent space. The freedom to vary N is the sections of its normal bundle in $\mathbb{R}^7$, which is 3 real functions. Thus, the deformation problem for coassociative 4-folds involves 4 *equations on* 3 *functions*, so it is an *overdetermined* problem.

To illustrate this, let $f : \mathbb{H} \to \mathbb{R}^3$ be a smooth function, written

$$f(x_0 + x_1 i + x_2 j + x_3 k) = (f_1,f_2,f_3)(x_0 + x_1 i + x_2 j + x_3 k).$$

Define a 4-submanifold N in $\mathbb{R}^7$ by

$$N = \bigl\{\bigl(f_1(x_0,\ldots,x_3),f_2(x_0,\ldots,x_3),f_3(x_0,\ldots,x_3),x_0,\ldots,x_3\bigr) : x_j \in \mathbb{R}\bigr\}.$$

Then Harvey and Lawson [12, §IV.2.B] calculate the conditions on f for N to be coassociative. With the conventions of §2.1, the equation is

$$i\partial f_1 + j\partial f_2 - k\partial f_3 = C(\partial f_1, \partial f_2, \partial f_3), \tag{19}$$

where the derivatives $\partial f_j = \partial f_j(x_0+x_1 i+x_2 j+x_3 k)$ are interpreted as functions $\mathbb{H} \to \mathbb{H}$, and C is as in (18). Here (19) is 4 equations on 3 functions, as we claimed, and is a *first order nonlinear overdetermined elliptic p.d.e.*

Cayley 4-folds. The set of all 4-planes in $\mathbb{R}^8$ has dimension 16, and the set of Cayley 4-planes in $\mathbb{R}^8$ has dimension 12 by Proposition 5.13, so the Cayley 4-planes are of *codimension* 4 in the set of all 4-planes. Therefore the condition for a 4-fold K in $\mathbb{R}^8$ to be Cayley is 4 equations on each tangent space. The freedom to vary K is the sections of its normal bundle in $\mathbb{R}^8$, which is 4 real functions. Thus, the deformation problem for Cayley 4-folds involves 4 *equations on* 4 *functions*, so it is a *determined* problem.

Let $f = f_0 + f_1 i + f_2 j + f_3 k = f(x_0 + x_1 i + x_2 j + x_3 k) : \mathbb{H} \to \mathbb{H}$ be smooth. Choosing signs for compatibility with (2), define a 4-submanifold K in $\mathbb{R}^8$ by

$$\begin{aligned} K = \bigl\{ \bigl(&-x_0, x_1, x_2, x_3, f_0(x_0 + x_1 i + x_2 j + x_3 k), -f_1(x_0 + x_1 i + x_2 j + x_3 k), \\ &-f_2(x_0 + x_1 i + x_2 j + x_3 k), f_3(x_0 + x_1 i + x_2 j + x_3 k)\bigr) : x_j \in \mathbb{R}\bigr\}. \end{aligned}$$

Following [12, §IV.2.C], the equation for K to be Cayley is

$$\frac{\partial f}{\partial x_0} + i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2} + k\frac{\partial f}{\partial x_3} = C(\partial f), \tag{20}$$

for $C : \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \to \mathbb{H}$ a homogeneous cubic polynomial. This is 4 equations on 4 functions, as we claimed, and is a first-order nonlinear elliptic p.d.e. on f. The linearization at $f = 0$ is equivalent to the *positive Dirac equation* on $\mathbb{R}^4$. More generally, first order deformations of a Cayley 4-fold K in a Spin(7)-manifold (M, Ω, g) correspond to solutions of a *twisted positive Dirac equation* on K.

6.2 Deformation theory of coassociative 4-folds

Here is the main result in the deformation theory of coassociative 4-folds, proved by McLean [28, Th. 4.5]. As our sign conventions for $\varphi_0, *\varphi_0$ in (1) are different to McLean's, we use self-dual 2-forms in place of McLean's anti-self-dual 2-forms.

Theorem 6.1. *Let (M, φ, g) be a G_2-manifold, and N a compact coassociative 4-fold in M. Then the moduli space $\mathcal{M}_N$ of coassociative 4-folds isotopic to N in M is a smooth manifold of dimension $b^2_+(N)$.*

Sketch proof. Suppose for simplicity that N is an embedded submanifold. There is a natural orthogonal decomposition $TM|_N = TN \oplus \nu$, where $\nu \to N$ is the *normal bundle* of N in M. There is a natural isomorphism $\nu \cong \Lambda^2_+ T^*N$, constructed as follows. Let $x \in N$ and $V \in \nu_x$. Then $V \in T_x M$, so $V \cdot \varphi|_x \in \Lambda^2 T^*_x M$, and $(V \cdot \varphi|_x)|_{T_x N} \in \Lambda^2 T^*_x N$. It turns out that $(V \cdot \varphi|_x)|_{T_x N}$

actually lies in $\Lambda^2_+ T^*_x N$, the bundle of *self-dual 2-forms* on N, and that the map $V \mapsto (V \cdot \varphi|_x)|_{T_x N}$ defines an *isomorphism* $\nu \xrightarrow{\cong} \Lambda^2_+ T^* N$.

Let T be a small *tubular neighbourhood* of N in M. Then we can identify T with a neighbourhood of the zero section in ν, using the exponential map. The isomorphism $\nu \cong \Lambda^2_+ T^* N$ then identifies T with a neighbourhood U of the zero section in $\Lambda^2_+ T^* N$. Let $\pi : T \to N$ be the obvious projection.

Under this identification, submanifolds N' in $T \subset M$ which are C^1 close to N are identified with the *graphs* $\Gamma(\alpha)$ of small smooth sections α of $\Lambda^2_+ T^* N$ lying in U. Write $C^\infty(U)$ for the subset of the vector space of smooth self-dual 2-forms $C^\infty(\Lambda^2_+ T^* N)$ on N lying in $U \subset \Lambda^2_+ T^* N$. Then for each $\alpha \in C^\infty(U)$ the graph $\Gamma(\alpha)$ is a 4-submanifold of U, and so is identified with a 4-submanifold of T. We need to know: which 2-forms α correspond to *coassociative* 4-folds $\Gamma(\alpha)$ in T?

Well, N' is coassociative if $\varphi|_{N'} \equiv 0$. Now $\pi|_{N'} : N' \to N$ is a diffeomorphism, so we can push $\varphi|_{N'}$ down to N, and regard it as a function of α. That is, we define

$$P : C^\infty(U) \longrightarrow C^\infty(\Lambda^3 T^* N) \quad \text{by} \quad P(\alpha) = \pi_*(\varphi|_{\Gamma(\alpha)}). \tag{21}$$

Then the moduli space $\mathcal{M}_N$ is locally isomorphic near N to the set of small self-dual 2-forms α on N with $\varphi|_{\Gamma(\alpha)} \equiv 0$, that is, to a neighbourhood of 0 in $P^{-1}(0)$.

To understand the equation $P(\alpha) = 0$, note that at $x \in N$, $P(\alpha)|_x$ depends on the tangent space to $\Gamma(\alpha)$ at $\alpha|_x$, and so on $\alpha|_x$ and $\nabla\alpha|_x$. Thus the functional form of P is

$$P(\alpha)|_x = F\bigl(x, \alpha|_x, \nabla\alpha|_x\bigr) \quad \text{for } x \in N,$$

where F is a smooth function of its arguments. Hence $P(\alpha) = 0$ is a *nonlinear first order p.d.e.* in α. The *linearization* $\mathrm{d}P(0)$ of P at $\alpha = 0$ turns out to be

$$\mathrm{d}P(0)(\beta) = \lim_{\epsilon \to 0} \bigl(\epsilon^{-1} P(\epsilon\beta)\bigr) = \mathrm{d}\beta.$$

Therefore $\mathrm{Ker}(\mathrm{d}P(0))$ is the vector space $\mathcal{H}^2_+$ of *closed self-dual 2-forms* β on N, which by Hodge theory is a finite-dimensional vector space isomorphic to $H^2_+(N, \mathbb{R})$, with dimension $b^2_+(N)$. This is the *Zariski tangent space* of $\mathcal{M}_N$ at N, the *infinitesimal deformation space* of N as a coassociative 4-fold.

To complete the proof we must show that $\mathcal{M}_N$ is locally isomorphic to its Zariski tangent space $\mathcal{H}^2_+$, and so is a smooth manifold of dimension $b^2_+(N)$. To do this rigorously requires some technical analytic machinery, which is passed over in a few lines in [28, p. 731]. Here is one way to do it.

Because $C^\infty(\Lambda^2_+ T^* N), C^\infty(\Lambda^3 T^* N)$ are not *Banach spaces*, we extend P in (21) to act on *Hölder spaces* $C^{k+1,\gamma}(\Lambda^2_+ T^* N), C^{k,\gamma}(\Lambda^3 T^* N)$ for $k \geqslant 1$ and $\gamma \in (0,1)$, giving

$$P_{k,\gamma} : C^{k+1,\gamma}(U) \longrightarrow C^{k,\gamma}(\Lambda^3 T^* N) \quad \text{defined by} \quad P_{k,\gamma}(\alpha) = \pi_*(\varphi|_{\Gamma(\alpha)}).$$

Then $P_{k,\gamma}$ is a smooth map of Banach manifolds. Let $V_{k,\gamma} \subset C^{k,\gamma}(\Lambda^3 T^* N)$ be the Banach subspace of *exact* $C^{k,\gamma}$ 3-forms on N.

As φ is closed, $\varphi|_N \equiv 0$, and $\Gamma(\alpha)$ is isotopic to N, we see that $\varphi|_{\Gamma(\alpha)}$ is an *exact* 3-form on $\Gamma(\alpha)$, so that $P_{k,\gamma}$ maps into $V_{k,\gamma}$. The linearization

$$\mathrm{d}P_{k,\gamma}(0) : C^{k+1,\gamma}(\Lambda^2_+ T^*N) \longrightarrow V_{k,\gamma}, \qquad \mathrm{d}P_{k,\gamma}(0) : \beta \longmapsto \mathrm{d}\beta$$

is then *surjective* as a map of Banach spaces. (To prove this requires a discursion, using elliptic regularity results for $\mathrm{d} + \mathrm{d}^*$.)

Thus, $P_{k,\gamma} : C^{k+1,\gamma}(U) \to V_{k,\gamma}$ is a smooth map of Banach manifolds, with $\mathrm{d}P_{k,\gamma}(0)$ surjective. The *Implicit Function Theorem for Banach spaces* now implies that $P^{-1}_{k,\gamma}(0)$ is near 0 a smooth submanifold of $C^{k+1,\gamma}(U)$, locally isomorphic to $\mathrm{Ker}(\mathrm{d}P_{k,\gamma}(0))$. But $P_{k,\gamma}(\alpha) = 0$ is an *overdetermined elliptic equation* for small α, and so elliptic regularity implies that solutions α are smooth. Therefore $P^{-1}_{k,\gamma}(0) = P^{-1}(0)$ near 0, and similarly $\mathrm{Ker}(\mathrm{d}P_{k,\gamma}(0)) = \mathrm{Ker}(\mathrm{d}P(0)) = \mathcal{H}^2_+$. This completes the proof. □

Here are some remarks on Theorem 6.1.

- This proof relies heavily on Proposition 5.5, that a 4-fold N in M is coassociative if and only if $\varphi|_N \equiv 0$, for φ a closed 3-form on M. The consequence of this is that the deformation theory of compact coassociative 4-folds is *unobstructed*, and the moduli space is *always* a smooth manifold with dimension given by a topological formula.

  Special Lagrangian m-folds of Calabi-Yau m-folds can also be defined in terms of the vanishing of closed forms, and their deformation theory is also unobstructed, as in [28, §3] and [11, §10.2]. However, associative 3-folds and Cayley 4-folds cannot be defined by the vanishing of closed forms, and we will see in §6.3 that this gives their deformation theory a different flavour.

- We showed in §6.1 that the condition for a 4-fold N in M to be coassociative is locally 4 equations on 3 functions, and so is *overdetermined*. However, Theorem 6.1 shows that coassociative 4-folds have *unobstructed* deformation theory, and often form *positive-dimensional* moduli spaces. This seems very surprising for an overdetermined equation.

  The explanation is that the condition $\mathrm{d}\varphi = 0$ acts as an *integrability condition* for the existence of coassociative 4-folds. That is, since closed 3-forms on N essentially depend locally only on 3 real parameters, not 4, as φ is closed the equation $\varphi|_N \equiv 0$ is in effect only 3 equations on N rather than 4, so we can think of the deformation theory as really controlled by a determined elliptic equation.

  Therefore $\mathrm{d}\varphi = 0$ is essential for Theorem 6.1 to work. In 'almost G_2-manifolds' (M, φ, g) with $\mathrm{d}\varphi \neq 0$, the deformation problem for coassociative 4-folds is overdetermined and obstructed, and generically there would be no coassociative 4-folds.

- In Example 5.11 we constructed an example of a compact coassociative 4-fold N diffeomorphic to T^4 in a compact G_2-manifold (M, φ, g). By

Theorem 6.1, N lies in a *smooth* 3*-dimensional family* of coassociative T^4's in M. Locally, these may form a *coassociative fibration* of M.

Now suppose $\{(M, \varphi_t, g_t) : t \in (-\epsilon, \epsilon)\}$ is a smooth 1-parameter family of G_2-manifolds, and N_0 a compact coassociative 4-fold in (M, φ_0, g_0). When can we extend N_0 to a smooth family of coassociative 4-folds N_t in (M, φ_t, g_t) for small t? By Corollary 5.6, a necessary condition is that $[\varphi_t|_{N_0}] = 0$ for all t. Our next result shows that locally, this is also a *sufficient* condition. It can be proved using similar techniques to Theorem 6.1, though McLean did not prove it.

Theorem 6.2. *Let* $\{(M, \varphi_t, g_t) : t \in (-\epsilon, \epsilon)\}$ *be a smooth* 1*-parameter family of* G_2*-manifolds, and* N_0 *a compact coassociative* 4*-fold in* (M, φ_0, g_0)*. Suppose that* $[\varphi_t|_{N_0}] = 0$ *in* $H^3(N_0, \mathbb{R})$ *for all* $t \in (-\epsilon, \epsilon)$*. Then* N_0 *extends to a smooth* 1*-parameter family* $\{N_t : t \in (-\delta, \delta)\}$*, where* $0 < \delta \leqslant \epsilon$ *and* N_t *is a compact coassociative* 4*-fold in* (M, φ_t, g_t)*.*

6.3 Deformations of associative 3-folds and Cayley 4-folds

Associative 3-folds and Cayley 4-folds cannot be defined in terms of the vanishing of closed forms, and this gives their deformation theory a different character to the coassociative case. Here is how the theories work, drawn mostly from McLean [28, §5–§6].

Let N be a compact associative 3-fold or Cayley 4-fold in a 7- or 8-manifold M. Then there are vector bundles $E, F \to N$ with $E \cong \nu$, the normal bundle of N in M, and a first-order elliptic operator $D_N : C^\infty(E) \to C^\infty(F)$ on N. The *kernel* $\operatorname{Ker} D_N$ is the set of *infinitesimal deformations* of N as an associative 3-fold or Cayley 4-fold. The *cokernel* $\operatorname{Coker} D_N$ is the *obstruction space* for these deformations.

Both are finite-dimensional vector spaces, and

$$\dim \operatorname{Ker} D_N - \dim \operatorname{Coker} D_N = \operatorname{ind}(D_N),$$

the *index* of D_N. It is a topological invariant, given in terms of characteristic classes by the *Atiyah–Singer Index Theorem*. In the associative case we have $E \cong F$, and D_N is anti-self-adjoint, so that $\operatorname{Ker}(D_N) \cong \operatorname{Coker}(D_N)$ and $\operatorname{ind}(D_N) = 0$ automatically. In the Cayley case we have

$$\operatorname{ind}(D_N) = \tau(N) - \tfrac{1}{2}\chi(N) - \tfrac{1}{2}[N] \cdot [N],$$

where τ is the signature, χ the Euler characteristic and $[N] \cdot [N]$ the self-intersection of N.

In a *generic* situation we expect $\operatorname{Coker} D_N = 0$, and then deformations of N will be unobstructed, so that the moduli space $\mathcal{M}_N$ of associative or Cayley deformations of N will locally be a smooth manifold of dimension $\operatorname{ind}(D_N)$. However, in nongeneric situations the obstruction space may be nonzero, and then the moduli space may not be smooth, or may have a larger than expected dimension.

This general structure is found in the deformation theory of other important mathematical objects — for instance, pseudo-holomorphic curves in almost complex manifolds, and instantons and Seiberg–Witten solutions on 4-manifolds. In each case, the moduli space is only smooth with topologically determined dimension under a *genericity assumption* which forces the obstructions to vanish.

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