Constructing compact 8-manifolds with holonomy Spin(7) from Calabi–Yau orbifolds

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published as pages 21–30 in C. Casuberta, R.M. Miró-Roig, J. Verdera and S. Xambo-Deschamps, editors, 'Proceedings of the 3rd European Congress of Mathematics, Barcelona, July 10-14, 2000', volume II, Progress in Mathematics 202, Birkhäuser, Basel, 2001.

Let M be an n-dimensional manifold and g a Riemannian metric on M. The *holonomy group* $\mathrm{Hol}(g)$ of g is the group of isometries of T_xM generated by parallel transport around closed loops based at x in M, using the Levi-Civita connection ∇ of g. We consider $\mathrm{Hol}(g)$ to be a subgroup of $O(n)$, defined up to conjugation by elements of $O(n)$. Then $\mathrm{Hol}(g)$ is independent of the base point x in M.

The classification of possible holonomy groups was due to Berger [1], although Alekseevskii later eliminated Spin(9) from Berger's list.

Theorem (Berger) *Let M be a simply-connected, n-dimensional manifold, and g an irreducible, nonsymmetric Riemannian metric on M. Then either*

(i) $\mathrm{Hol}(g) = SO(n)$,

(ii) $n = 2m$ *and* $\mathrm{Hol}(g) = SU(m)$ *or* $U(m)$,

(iii) $n = 4m$ *and* $\mathrm{Hol}(g) = \mathrm{Sp}(m)$ *or* $\mathrm{Sp}(m)\,\mathrm{Sp}(1)$,

(iv) $n = 7$ *and* $\mathrm{Hol}(g) = G_2$, *or*

(v) $n = 8$ *and* $\mathrm{Hol}(g) = \mathrm{Spin}(7)$.

The exceptional cases G_2 and Spin(7) in this classification are called the *exceptional holonomy groups.* For some time after Berger's classification, they remained a mystery. In 1987, Bryant [2] used the theory of exterior differential systems to show that locally there exist many metrics with holonomy G_2 and Spin(7), and gave some explicit, incomplete examples. Then in 1989, Bryant and Salamon [3] found explicit, *complete* metrics with holonomy G_2 and Spin(7) on noncompact manifolds.

In 1994-5, the author found examples of metrics with holonomy G_2 and Spin(7) on *compact* manifolds [6, 7], by resolving the singularities of orbifolds T^7/Γ and T^8/Γ. These are described at length in Chapters 11–14 of the author's book [9], which also makes the construction substantially more powerful by incorporating Calabi Conjecture methods, and gives many more examples.

The subject of this paper is a *second* construction of compact 8-manifolds with holonomy Spin(7), which was published by the author in [8] and [9, Ch. 15]. As in the first construction, we begin with a compact Spin(7)-orbifold, and resolve its singularities to get a Spin(7)-manifold.

However, in this case the initial orbifold is not T^8/Γ but a quotient $Y/\langle\sigma\rangle$ of a Calabi–Yau 4-orbifold Y by an antiholomorphic involution σ. It turns out that if Y and σ are carefully chosen so that the singularities of $Y/\langle\sigma\rangle$ are of a very special kind, then we can resolve $Y/\langle\sigma\rangle$ with holonomy Spin(7).

1. The holonomy group Spin(7)

Let $\mathbb{R}^8$ have coordinates $(x_1,\ldots,x_8)$. Write $\mathrm{d}\mathbf{x}_{ijkl}$ for the 4-form $\mathrm{d}x_i\wedge\mathrm{d}x_j\wedge\mathrm{d}x_k\wedge\mathrm{d}x_l$ on $\mathbb{R}^8$. Define a 4-form Ω_0 on $\mathbb{R}^8$ by

$$\begin{aligned}\Omega_0=&\mathrm{d}\mathbf{x}_{1234}+\mathrm{d}\mathbf{x}_{1256}+\mathrm{d}\mathbf{x}_{1278}+\mathrm{d}\mathbf{x}_{1357}-\mathrm{d}\mathbf{x}_{1368}\\ &-\mathrm{d}\mathbf{x}_{1458}-\mathrm{d}\mathbf{x}_{1467}-\mathrm{d}\mathbf{x}_{2358}-\mathrm{d}\mathbf{x}_{2367}-\mathrm{d}\mathbf{x}_{2457}\\ &+\mathrm{d}\mathbf{x}_{2468}+\mathrm{d}\mathbf{x}_{3456}+\mathrm{d}\mathbf{x}_{3478}+\mathrm{d}\mathbf{x}_{5678}.\end{aligned}\tag{1}$$

The subgroup of $\mathrm{GL}(8,\mathbb{R})$ preserving Ω_0 is the holonomy group Spin(7). This group also preserves the orientation on $\mathbb{R}^8$ and the Euclidean metric $g_0=\mathrm{d}x_1^2+\cdots+\mathrm{d}x_8^2$. It is a compact, semisimple, 21-dimensional Lie group, a subgroup of SO(8).

A Spin(7)-structure on an 8-manifold M gives rise to a 4-form Ω and a metric g on M, such that each tangent space of M admits an isomorphism with $\mathbb{R}^8$ identifying Ω and g with Ω_0 and g_0 respectively. By an abuse of notation we will refer to the pair (Ω,g) as a Spin(7)-structure.

Proposition 1.1 *Let M be a compact 8-manifold and (Ω,g) a* Spin(7)*-structure on M. Then the following are equivalent:*

(i) $\mathrm{Hol}(g)\subseteq\mathrm{Spin}(7)$*, and Ω is the induced 4-form,*

(ii) $\nabla\Omega=0$ *on M, where ∇ is the Levi-Civita connection of g, and*

(iii) $\mathrm{d}\Omega=0$ *on M.*

We call $\nabla\Omega$ the *torsion* of the Spin(7)-structure (Ω,g), and (Ω,g) *torsion-free* if $\nabla\Omega=0$. A triple (M,Ω,g) is called a Spin(7)*-manifold* (or Spin(7)*-orbifold*) if M is an 8-manifold (or 8-orbifold) and (Ω,g) a torsion-free Spin(7)-structure on M. If g has holonomy $\mathrm{Hol}(g)\subseteq\mathrm{Spin}(7)$, then g is Ricci-flat.

Here is a result on *compact* 8-manifolds with holonomy Spin(7).

Theorem 1.2 *Let (M,Ω,g) be a compact* Spin(7)*-manifold. Then* $\mathrm{Hol}(g)=\mathrm{Spin}(7)$ *if and only if M is simply-connected, and $b^3(M)+b^4_+(M)=b^2(M)+2b^4_-(M)+25$. In this case the moduli space of metrics with holonomy* Spin(7) *on M, up to diffeomorphisms isotopic to the identity, is a smooth manifold of dimension $1+b^4_-(M)$.*

Next we sketch the construction in [7] of compact 8-manifolds with holonomy Spin(7). One begins with a flat Spin(7)-structure (Ω_0, g_0) on the 8-torus T^8, and a finite group Γ of isometries of T^8 preserving (Ω_0, g_0). Then T^8/Γ is an *orbifold*, a singular manifold with only quotient singularities.

For certain Γ one can resolve the singularities of T^8/Γ in a natural way, using complex geometry. This gives a nonsingular, compact 8-manifold M, and a projection $\pi : M \to T^8/\Gamma$. We write down a family of Spin(7)-structures (Ω_t, g_t) on M for $t \in (0, \epsilon)$, such that (Ω_t, g_t) has small torsion for small t, and converges to the singular Spin(7)-structure $\pi^*(\Omega_0, g_0)$ as $t \to 0$.

Finally, we prove using analysis that for small t, the Spin(7)-structure (Ω_t, g_t) can be deformed to a nearby Spin(7)-structure $(\tilde{\Omega}_t, \tilde{g}_t)$ on M, with zero torsion. If M satisfies the topological conditions of Theorem 1.2, then $\tilde{g}_t$ has holonomy Spin(7).

Here is the analytic result used in the last stage of this construction. It is proved in [7, Th. A & Th. B], and also in [9, Ch. 13].

Theorem 1.3 *Let λ, μ, ν be positive constants. Then there exist positive constants κ, K such that whenever $0 < t \leq \kappa$, the following is true.*

Let M be a compact 8-manifold, and (Ω, g) a Spin(7)*-structure on M. Suppose that ϕ is a smooth 4-form on M with $\mathrm{d}\Omega + \mathrm{d}\phi = 0$, and*

(i) $\|\phi\|_{L^2} \leq \lambda t^{9/2}$ *and* $\|\mathrm{d}\phi\|_{L^{10}} \leq \lambda t$,

(ii) *the injectivity radius $\delta(g)$ satisfies $\delta(g) \geq \mu t$, and*

(iii) *the Riemann curvature $R(g)$ satisfies $\|R(g)\|_{C^0} \leq \nu t^{-2}$.*

Then there exists a smooth, torsion-free Spin(7)*-structure $(\tilde{\Omega}, \tilde{g})$ on M with $\|\tilde{\Omega} - \Omega\|_{C^0} \leq K t^{1/2}$.*

Here is how to interpret this theorem. As $\nabla\Omega = 0$ if and only if $\mathrm{d}\Omega = 0$, and $\mathrm{d}\phi + \mathrm{d}\Omega = 0$, the torsion $\nabla\Omega$ is determined by $\mathrm{d}\phi$. Thus we can think of ϕ as a *first integral of the torsion* of (Ω, g). So $\|\phi\|_{L^2}$ and $\|\mathrm{d}\phi\|_{L^{10}}$ are both measures of the torsion of (Ω, g). As t is small, part (i) of the theorem says that (Ω, g) has *small torsion* in a certain sense.

Parts (ii) and (iii) say that the injectivity radius of g should not be too small, and its curvature not too large. When a metric becomes singular, in general its injectivity radius becomes zero and its curvature infinite. So we can interpret (ii) and (iii) to mean that g is not too close to being singular.

Thus, the theorem as a whole says that if the torsion of (Ω, g) is small enough, and g is not too singular, then we can deform (Ω, g) to a nearby, torsion-free Spin(7)-structure $(\tilde{\Omega}, \tilde{g})$ on M.

We prove Theorem 1.3 using analysis: we write the condition that $(\tilde{\Omega}, \tilde{g})$ be torsion-free as a nonlinear elliptic p.d.e., which can be approximated by a linear elliptic p.d.e. when $\tilde{\Omega} - \Omega$ is small. Then we use tools such as Sobolev spaces, the Sobolev Embedding Theorem and elliptic regularity to show that this nonlinear elliptic p.d.e. has a smooth solution.

2. Calabi–Yau orbifolds

We now give a brief introduction to Calabi–Yau geometry, and the relation between Calabi–Yau 4-folds and Spin(7)-manifolds. A suitable reference is [9, Ch. 6]. For convenience we work in the orbifold category.

A *Calabi–Yau orbifold* is a compact Kähler orbifold (Y, J, g) of dimension m, with $\mathrm{Hol}(g) \subseteq \mathrm{SU}(m)$. Calabi–Yau orbifolds are automatically Ricci-flat. Using Yau's proof of the Calabi conjecture [10], one can show that suitable complex orbifolds admit metrics making them Calabi–Yau.

Theorem 2.1 *Let (Y, J) be a compact complex orbifold admitting Kähler metrics, with trivial canonical bundle. Then there is a unique Ricci-flat Kähler metric in each Kähler class, making Y into a Calabi–Yau orbifold.*

By elementary properties of holonomy groups, every Calabi–Yau orbifold (Y, J, g) admits a Kähler form ω and a complex volume form θ, both constant under the Levi-Civita connection ∇, such that near every point $p \in Y$ we can choose complex coordinates $(z_1, \ldots, z_m)$ in which

$$\begin{aligned} g = |\mathrm{d}z_1|^2 + \cdots + |\mathrm{d}z_m|^2, \quad \omega = \frac{i}{2}(\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \cdots + \mathrm{d}z_m \wedge \mathrm{d}\bar{z}_m), \\ \text{and} \quad \theta = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_m \quad \text{at } p. \end{aligned} \tag{2}$$

A Calabi–Yau 4-orbifold (Y, J, g) has $\mathrm{Hol}(g) \subseteq \mathrm{SU}(4)$. But $\mathrm{SU}(4) \subset \mathrm{Spin}(7) \subset \mathrm{SO}(8)$. Therefore $\mathrm{Hol}(g) \subset \mathrm{Spin}(7)$, and it follows from general principles of holonomy that g must be part of a torsion-free Spin(7)-structure (Ω, g) on Y. We shall show that we may take Ω to be $\frac{1}{2}\omega \wedge \omega + \mathrm{Re}(\theta)$.

Let $p \in Y$, and choose complex coordinates $(z_1, \ldots, z_4)$ near p satisfying (2). Define real coordinates $(x_1, \ldots, x_8)$ near p by $(z_1, \ldots, z_4) = (x_1 + ix_2, x_3 + ix_4, x_5 + ix_6, x_7 + ix_8)$. Then from (2) we see that at p we have

$$g = \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_8^2, \quad \omega = \mathrm{d}\mathbf{x}_{12} + \mathrm{d}\mathbf{x}_{34} + \mathrm{d}\mathbf{x}_{56} + \mathrm{d}\mathbf{x}_{78} \quad \text{and}$$
$$\mathrm{Re}(\theta) = \mathrm{d}\mathbf{x}_{1357} - \mathrm{d}\mathbf{x}_{1368} - \mathrm{d}\mathbf{x}_{1458} - \mathrm{d}\mathbf{x}_{1467} - \mathrm{d}\mathbf{x}_{2358} - \mathrm{d}\mathbf{x}_{2367} - \mathrm{d}\mathbf{x}_{2457} + \mathrm{d}\mathbf{x}_{2468},$$

where $\mathrm{d}\mathbf{x}_{ij\ldots l} = \mathrm{d}x_i \wedge \mathrm{d}x_j \wedge \cdots \wedge \mathrm{d}x_l$.

It follows from this equation that $\Omega = \frac{1}{2}\omega \wedge \omega + \mathrm{Re}(\theta)$ coincides with the 4-form Ω_0 defined in (1). As this holds for all $p \in Y$, we see that (Ω, g) is a Spin(7)-*structure* on Y. But $\nabla\omega = \nabla\theta = 0$, so $\nabla\Omega = 0$, and (Ω, g) is torsion-free. This shows that Calabi-Yau 4-orbifolds are also Spin(7)-orbifolds.

3. ALE Spin(7)-manifolds

Asymptotically Locally Euclidean manifolds, or *ALE manifolds* for short, are a class of noncompact Riemannian manifolds with one end modelled asymptotically on a quotient singularity $\mathbb{R}^n/G$.

Definition 3.1 Let (X, Ω, g) be a Spin(7)-manifold, and G a finite subgroup of Spin(7) which acts freely on $\mathbb{R}^8 \setminus \{0\}$. We call (X, Ω, g) an *ALE* Spin(7)*-manifold* *asymptotic to* $\mathbb{R}^8/G$ if there exists a surjective, continuous map $\pi : X \to \mathbb{R}^8/G$ such that $\pi : X \setminus \pi^{-1}(0) \to (\mathbb{R}^8/G) \setminus \{0\}$ is a diffeomorphism, and $\pi^{-1}(0)$ is compact, and

$$\begin{array}{rl} & \nabla^l(\pi_*(g) - g_0) = O(r^{-8-l}) \\ \text{and} & \nabla^l(\pi_*(\Omega) - \Omega_0) = O(r^{-8-l}) \end{array} \quad \text{on } \{x \in \mathbb{R}^8/G : r(x) > 1\}, \text{ for all } l \geq 0.$$

Here $g_0 = \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_8^2$ and ∇ its Levi-Civita connection, Ω_0 is the Spin(7) form on $\mathbb{R}^8$ given in (1), and r is the radius function on $\mathbb{R}^8/G$.

The point of this is that if (X, Ω_X, g_X) is an ALE Spin(7)-manifold asymptotic to $\mathbb{R}^8/G$, and (Y, Ω_Y, g_Y) is a Spin(7)-orbifold with just one singular point p modelled on $\mathbb{R}^8/G$, then we can join X and Y together to get a nonsingular 8-manifold M. Moreover, it may be possible to glue the Spin(7)-structures (Ω_X, g_X) and (Ω_Y, g_Y) together to get a torsion-free Spin(7)-structure (Ω, g) on M.

We now study an example. Define $\alpha, \beta : \mathbb{R}^8 \to \mathbb{R}^8$ by

$$\begin{aligned} \alpha &: (x_1, \ldots, x_8) \mapsto (-x_2, x_1, -x_4, x_3, -x_6, x_5, -x_8, x_7), \\ \beta &: (x_1, \ldots, x_8) \mapsto (x_3, -x_4, -x_1, x_2, x_7, -x_8, -x_5, x_6). \end{aligned}$$

Then α, β preserve the standard Spin(7)-structure (Ω_0, g_0) on $\mathbb{R}^8$ given in §1, so they lie in Spin(7). Also α, β satisfy $\alpha^4 = \beta^4 = 1$, $\alpha^2 = \beta^2$ and $\alpha\beta = \beta\alpha^3$. Let $G = \langle \alpha, \beta \rangle$. Then G is a finite nonabelian subgroup of Spin(7) of order 8, which acts freely on $\mathbb{R}^8 \setminus \{0\}$.

In the next two examples we shall construct two topologically distinct ALE Spin(7)-manifolds (X_1, Ω_1, g_1) and (X_2, Ω_2, g_2) asymptotic to $\mathbb{R}^8/G$.

Example 3.2 Define complex coordinates $(z_1, \ldots, z_4)$ on $\mathbb{R}^8$ by

$$(z_1, z_2, z_3, z_4) = (x_1 + ix_2, x_3 + ix_4, x_5 + ix_6, x_7 + ix_8),$$

Then $g_0 = |\mathrm{d}z_1|^2 + \cdots + |\mathrm{d}z_4|^2$, and $\Omega_0 = \frac{1}{2}\omega_0 \wedge \omega_0 + \mathrm{Re}(\theta_0)$, where ω_0 and θ_0 are the usual Kähler form and complex volume form on $\mathbb{C}^4$. In these coordinates, α and β are given by

$$\begin{aligned} \alpha &: (z_1, \ldots, z_4) \mapsto (iz_1, iz_2, iz_3, iz_4), \\ \beta &: (z_1, \ldots, z_4) \mapsto (\bar{z}_2, -\bar{z}_1, \bar{z}_4, -\bar{z}_3). \end{aligned} \tag{3}$$

Now $\mathbb{C}^4/\langle \alpha \rangle$ is a complex singularity, as $\alpha \in \mathrm{SU}(4)$. Let (Y_1, π_1) be the blow-up of $\mathbb{C}^4/\langle \alpha \rangle$ at 0. Then Y_1 is the unique crepant resolution of $\mathbb{C}^4/\langle \alpha \rangle$. The action of β on $\mathbb{C}^4/\langle \alpha \rangle$ lifts to a *free* antiholomorphic map $\beta : Y_1 \to Y_1$ with $\beta^2 = 1$. Define $X_1 = Y_1/\langle \beta \rangle$. Then X_1 is a nonsingular 8-manifold, and the projection $\pi_1 : Y_1 \to \mathbb{C}^4/\langle \alpha \rangle$ pushes down to $\pi_1 : X_1 \to \mathbb{R}^8/G$.

There exist ALE Kähler metrics g_1 on Y_1 with holonomy SU(4), which were written down explicitly by Calabi [4, p. 285]. Each such g_1 is invariant under the action of β on Y_1. Let ω_1 be the Kähler form of g_1, and $\theta_1 = \pi_1^*(\theta_0)$ the holomorphic volume form on Y_1. Define $\Omega_1 = \frac{1}{2}\omega_1 \wedge \omega_1 + \mathrm{Re}(\theta_1)$. Then (Ω_1, g_1) is a torsion-free Spin(7)-structure on Y_1.

As $\beta^*(\omega_1) = -\omega_1$ and $\beta^*(\theta_1) = \bar{\theta}_1$, we see that β preserves (Ω_1, g_1). Thus (Ω_1, g_1) pushes down to a torsion-free Spin(7)-structure (Ω_1, g_1) on X_1. Then (X_1, Ω_1, g_1) is an *ALE* Spin(7)-*manifold* asymptotic to $\mathbb{R}^8/G$. The Betti numbers of X_1 are $b^1 = b^2 = b^3 = 0$ and $b^4 = 1$, and $\pi_1(X_1) = \mathbb{Z}_2$.

Example 3.3 Define new complex coordinates $(w_1, \ldots, w_4)$ on $\mathbb{R}^8$ by

$$(w_1, w_2, w_3, w_4) = (-x_1 + ix_3, x_2 + ix_4, -x_5 + ix_7, x_6 + ix_8).$$

Again we find that $g_0 = |\mathrm{d}w_1|^2 + \cdots + |\mathrm{d}w_4|^2$ and $\Omega_0 = \frac{1}{2}\omega_0 \wedge \omega_0 + \mathrm{Re}(\theta_0)$. In these coordinates, α and β are given by

$$\begin{aligned} \alpha &: (w_1, \ldots, w_4) \mapsto (\bar{w}_2, -\bar{w}_1, \bar{w}_4, -\bar{w}_3), \\ \beta &: (w_1, \ldots, w_4) \mapsto (iw_1, iw_2, iw_3, iw_4). \end{aligned} \tag{4}$$

Observe that (3) and (4) are the same, except that the rôles of α, β are reversed. Therefore we can use the ideas above again.

Let Y_2 be the crepant resolution of $\mathbb{C}^4/\langle\beta\rangle$. The action of α on $\mathbb{C}^4/\langle\beta\rangle$ lifts to a free antiholomorphic involution of Y_2. Let $X_2 = Y_2/\langle\alpha\rangle$. Then X_2 is nonsingular, and carries a torsion-free Spin(7)-structure (Ω_2, g_2), making (X_2, Ω_2, g_2) into an ALE Spin(7)-manifold asymptotic to $\mathbb{R}^8/G$.

In effect, the two sets of complex coordinates z_j and w_j define two *different* complex structures on $\mathbb{R}^8$, and two *different* embeddings of SU(4) in Spin(7). In the first example $\alpha \in \mathrm{SU}(4)$ but $\beta \notin \mathrm{SU}(4)$, and we use complex geometry with respect to the first complex structure to resolve $\mathbb{C}^4/\langle\alpha\rangle$, in a β-equivariant way. In the second example $\beta \in \mathrm{SU}(4)$ but $\alpha \notin \mathrm{SU}(4)$, and we use the second complex structure to resolve $\mathbb{C}^4/\langle\beta\rangle$, in an α-equivariant way.

Now (X_1, Ω_1, g_1), (X_2, Ω_2, g_2) are clearly isomorphic as Spin(7)-manifolds, but they should be regarded as *topologically distinct* ALE manifolds, because the isomorphism between them acts nontrivially on $\mathbb{R}^8/G$. Thus, we have found two topologically distinct ALE Spin(7)-manifolds (X_1, Ω_1, g_1), (X_2, Ω_2, g_2) asymptotic to the same singularity $\mathbb{R}^8/G$.

Let i be 1 or 2, and let $t > 0$. Then $(t^4\Omega_i, t^2 g_i)$ is also a torsion-free Spin(7)-structure on X_i, which is homothetic to (Ω_i, g_i). That is, we obtain $(t^4\Omega_i, t^2 g_i)$ by shrinking distances on X_i by a factor t. Thus, $(X_i, t^4\Omega_i, t^2 g_i)$ is a Spin(7)-*manifold* for each $t > 0$.

Furthermore, it is an *ALE* Spin(7)-manifold. For if $\pi_i : X_i \to \mathbb{R}^8/G$ satisfies the conditions of Definition 3.1 for (Ω_i, g_i), then $t\,\pi_i : X_i \to \mathbb{R}^8/G$ satisfies the conditions of Definition 3.1 for $(t^4\Omega_i, t^2 g_i)$. This means that X_i carries not just one, but a whole 1-parameter family of ALE Spin(7)-structures $(t^4\Omega_i, t^2 g_i)$.

4. The construction

Starting with a Calabi–Yau 4-orbifold Y with isolated singularities of a certain kind, and an antiholomorphic involution σ on Y, we will now explain how to construct a family of compact 8-manifolds $M_{\mathbf{i}}$ by resolving $Z = Y/\langle\sigma\rangle$, and show that there exist torsion-free Spin(7)-structures $(\tilde{\Omega}, \tilde{g})$ on $M_{\mathbf{i}}$, which have holonomy Spin(7) if $M_{\mathbf{i}}$ is simply-connected. The following conditions set out the ingredients for our construction.

Condition 4.1 Let (Y, J) be a compact complex 4-orbifold with trivial canonical bundle, admitting Kähler metrics. Let σ be an antiholomorphic involution on Y. Define $\alpha : \mathbb{C}^4 \to \mathbb{C}^4$ by

$$\alpha : (z_1, z_2, z_3, z_4) \longmapsto (iz_1, iz_2, iz_3, iz_4). \tag{5}$$

Then $\langle\alpha\rangle \cong \mathbb{Z}_4$, and $\mathbb{C}^4/\langle\alpha\rangle$ has an isolated singular point at 0. Suppose that the singular set of Y is k isolated points $p_1, \dots, p_k$ for some $k \geq 1$, each modelled on $\mathbb{C}^4/\langle\alpha\rangle$, and that the fixed set of σ in Y is exactly $\{p_1, \dots, p_k\}$. Suppose also that $Y \setminus \{p_1, \dots, p_k\}$ is simply-connected, and $h^{2,0}(Y) = 0$.

Assume these conditions hold. Choose a σ-invariant Kähler class on Y. By Theorem 2.1, there is a unique g_Y in this Kähler class making (Y, J, g_Y) into a Calabi–Yau 4-orbifold. Clearly g_Y is σ-invariant. Let ω_Y be the Kähler form of g_Y. Then $\sigma^*(\omega_Y) = -\omega_Y$, as σ is antiholomorphic.

As in §2, there exists a holomorphic volume form θ_Y on Y such that g_Y, ω_Y and θ_Y may be written in the form (2) at each point of Y. Multiplying θ_Y by a phase $e^{i\phi}$ if necessary, we can also ensure that $\sigma^*(\theta_Y) = \bar{\theta}_Y$.

Define $\Omega_Y = \frac{1}{2}\omega_Y \wedge \omega_Y + \mathrm{Re}(\theta_Y)$. Then (Ω_Y, g_Y) is torsion-free Spin(7)-structure on Y, as in §2. Furthermore, the equations $\sigma^*(\omega_Y) = -\omega_Y$ and $\sigma^*(\theta_Y) = \bar{\theta}_Y$ imply that $\sigma^*(\Omega_Y) = \Omega_Y$. Thus (Ω_Y, g_Y) is σ-invariant.

Define $Z = Y/\langle\sigma\rangle$. Then Z is a compact, real 8-orbifold. As (Ω_Y, g_Y) is σ-invariant, it pushes down to give a torsion-free Spin(7)-structure (Ω_Z, g_Z) on Z. Thus, (Z, Ω_Z, g_Z) is a Spin(7)-*orbifold*.

Condition 4.1 implies that the singularities of Z are k points $p_1, \dots, p_k$. Furthermore, it can be shown that each singularity p_j is modelled on the singularity $\mathbb{R}^8/G$ considered in §3, in a way which identifies the Spin(7) structures on $T_{p_j}Z$ and $\mathbb{R}^8/G$.

Now Examples 3.2 and 3.3 defined two distinct ALE Spin(7)-manifolds (X_i, Ω_i, g_i) asymptotic to $\mathbb{R}^8/G$. For $j = 1, \dots, k$, choose $i_j = 1$ or 2, and let $\mathbf{i} = (i_1, \dots, i_k)$. Define a compact, nonsingular 8-manifold $M_{\mathbf{i}}$ by resolving each singular point p_j in Z using the ALE 8-manifold X_{i_j}. We explain how to do this in [8, §5.3] and [9, §15.2.2]; the basic idea is to identify a small ball about p_j in Z with a small ball about 0 in $\mathbb{R}^8/G$, and then glue together open sets in Z and X_{i_j} along an annulus around 0 in $\mathbb{R}^8/G$.

As there are 2^k possible choices for the i_j, this gives 2^k 8-manifolds $M_{\mathbf{i}}$ resolving Z. Calculation shows that one of these 8-manifolds (with $i_j = 1$ for all j) has $\pi_1(M_{\mathbf{i}}) = \mathbb{Z}_2$, and the rest are all simply-connected. Also, all the $M_{\mathbf{i}}$ have the same Betti numbers b^k, the sum of $b^k(Z)$ and the $b^k(X_{i_j})$ for $0 < k < 8$, which satisfy $b^3(M_{\mathbf{i}}) + b^4_+(M_{\mathbf{i}}) = b^2(M_{\mathbf{i}}) + b^4_-(M_{\mathbf{i}}) + 25$.

Next, we define a 1-parameter family of Spin(7)-structures (Ω^t, g^t) on $M_{\mathbf{i}}$ depending on $t \in (0, \epsilon]$, for some small $\epsilon > 0$, and a corresponding family of

4-forms ϕ^t on $M_{\mathbf{i}}$ such that $\mathrm{d}\Omega^t + \mathrm{d}\phi^t = 0$. The general idea is that ϕ^t is a measure of the *torsion* of (Ω^t, g^t), and the definitions are set up very carefully to ensure that ϕ^t is small when t is small. That is, when t is small, (Ω^t, g^t) has *small torsion.*

The details of the definition of (Ω^t, g^t) and ϕ^t are rather complicated, but here is a rough sketch. Recall from §3 that X_{i_j} carries a 1-parameter family of Spin(7)-structures $(t^4\Omega_{i_j}, t^2 g_{i_j})$ depending on $t > 0$. We make (Ω^t, g^t) by gluing together the torsion-free Spin(7)-structures (Ω_Z, g_Z) on Z and $(t^4\Omega_{i_j}, t^2 g_{i_j})$ on X_{i_j}, using a partition of unity on the $k+1$ patches used to define $M_{\mathbf{i}}$.

Because Ω^t must be a Spin(7) form, which does not hold for generic 4-forms but implies a pointwise restriction, we are unable to make Ω^t closed. Instead, we construct a 4-form ϕ^t such that $\Omega^t + \phi^t$ is closed, and try and arrange that ϕ^t is as small as possible.

We then prove the following estimates on (Ω^t, g^t) and ϕ^t.

Theorem 4.2 *In the situation above, there exist constants $\lambda, \mu, \nu > 0$ such that for all $t \in (0, \epsilon]$ we have*

(i) $\|\phi^t\|_{L^2} \leq \lambda t^{24/5}$ *and* $\|\mathrm{d}\phi^t\|_{L^{10}} \leq \lambda t^{36/25}$*;*

(ii) *the injectivity radius $\delta(g^t)$ satisfies $\delta(g^t) \geq \mu t$; and*

(iii) *the Riemann curvature $R(g^t)$ satisfies $\|R(g^t)\|_{C^0} \leq \nu t^{-2}$.*

Here all norms are calculated using the metric g^t on $M_{\mathbf{i}}$.

This should be compared with Theorem 1.3. Parts (ii) and (iii) of both results are the same, and the powers of t in part (i) of Theorem 1.3 are smaller than those in part (i) above, which means that part (i) of Theorem 4.2 is stronger for small t. So combining the two theorems shows that for small t there exists a torsion-free Spin(7)-structure $(\tilde{\Omega}, \tilde{g})$ on $M_{\mathbf{i}}$ close to (Ω^t, g^t).

Thus, $(M_{\mathbf{i}}, \tilde{\Omega}, \tilde{g})$ is a compact Spin(7)-manifold. We can then use Theorem 1.2 to determine the holonomy group of $\tilde{g}$. If $M_{\mathbf{i}}$ is simply-connected, which holds for $2^k - 1$ choices of $\mathbf{i}$, then $\mathrm{Hol}(\tilde{g}) = \mathrm{Spin}(7)$. If $\pi_1(M_{\mathbf{i}}) = \mathbb{Z}_2$, which holds for the remaining one choice of $\mathbf{i}$, it turns out that $\mathrm{Hol}(\tilde{g}) = \mathbb{Z}_2 \ltimes \mathrm{SU}(4)$. This proves our main result.

Theorem 4.3 *Suppose Condition 4.1 holds. Then there are $2^k - 1$ simply-connected, compact 8-manifolds $M_{\mathbf{i}}$ resolving $Z = Y/\langle\sigma\rangle$, which carry metrics with holonomy* Spin(7)*.*

We do not claim here that the 8-manifolds $M_{\mathbf{i}}$ are distinct as smooth 8-manifolds, but only that they are topologically distinct as resolutions of Z.

5. An example

Next we give an example of this construction. The main idea we shall use is borrowed from Candelas, Lynker and Schrimmrigk [5], who constructed a large number of Calabi–Yau 3-folds as crepant resolutions of hypersurfaces in weighted projective spaces $\mathbb{CP}^4_{a_0,\ldots,a_4}$. Here *weighted projective spaces* are an important class of complex orbifolds.

Definition 5.1 Let $m \geq 1$ be an integer, and $a_0, a_1, \ldots, a_m$ positive integers with highest common factor 1. Let $\mathbb{C}^{m+1}$ have complex coordinates on $(z_0, \ldots, z_m)$, and define an action of the complex Lie group $\mathbb{C}^*$ on $\mathbb{C}^{m+1}$ by

$$(z_0, \ldots, z_m) \stackrel{u}{\longmapsto} (u^{a_0} z_0, \ldots, u^{a_m} z_m), \qquad \text{for } u \in \mathbb{C}^*.$$

The *weighted projective space* $\mathbb{CP}^m_{a_0,\ldots,a_m}$ is $\bigl(\mathbb{C}^{m+1} \setminus \{0\}\bigr)/\mathbb{C}^*$.

Let Y be the hypersurface of degree 12 in $\mathbb{CP}^5_{1,1,1,1,4,4}$ given by

$$Y = \bigl\{[z_0, \ldots, z_5] \in \mathbb{CP}^5_{1,1,1,1,4,4} : z_0^{12} + z_1^{12} + z_2^{12} + z_3^{12} + z_4^3 + z_5^3 = 0\bigr\}.$$

Calculation shows that Y has trivial canonical bundle and three singular points $p_1 = [0,0,0,0,1,-1]$, $p_2 = [0,0,0,0,1,e^{\pi i/3}]$ and $p_3 = [0,0,0,0,1,e^{-\pi i/3}]$, all modelled on $\mathbb{C}^4/\langle\alpha\rangle$, where α is given in (5).

Now define a map $\sigma : Y \to Y$ by

$$\sigma : [z_0, \ldots, z_5] \longmapsto [\bar{z}_1, -\bar{z}_0, \bar{z}_3, -\bar{z}_2, \bar{z}_5, \bar{z}_4].$$

Note that $\sigma^2 = 1$, though this is not immediately obvious, because of the geometry of $\mathbb{CP}^5_{1,1,1,1,4,4}$. It can be shown that Condition 4.1 holds for Y and σ. So we can apply the construction of §4, and resolve $Z = Y/\langle\sigma\rangle$ to get seven compact 8-manifolds $M_{\mathbf{i}}$ admitting metrics with holonomy Spin(7). A long computation proves that the $M_{\mathbf{i}}$ have Betti numbers

$$b^0 = 1,\ \ b^1 = b^2 = b^3 = 0,\ \ b^4 = 2446,\ \ b^4_+ = 1639 \ \text{ and } \ b^4_- = 807,$$

and so by Theorem 1.2 the metrics with holonomy Spin(7) on each $M_{\mathbf{i}}$ form a smooth family of dimension 808.

6. Conclusions

Finding examples of complex orbifolds (Y, J) satisfying Condition 4.1 is not that easy, and requires a degree of ingenuity. The requirements that Y have only $\mathbb{Z}_4$ singularities of the given type, and that σ fixes these points and no others, are very restrictive. Nonetheless, by considering hypersurfaces and complete intersections in weighted projective spaces, and using various tricks, the author was able in [8] and [9, Ch. 15] to construct compact 8-manifolds with holonomy Spin(7) realizing 14 distinct sets of Betti numbers, which are given in Table 1. Probably there are many other examples which can be produced by the same methods.

Comparing these Betti numbers with those of the compact 8-manifolds constructed in [7] and [9, Ch. 14] by resolving torus orbifolds T^8/Γ, we see that these examples the middle Betti number b^4 is much bigger, as much as 11 662 in one case.

Table 1: Betti numbers (b^2, b^3, b^4) of compact Spin(7)-manifolds

$(4, 33, 200)$	$(3, 33, 202)$	$(2, 33, 204)$	$(1, 33, 206)$	$(0, 33, 208)$
$(1, 0, 908)$	$(0, 0, 910)$	$(1, 0, 1292)$	$(0, 0, 1294)$	$(1, 0, 2444)$
$(0, 0, 2446)$	$(0, 6, 3730)$	$(0, 0, 4750)$	$(0, 0, 11\,662)$	

Given that the two constructions of compact 8-manifolds with holonomy Spin(7) that we know appear to produce sets of 8-manifolds with rather different 'geography', it is tempting to speculate that the set of all compact 8-manifolds with holonomy Spin(7) may be rather large, and that those constructed so far are a small sample with atypical behaviour.

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