Shifted Symplectic Derived Algebraic Geometry and generalizations of Donaldson–Thomas Theory

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Plan of talk:

1. Derived Algebraic Geometry
References for §1

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1.1. Derived Algebraic Geometry for dummies

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, e.g. $\mathbb{K} = \mathbb{C}$. Work in the context of Toën and Vezzosi's theory of *Derived Algebraic Geometry* (*DAG*). This gives ∞-categories of *derived* $\mathbb{K}$*-schemes* $\mathbf{dSch}_{\mathbb{K}}$ and *derived stacks* $\mathbf{dSt}_{\mathbb{K}}$. In this talk, for simplicity, we will mostly discuss derived schemes, though the results also extend to derived stacks.

This is a very technical subject. It is not easy to motivate DAG, or even to say properly what a derived scheme is, in an elementary talk. So I will lie a little bit.

What is a derived scheme?

$\mathbb{K}$-schemes in classical algebraic geometry are geometric spaces X which can be covered by Zariski open sets $Y \subseteq X$ with $Y \cong \operatorname{Spec} A$ for A a commutative $\mathbb{K}$-algebra. General $\mathbb{K}$-schemes are very singular, but *smooth* $\mathbb{K}$*-schemes* X are very like smooth manifolds over $\mathbb{K}$, many differential geometric ideas like cotangent bundles TX, T^*X work nicely for them.

Think of a derived $\mathbb{K}$-scheme $\boldsymbol{X}$ as a geometric space which can be covered by Zariski open sets $\boldsymbol{Y} \subseteq \boldsymbol{X}$ with $\boldsymbol{Y} \simeq \operatorname{Spec} A^\bullet$ for $A^\bullet = (A, \mathrm{d})$ a commutative differential graded algebra (cdga) over $\mathbb{K}$, in degrees $\leqslant 0$.

We require $\boldsymbol{X}$ to be *locally finitely presented*, that is, we can take the $A^\bullet$ to be finitely presented, a strong condition.

Why derived algebraic geometry?

One reason derived algebraic geometry can be a powerful tool, is the combination of two facts:

(A) Many algebro-geometric spaces one wants to study, such as moduli spaces of coherent sheaves, or complexes, or representations, etc., which in classical algebraic geometry may be very singular, also have an incarnation as (locally finitely presented) derived schemes (or derived stacks).

(B) Within the framework of DAG, one can treat (locally finitely presented) derived schemes or stacks very much like smooth, nonsingular objects (Kontsevich's 'hidden smoothness philosophy'). Some nice things work in the derived world, which do not work in the classical world.

1.2. Tangent and cotangent complexes

In going from classical to derived geometry, we always replace vector bundles, sheaves, representations, ..., by *complexes* of vector bundles, ... A classical smooth $\mathbb{K}$-scheme X has a tangent bundle TX and dual cotangent bundle T^*X, which are vector bundles on X, of rank the dimension $\dim X \in \mathbb{N}$.

Similarly, a derived $\mathbb{K}$-scheme $\boldsymbol{X}$ has a *tangent complex* $\mathbb{T}_{\boldsymbol{X}}$ and a dual *cotangent complex* $\mathbb{L}_{\boldsymbol{X}}$, which are perfect complexes of coherent sheaves on $\boldsymbol{X}$, of rank the virtual dimension $\mathrm{vdim}\, \boldsymbol{X} \in \mathbb{Z}$. A complex $\mathcal{E}^\bullet$ on $\boldsymbol{X}$ is called *perfect in the interval* $[a, b]$ if locally on $\boldsymbol{X}$ it is quasi-isomorphic to a complex $\cdots 0 \to E_a \to E_{a+1} \to \cdots \to E_b \to 0 \to \cdots$, with E_i a vector bundle in position i. For $\boldsymbol{X}$ a derived scheme, $\mathbb{T}_{\boldsymbol{X}}$ is perfect in $[0, \infty)$ and $\mathbb{L}_{\boldsymbol{X}}$ perfect in $(-\infty, 0]$; for $\boldsymbol{X}$ a derived Artin stack, $\mathbb{T}_{\boldsymbol{X}}$ is perfect in $[-1, \infty)$ and $\mathbb{L}_{\boldsymbol{X}}$ perfect in $(-\infty, 1]$.

Tangent complexes of moduli stacks

Suppose X is a smooth projective scheme, and $\boldsymbol{\mathcal{M}}$ is a derived moduli stack of coherent sheaves E on X. Then for each point $[E]$ in $\boldsymbol{\mathcal{M}}$ and each $i \in \mathbb{Z}$ we have natural isomorphisms

$$H^i(\mathbb{T}_{\boldsymbol{\mathcal{M}}}|_{[E]}) \cong \mathrm{Ext}^{i-1}(E, E). \tag{1.1}$$

In effect, the derived stack $\boldsymbol{\mathcal{M}}$ remembers the entire deformation theory of sheaves on X. In contrast, if $\mathcal{M} = t_0(\boldsymbol{\mathcal{M}})$ is the corresponding classical moduli scheme, (1.1) holds when $i \leqslant 1$ only. This shows that the derived structure on a moduli scheme/stack can remember useful information forgotten by the classical moduli scheme/stack, e.g. the Ext groups $\mathrm{Ext}^i(E, E)$ for $i \geqslant 2$.
If X has dimension n then (1.1) implies that $H^i(\mathbb{T}_{\boldsymbol{\mathcal{M}}}|_{[E]}) = 0$ for $i \geqslant n$, so $\mathbb{T}_{\boldsymbol{\mathcal{M}}}$ is perfect in $[-1, n-1]$.

Quasi-smooth derived schemes and virtual cycles

A derived scheme $\boldsymbol{X}$ is called *quasi-smooth* if $\mathbb{T}_{\boldsymbol{X}}$ is perfect in $[0,1]$, or equivalently $\mathbb{L}_{\boldsymbol{X}}$ is perfect in $[-1,0]$.

A proper quasi-smooth derived scheme $\boldsymbol{X}$ has a *virtual cycle* $[\boldsymbol{X}]_{\rm virt}$ in the Chow homology $A_*(X)$, where $X = t_0(\boldsymbol{X})$ is the classical truncation. This is because the natural morphism $\mathbb{L}_{\boldsymbol{X}}|_X \to \mathbb{L}_X$ induced by the inclusion $X \hookrightarrow \boldsymbol{X}$ is a 'perfect obstruction theory' in the sense of Behrend and Fantechi.

Most theories of invariants in algebraic geometry – e.g. Gromov–Witten invariants, Mochizuki invariants counting sheaves on surfaces, Donaldson–Thomas invariants – can be traced back to the existence of quasi-smooth derived moduli schemes.

For an (ordinary) derived moduli scheme $\boldsymbol{\mathcal{M}}$ of coherent sheaves E on X to be quasi-smooth, we need $\mathrm{Ext}^i(E,E)=0$ for $i \geqslant 3$. This is automatic if $\dim X \leqslant 2$. For Calabi–Yau 3-folds X, you would expect a problem with $\mathrm{Ext}^3(E,E)\neq 0$, but stable sheaves E with fixed determinant have trace-free Ext groups $\mathrm{Ext}^3(E,E)_0=0$.

An example of nice behaviour in the derived world

Here is an example of the 'hidden smoothness philosophy'. Suppose we have a Cartesian square of smooth $\mathbb{K}$-schemes (or indeed, smooth manifolds)

$$\begin{array}{ccc} W & \xrightarrow[f]{} & Y \\ \big\downarrow{\scriptstyle e} & & {\scriptstyle h}\big\downarrow \\ X & \xrightarrow{g} & Z, \end{array}$$

with g, h transverse. Then we have an exact sequence of vector bundles on W, which we can use to compute TW:

$$0 \to TW \xrightarrow{Te \oplus Tf} e^*(TX) \oplus f^*(TY) \xrightarrow{e^*(Tg) \oplus -f^*(Th)} (g \circ e)^*(TZ) \to 0.$$

Similarly, if we have a homotopy Cartesian square of derived $\mathbb{K}$-schemes

$$\begin{array}{ccc} W & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle h} \\ X & \xrightarrow{g} & Z, \end{array}$$

with no transversality, we have a distinguished triangle on W

$$\mathbb{T}_W \xrightarrow{\mathbb{T}_e \oplus \mathbb{T}_f} e^*(\mathbb{T}_X) \oplus f^*(\mathbb{T}_Y) \xrightarrow{e^*(\mathbb{T}_g) \oplus -f^*(\mathbb{T}_h)} (g \circ e)^*(\mathbb{T}_Z) \to \mathbb{T}_W[+1],$$

which we can use to compute $\mathbb{T}_W$. This is false for classical schemes. So, derived schemes with arbitrary morphisms, have good behaviour analogous to smooth classical schemes with transverse morphisms, and are better behaved than classical schemes.

2. Shifted symplectic geometry
References for §2

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Classical symplectic geometry

Let M be a smooth manifold. Then M has a tangent bundle and cotangent bundle T^*M. We have k-forms $\omega \in C^\infty(\Lambda^k T^*M)$, and the de Rham differential $\mathrm{d}_{dR} : C^\infty(\Lambda^k T^*M) \to C^\infty(\Lambda^{k+1} T^*M)$. A k-form ω is *closed* if $\mathrm{d}_{dR}\omega = 0$.

A 2-form ω on M is *nondegenerate* if $\omega\cdot : TM \to T^*M$ is an isomorphism. This is possible only if $\dim M = 2n$ for $n \geqslant 0$. A *symplectic structure* is a closed, nondegenerate 2-form ω on M.

Symplectic geometry is the study of symplectic manifolds (M, ω).

A *Lagrangian* in (M, ω) is a submanifold $i : L \to M$ such that $\dim L = n$ and $i^*(\omega) = 0$.

2.1. PTVV's shifted symplectic geometry

Pantev, Toën, Vaquié and Vezzosi (arXiv:1111.3209) defined a version of symplectic geometry in the derived world.

Let $\boldsymbol{X}$ be a derived $\mathbb{K}$-scheme. The cotangent complex $\mathbb{L}_{\boldsymbol{X}}$ has exterior powers $\Lambda^p\mathbb{L}_{\boldsymbol{X}}$. The *de Rham differential* $\mathrm{d}_{dR} : \Lambda^p\mathbb{L}_{\boldsymbol{X}} \rightarrow \Lambda^{p+1}\mathbb{L}_{\boldsymbol{X}}$ is a morphism of complexes. Each $\Lambda^p\mathbb{L}_{\boldsymbol{X}}$ is a complex, so has an internal differential $\mathrm{d} : (\Lambda^p\mathbb{L}_{\boldsymbol{X}})^k \rightarrow (\Lambda^p\mathbb{L}_{\boldsymbol{X}})^{k+1}$. We have $\mathrm{d}^2 = \mathrm{d}_{dR}^2 = \mathrm{d} \circ \mathrm{d}_{dR} + \mathrm{d}_{dR} \circ \mathrm{d} = 0$.

A *p-form of degree k* on $\boldsymbol{X}$ for $k \in \mathbb{Z}$ is an element $[\omega^0]$ of $H^k\big(\Lambda^p\mathbb{L}_{\boldsymbol{X}}, \mathrm{d}\big)$. A *closed p-form of degree k* on $\boldsymbol{X}$ is an element

$$[(\omega^0, \omega^1, \ldots)] \in H^k\big(\textstyle\prod_{i=0}^{\infty} \Lambda^{p+i}\mathbb{L}_{\boldsymbol{X}}[i], \mathrm{d} + \mathrm{d}_{dR}\big).$$

There is a projection $\pi : [(\omega^0, \omega^1, \ldots)] \mapsto [\omega^0]$ from closed p-forms $[(\omega^0, \omega^1, \ldots)]$ of degree k to p-forms $[\omega^0]$ of degree k.

Nondegenerate 2-forms and symplectic structures

Let $[\omega^0]$ be a 2-form of degree k on $\boldsymbol{X}$. Then $[\omega^0]$ induces a morphism $\omega^0 : \mathbb{T}_{\boldsymbol{X}} \to \mathbb{L}_{\boldsymbol{X}}[k]$, where $\mathbb{T}_{\boldsymbol{X}} = \mathbb{L}_{\boldsymbol{X}}^\vee$ is the tangent complex of $\boldsymbol{X}$. We call $[\omega^0]$ *nondegenerate* if $\omega^0 : \mathbb{T}_{\boldsymbol{X}} \to \mathbb{L}_{\boldsymbol{X}}[k]$ is a quasi-isomorphism.

If $\boldsymbol{X}$ is a derived scheme then the complex $\mathbb{L}_{\boldsymbol{X}}$ lives in degrees $(-\infty, 0]$ and $\mathbb{T}_{\boldsymbol{X}}$ in degrees $[0, \infty)$. So $\omega^0 : \mathbb{T}_{\boldsymbol{X}} \to \mathbb{L}_{\boldsymbol{X}}[k]$ can be a quasi-isomorphism only if $k \leqslant 0$, and then $\mathbb{L}_{\boldsymbol{X}}$ lives in degrees $[k, 0]$ and $\mathbb{T}_{\boldsymbol{X}}$ in degrees $[0, -k]$. If $k = 0$ then $\boldsymbol{X}$ is a smooth classical $\mathbb{K}$-scheme, and if $k = -1$ then $\boldsymbol{X}$ is quasi-smooth.

A closed 2-form $\omega = [(\omega^0, \omega^1, \ldots)]$ of degree k on $\boldsymbol{X}$ is called a *k-shifted symplectic structure* if $[\omega^0] = \pi(\omega)$ is nondegenerate.

Although the details are complex, PTVV are following a simple recipe for translating some piece of geometry from smooth manifolds/smooth classical schemes to derived schemes:

(i) replace manifolds/smooth schemes X by derived schemes $\boldsymbol{X}$.

(ii) replace vector bundles $TX, T^*X, \Lambda^p T^*X, \ldots$ by complexes $\mathbb{T}_{\boldsymbol{X}}, \mathbb{L}_{\boldsymbol{X}}, \Lambda^p\mathbb{L}_{\boldsymbol{X}}, \ldots$.

(iii) replace sections of $TX, T^*X, \Lambda^p T^*X, \ldots$ by cohomology classes of the complexes $\mathbb{T}_{\boldsymbol{X}}, \mathbb{L}_{\boldsymbol{X}}, \Lambda^p\mathbb{L}_{\boldsymbol{X}}, \ldots$, in degree $k \in \mathbb{Z}$.

(iv) replace isomorphisms of vector bundles by quasi-isomorphisms of complexes.

Note that in (iii), we can specify the degree $k \in \mathbb{Z}$ of the cohomology class (e.g. $[\omega] \in H^k(\Lambda^p\mathbb{L}_{\boldsymbol{X}})$), which doesn't happen at the classical level.

Calabi–Yau moduli schemes and moduli stacks

PTVV prove that if Y is a Calabi–Yau m-fold over $\mathbb{K}$ and $\boldsymbol{\mathcal{M}}$ is a derived moduli scheme or stack of (complexes of) coherent sheaves on Y, then $\boldsymbol{\mathcal{M}}$ has a $(2-m)$-shifted symplectic structure ω.

This suggests applications — lots of interesting geometry concerns Calabi–Yau moduli schemes, e.g. Donaldson–Thomas theory.

We can understand the associated nondegenerate 2-form $[\omega^0]$ in terms of *Serre duality*. At a point $[E]\in\boldsymbol{\mathcal{M}}$, we have $h^i(\mathbb{T}_{\boldsymbol{\mathcal{M}}})|_{[E]}\cong \mathrm{Ext}^{i-1}(E,E)$ and $h^i(\mathbb{L}_{\boldsymbol{\mathcal{M}}})|_{[E]}\cong \mathrm{Ext}^{1-i}(E,E)^*$. The Calabi–Yau condition gives $\mathrm{Ext}^i(E,E)\cong \mathrm{Ext}^{m-i}(E,E)^*$, which corresponds to $h^{i+1}(\mathbb{T}_{\boldsymbol{\mathcal{M}}})|_{[E]}\cong h^{i+1}(\mathbb{L}_{\boldsymbol{\mathcal{M}}}[2-m])|_{[E]}$. This is the cohomology at $[E]$ of the quasi-isomorphism $\omega^0:\mathbb{T}_{\boldsymbol{\mathcal{M}}}\to\mathbb{L}_{\boldsymbol{\mathcal{M}}}[2-m]$.

Lagrangians and Lagrangian intersections

Let $(\boldsymbol{X},\omega)$ be a k-shifted symplectic derived scheme or stack. Then Pantev et al. define a notion of *Lagrangian* $\boldsymbol{L}$ in $(\boldsymbol{X},\omega)$, which is a morphism $\boldsymbol{i}:\boldsymbol{L}\to\boldsymbol{X}$ of derived schemes or stacks together with a homotopy $i^*(\omega)\sim 0$ satisfying a nondegeneracy condition, implying that $\mathbb{T}_{\boldsymbol{L}}\simeq\mathbb{L}_{\boldsymbol{L}/\boldsymbol{X}}[k-1]$.

If $\boldsymbol{L},\boldsymbol{M}$ are Lagrangians in $(\boldsymbol{X},\omega)$, then the fibre product $\boldsymbol{L}\times_{\boldsymbol{X}}\boldsymbol{M}$ has a natural $(k-1)$-shifted symplectic structure.

If (S,ω) is a classical smooth symplectic scheme, then it is a 0-shifted symplectic derived scheme in the sense of PTVV, and if $L,M\subset S$ are classical smooth Lagrangian subschemes, then they are Lagrangians in the sense of PTVV. Therefore the (derived) Lagrangian intersection $L\cap M=L\times_S M$ is a -1-shifted symplectic derived scheme.

Examples of Lagrangians

Let $(\boldsymbol{X},\omega)$ be k-shifted symplectic, and $\boldsymbol{i}_a:\boldsymbol{L}_a\to\boldsymbol{X}$ be Lagrangian in $\boldsymbol{X}$ for $a=1,\ldots,d$. Then Ben-Bassat (arXiv:1309.0596) shows

$$\boldsymbol{L}_1\times_{\boldsymbol{X}}\boldsymbol{L}_2\times_{\boldsymbol{X}}\cdots\times_{\boldsymbol{X}}\boldsymbol{L}_d\longrightarrow(\boldsymbol{L}_1\times_{\boldsymbol{X}}\boldsymbol{L}_2)\times\cdots\times(\boldsymbol{L}_{d-1}\times_{\boldsymbol{X}}\boldsymbol{L}_d)\times(\boldsymbol{L}_d\times_{\boldsymbol{X}}\boldsymbol{L}_1)$$

is Lagrangian, where the r.h.s. is $(k-1)$-shifted symplectic by PTVV. This is relevant to defining 'Fukaya categories' of complex symplectic manifolds.

Let Y be a Calabi–Yau m-fold, so that the derived moduli stack $\boldsymbol{\mathcal{M}}$ of coherent sheaves (or complexes) on Y is $(2-m)$-shifted symplectic by PTVV, with symplectic form ω. Then

$$\boldsymbol{\mathcal{E}xact}\xrightarrow{\pi_1\times\pi_2\times\pi_3}(\boldsymbol{\mathcal{M}},\omega)\times(\boldsymbol{\mathcal{M}},-\omega)\times(\boldsymbol{\mathcal{M}},\omega)$$

is Lagrangian, where $\boldsymbol{\mathcal{E}xact}$ is the derived moduli stack of short exact sequences in $\mathrm{coh}(Y)$ (or distinguished triangles in $D^b\,\mathrm{coh}(Y)$). This is relevant to Cohomological Hall Algebras.

Summary of the story so far

- Derived schemes behave better than classical schemes in some ways – they are analogous to smooth schemes, or manifolds. So, we can extend stories in smooth geometry to derived schemes. This introduces an extra degree $k \in \mathbb{Z}$.
- PTVV define a version of ('k-shifted') symplectic geometry for derived schemes. This is a new geometric structure.
- 0-shifted symplectic derived schemes are just classical smooth symplectic schemes.
- Calabi–Yau m-fold moduli schemes and stacks are $(2 - m)$-shifted symplectic. This gives a *new geometric structure* on Calabi–Yau moduli spaces – relevant to Donaldson–Thomas theory and its generalizations.
- One can go from k-shifted symplectic to $(k - 1)$-shifted symplectic by taking intersections of Lagrangians.

2.2. A 'Darboux theorem' for shifted symplectic schemes

Theorem 2.1 (Brav, Bussi and Joyce arXiv:1305.6302)

Suppose $(\boldsymbol{X},\omega)$ *is a* k*-shifted symplectic derived* $\mathbb{K}$*-scheme for* $k<0$. *If* $k\not\equiv 2 \mod 4$, *then each* $x\in\boldsymbol{X}$ *admits a Zariski open neighbourhood* $\boldsymbol{Y}\subseteq\boldsymbol{X}$ *with* $\boldsymbol{Y}\simeq\mathrm{Spec}(A,\mathrm{d})$ *for* (A,d) *an explicit cdga generated by graded variables* x_j^{-i},y_j^{k+i} *for* $0\leqslant i\leqslant -k/2$, *and* $\omega|_{\boldsymbol{Y}}=[(\omega^0,0,0,\ldots)]$ *where* x_j^l,y_j^l *have degree* l, *and*

$$\omega^0=\sum_{i=0}^{[-k/2]}\sum_{j=1}^{m_i}\mathrm{d}_{dR}y_j^{k+i}\mathrm{d}_{dR}x_j^{-i}.$$

Also the differential d *in* (A,d) *is given by Poisson bracket with a Hamiltonian* H *in* A *of degree* $k+1$.
If $k\equiv 2 \mod 4$, *we have two statements, one étale local with* ω^0 *standard, and one Zariski local with the components of* ω^0 *in the degree* $k/2$ *variables depending on some invertible functions.*

In Ben-Bassat, Bussi, Brav and Joyce arXiv:1312.0090 we extend this from (derived) schemes to (derived) Artin stacks.

Sketch of the proof of Theorem 2.1

Suppose $(\boldsymbol{X},\omega)$ is a k-shifted symplectic derived $\mathbb{K}$-scheme for $k<0$, and $x\in\boldsymbol{X}$. Then $\mathbb{L}_{\boldsymbol{X}}$ lives in degrees $[k,0]$. We first show that we can build Zariski open $x\in\boldsymbol{Y}\subseteq\boldsymbol{X}$ with $\boldsymbol{Y}\simeq\mathrm{Spec}(A,\mathrm{d})$, for $A=\bigoplus_{i\leqslant 0}A^i,\mathrm{d}$ a cdga over $\mathbb{K}$ with A^0 a smooth $\mathbb{K}$-algebra, and such that A is freely generated over A^0 by graded variables x_j^{-i},y_j^{k+i} in degrees $-1,-2,\ldots,k$. We take $\dim A^0$ and the number of x_j^{-i},y_j^{k+i} to be minimal at x.

Using theorems about periodic cyclic cohomology, we show that on $Y\simeq\mathrm{Spec}(A,\mathrm{d})$ we can write $\omega|_Y=[(\omega^0,0,0,\ldots)]$, for ω^0 a 2-form of degree k with $\mathrm{d}\omega^0=\mathrm{d}_{dR}\omega^0=0$. Minimality at x implies ω^0 is strictly nondegenerate near x, so we can change variables to write $\omega^0=\sum_{i,j}\mathrm{d}_{dR}y_j^{k+i}\mathrm{d}_{dR}x_j^{-i}$. Finally, we show d in (A,d) is a symplectic vector field, which integrates to a Hamiltonian H.

The case of -1-shifted symplectic derived schemes

When $k=-1$ the Hamiltonian H in the theorem has degree 0. Then Theorem 2.1 reduces to:

Corollary 2.2

Suppose $(\boldsymbol{X},\omega)$ is a -1-shifted symplectic derived $\mathbb{K}$-scheme. Then $(\boldsymbol{X},\omega)$ is Zariski locally equivalent to a derived critical locus $\mathbf{Crit}(H:U\rightarrow\mathbb{A}^1)$, for U a smooth classical $\mathbb{K}$-scheme and $H:U\rightarrow\mathbb{A}^1$ a regular function. Hence, the underlying classical $\mathbb{K}$-scheme $X=t_0(\boldsymbol{X})$ is Zariski locally isomorphic to a classical critical locus $\mathrm{Crit}(H:U\rightarrow\mathbb{A}^1)$.

3. Applications to Donaldson–Thomas theory (sketch)
References for §3

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3.1. Donaldson–Thomas invariants of Calabi–Yau 3-folds

Let X be a Calabi–Yau 3-fold. Fix a Chern character α in $H^{\mathrm{even}}(X;\mathbb{Q})$. Then one can define *coarse moduli schemes* $\mathfrak{M}_{\mathrm{st}}^{\alpha}(\tau)$, $\mathfrak{M}_{\mathrm{ss}}^{\alpha}(\tau)$ parametrizing equivalence classes of τ-(semi)stable coherent sheaves on X with Chern character α.

- $\mathfrak{M}_{\mathrm{ss}}^{\alpha}(\tau)$ is a projective $\mathbb{C}$-scheme, so in particular it is proper.
- $\mathfrak{M}_{\mathrm{st}}^{\alpha}(\tau)$ is an open subset in $\mathfrak{M}_{\mathrm{ss}}^{\alpha}(\tau)$, and has an extra structure, a *symmetric obstruction theory*, which does not extend to $\mathfrak{M}_{\mathrm{ss}}^{\alpha}(\tau)$.

If $\mathfrak{M}_{\mathrm{ss}}^{\alpha}(\tau)=\mathfrak{M}_{\mathrm{st}}^{\alpha}(\tau)$, then $\mathfrak{M}_{\mathrm{st}}^{\alpha}(\tau)$ is proper with a symmetric obstruction theory. Thomas used Behrend–Fantechi virtual classes to define the 'number' $DT^{\alpha}(\tau)\in\mathbb{Z}$ of points in $\mathfrak{M}_{\mathrm{st}}^{\alpha}(\tau)$, and showed $DT^{\alpha}(\tau)$ is unchanged under deformations of X. Behrend showed that $DT^{\alpha}(\tau)$ is a weighted Euler characteristic $\chi(\mathfrak{M}_{\mathrm{st}}^{\alpha}(\tau),\nu)$. Joyce–Song 2008 extended this to define $DT^{\alpha}(\tau)\in\mathbb{Q}$ for all Chern characters α, using Artin stacks, and showed the $DT^{\alpha}(\tau)$ satisfy a wall crossing formula under change of τ.

PTVV and generalizations of Donaldson–Thomas theory

When PTVV came out in 2011, it gave us a new geometric structure on (derived) moduli spaces of coherent sheaves on Calabi–Yau 3-folds, -1-shifted symplectic structures. It was natural to look for applications of this in Donaldson–Thomas theory.

There are three main ways to generalize D–T invariants, and PTVV is important for all three:

- *Categorification*: finding (graded) vector spaces with (graded) dimension $DT^\alpha(\tau)$. Putting algebraic structures on these vector spaces.
- *Motivic invariants*: upgrade $DT^\alpha(\tau)$ from $\mathbb{Q}$ to some larger ring. For example, $DT^\alpha(\tau)$ is a weighted Euler characteristic of $\mathfrak{M}_{\rm ss}^\alpha(\tau)$, we could upgrade it to a weighted virtual Poincaré polynomial of $\mathfrak{M}_{\rm ss}^\alpha(\tau)$. These are called *refined Donaldson–Thomas invariants*.
- Conjecturally there may exist a second categorification, involving constructing ∞-sheaves of categories on 3-CY moduli spaces, locally modelled on categories of matrix factorizations.

Combining PTVV and BBJ shows that moduli spaces of coherent sheaves on Calabi–Yau 3-folds are locally modelled on critical loci. There is some interesting geometry associated with critical loci:

- Perverse sheaves of vanishing cycles.
- Motivic Milnor fibres.
- Categories of matrix factorizations.

Bussi–Brav–Dupont–Joyce–Szendrői and Bussi–Joyce–Meinhardt showed that given some extra 'orientation data' on a CY3 moduli space $\mathfrak{M}$ (roughly, a spin structure), one can glue these local perverse sheaves and motives to give a global perverse sheaf $\mathcal{P}^\bullet_{\mathfrak{M}}$ and a global motive $[\mathfrak{M}]_{\rm mot}$ on $\mathfrak{M}$. When $\mathfrak{M}$ is a finite type $\mathbb{C}$-scheme, the hypercohomology $\mathbb{H}^*(\mathcal{P}^\bullet_{\mathfrak{M}})$ has graded dimension the Donaldson–Thomas invariant $DT(\mathfrak{M})$, and so provides a categorification of $DT(\mathfrak{M})$. The motive $[\mathfrak{M}]_{\rm mot}$ is in effect the motivic D–T invariant $DT_{\rm mot}(\mathfrak{M})$. Motivic D–T invariants satisfy a wall crossing formula. The gluing problem for matrix factorizations on CY3 moduli spaces is still unsolved.

3.2. D–T style invariants for Calabi–Yau 4-folds

Let X be a Calabi–Yau 4-fold over $\mathbb{C}$, and $\mathfrak{M}$ a derived moduli space of coherent sheaves on X. Then PTVV shows that $\mathfrak{M}$ is -2-shifted symplectic, and BBJ/BBBJ give local models for $\mathfrak{M}$. If $\mathfrak{M}$ is a -2-shifted symplectic derived scheme then the cotangent complex $\mathbb{L}_{\mathfrak{M}}$ is perfect in $[-2,0]$. So it is not quasi-smooth (which means $\mathbb{L}_{\mathfrak{M}}$ is perfect in $[-1,0]$), and it does not have a natural perfect obstruction theory, or a Behrend–Fantechi virtual class.

Using the BBJ local models, Borisov–Joyce 2015 showed that if $\mathfrak{M}$ is a separated -2-shifted symplectic derived $\mathbb{C}$-scheme then the underlying complex analytic topological space $\mathfrak{M}_{\rm an}$ has the structure of a *derived smooth manifold*, of real dimension $\dim_{\mathbb{C}} \mathfrak{M} = \frac{1}{2} \dim_{\mathbb{R}} \mathfrak{M}$. Note the halving of dimension; heuristically this is because we have a real -2-Lagrangian fibration $\mathfrak{M} \to \mathfrak{M}_{\rm an}$, and the base of a Lagrangian fibration of a symplectic manifold has half the dimension of the symplectic manifold.

If we are given an orientation for $\mathfrak{M}$, and $\mathfrak{M}$ is proper, then $\mathfrak{M}_{\rm an}$ is a compact oriented derived smooth manifold, and has a virtual class $[\mathfrak{M}_{\rm an}]_{\rm virt}$ in $H_*(\mathfrak{M},\mathbb{Z})$. We regard this as a *virtual class* for $\mathfrak{M}$.
Nobody could do anything with Borisov–Joyce, as noone in the world except me likes derived smooth manifolds. But in 2020, Oh–Thomas found an alternative algebro-geometric construction of the Borisov–Joyce virtual class, in the style of Behrend–Fantechi. Now everyone uses Oh–Thomas virtual classes.
This is the beginning of a theory of *Donaldson–Thomas type invariants* of Calabi–Yau 4-folds, often called *DT4 invariants*, which is under development. They are invariant under deformations of the complex structure of X, by properties of virtual classes. Some important authors are Yalong Cao and Hyeonjun Park.
Eventually one should expect a theory of DT4 invariants and wall-crossing with the universal structure in my paper arXiv:2111.04694.