Lukas Brantner 14/3/25

Deformation quantization (in char p)

[follows Bezrukavnikov – Kaledin. 2 papers early 2000s.]

Example. $A = k[[x_1,\dots,x_n,y_1,\dots,y_n]]$

$\Omega = \sum dx_i \wedge dy_i$

input data

Then Ω gives rise to a Poisson bracket on A.

Given a vector X, get a 1-form

$$\iota_X \Omega = \Omega(X,-)$$

– For a function f with $df = \frac{\partial f}{\partial x_i} dx_i + \frac{\partial f}{\partial y_i} dy_i$

the vector field $X_f = \sum \frac{\partial f}{\partial y_i}\frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial}{\partial y_i}$

Hamiltonian vector field.

satisfies $\iota_{X_f}\omega = df$.

Poisson bracket $\{f,g\} = \Omega(H_f, H_g)$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial y_i}$$

$\Rightarrow$ A is a Poisson algebra.

A, Λ is input data.

Output of quantization is the formal Weyl algebra

non commutative algebra

point

$$D = k[[x_1, \dots, x_n, y_1, \dots, y_n, \hbar]], \ *$$

with: $[x_i, x_j] = [y_i, y_j] = [x_i, \hbar] = [y_j, \hbar] = 0,$

$$[x_i, y_j] = \delta_{ij}\, \hbar$$

$$x_i * y_i = x_i \otimes y_i = \underbrace{\frac{x_i \otimes y_i + y_i \otimes x_i}{2}}_{x_i \cdot y_i} + \underbrace{\frac{x_i \otimes y_i - y_i \otimes x_i}{2}}_{\hbar}$$

Satisfying: (1) D is flat over $k[[\hbar]]$

(2) $D/\hbar = A$

(3) $\dfrac{f * g - g * f}{\hbar} \equiv \{f, g\} \mod \hbar.$

Deformation quantization: aims to generalize this to geometric situations.

Goal: Geometrize this!

Setup: Let S be a scheme of finite type over a field k of characteristic zero.

"S-manifold": scheme X of finite type, smooth $/S$.

$\to \Omega^\bullet_{X/S}$ algebraic de Rham complex.

$H^*_{dR}(X/S)$ hypercohomology

Fix a symplectic 2-form $\Omega \in \Omega^2(X/S)$

$\Rightarrow$ A Poisson structure $\{\,,\}$ on $\mathcal{O}_X$.

$$\{f, g\} = \Omega(X_f, X_g)$$

Defn A deformation quantization of (X, Ω)

i) a sheaf D of associative flat $\pi^{-1}\mathcal{O}_S((t))$-algebras on X, complete w.r.t. t, equipped with isomorphism

$D/tD \cong \mathcal{O}_X$, such that the commutator in D reduces to $\{-,-\}$ mod t.

Question: Do quantizations exist?

If so, how many?

We'll now assume X/S is **admissible**,

i.e. $H^i_{dR}(X) \to H^i(X, \mathcal{O}_X)$ is

surjective for $i=1,2$.

Example: For $S = \operatorname{Spec} k$, this holds for projective X (because of Hodge-deRham degeneration.)

Theorem X be admissible of dimension $2d$ over S as above.

Fix $\Omega \in H^0(X, \Omega^2_{X/S})$ a symplectic 2-form.

Write $Q(X, \Omega)$ for the set of (iso classes of) quantizations.

Then there is a natural injection

$\mathrm{Per}: \mathcal{Q}(X, \Omega) \longrightarrow H^2_{dR}(X/S)((\hbar))$.

Image: As X is admissible,

$H^2_F(X) := H^2(X, \Omega^{\geq 1}_X) =$

$\ker(H^2_{dR}(X) \to H^2(X, \mathcal{O}_X))$

$\subset H^2_{dR}(X)$

Choose a splitting $P: H^2_{dR}(X) \to H^2(X, \Omega^{\geq 1}_X)$

Then the composite

$\mathrm{Per}_P: \mathcal{Q}(X, \Omega) \longrightarrow H^2_F(X/S)((\hbar))$

is injective with image

$P([\Omega]) + \hbar\, H^2(X/S)((\hbar))$.

Towards deformation quantization in characteristic p

Let k be a field of characteristic $p>2$. X/k a reduced Gorenstein noetherian k-scheme (crucial is a simplified base scheme.) ($S = \operatorname{Spec} k$.)

Assumptions imply Ω_X is locally free, although not necessarily smooth.

$\Omega \in H^0(X, \Omega^2_{X/k})$ a symplectic structure.

As in char 0, Ω gives rise to a Poisson bracket $\{-,-\}$ on $\mathcal{O}_X$.

Restricted structures

Recall. A restricted Lie algebra,

$$(\mathfrak{g}, [-,-], (\,)^{(p)}) \text{ in char } p>2$$

is a Lie algebra with a map $(\,)^{(p)} : \mathfrak{g} \to \mathfrak{g}$

satisfying (1) $(\lambda D)^{(p)} = \lambda^p D^{(p)}$

(2) $ad(D) = [D, -]$, $ad(D^{(p)}) = ad(D)^{op}$

(3) $(D_1 + D_2)^{(p)} = D_1^{(p)} + \sum_{i=1}^{p-1} s_i(D_1, D_2) + D_2^{(p)}$

for $s_i(D_1, D_2)$ the coeff of t^{i-1} in

$ad(t D_1 + D_2)^{p-1}(D_1)$

A restricted Poisson algebra: Poisson algebra $(A, \cdot, \{-,-\}, (\ )^{(p)}_{A \to A})$

such that: $(A, \cdot, \{-,-\})$ Poisson algebra

. $(A, \{-,-\}, (\ )^{(p)})$ is a restricted Lie algebra

- $(x^2)^{(p)} = 2x^p x^{(p)}$

Remark: Can also write last axiom as

$$(xy)^{(p)} = x^p y^{(p)} + x^{(p)} y^p + P_T(x,y)$$

some polynomial they specify.

(Aside: for a homotopy theorist, Poisson algebras are what you get when take $\mathbb{E}_2$-algebras & take its homotopy theory. So, this should be what you get from $\mathbb{E}_2$-algebras in char. p.)

— Want to regard $\mathbb{T}_X$ as having more structure than a Lie algebra in char p

— want it to be a restricted Lie algebra / partition Lie algebra...

Quantized algebra:

Flat $k[[\hbar]]$ algebra $A_\hbar$ such that $A_\hbar / \hbar A_\hbar$ is commutative.

$\Rightarrow$ Poisson bracket $\dfrac{ab-ba}{\hbar}$ defined uniquely

<u>Restricted quantized algebra</u>: quantized algebra

and operator $(-)^{(p)}$.

such that $\left(A, \langle -,- \rangle, (-)^{(p)}\right)$ is a restricted Lie alg,

$$\hbar^{(p)} = \hbar$$

$$(xy)^{(p)} = x^p y^{(p)} + x^{(p)} y^p - \hbar^{p-1} x^{(p)} y^{(p)} + P(x,y)$$

<u>Exercise</u>: $(\hbar x)^{(p)} = \hbar x^p$.

Given a restricted quantized algebra $A_\hbar$

define a map $s: A_\hbar^{(1)} = A_\hbar \otimes_k k \longrightarrow A_\hbar$

map $k \to k$ is Frobenius

$$x \mapsto x^p - \hbar^{p-1} x^{(p)}.$$

The s is $k((\hbar))$-linear, satisfies

$s(\hbar) = 0$, is multiplicative, Im $s \subseteq A_\hbar$ is central, and $s(t) \equiv a^p \mod t^{p-1}$.

Get a factorization

$$\begin{tikzcd} A_h^{(1)} \arrow[r, "s"] \arrow[dr] & A_h \\ & A^{(1)} = A_h^{(1)}/h \arrow[u, "s'"] \end{tikzcd}$$

We can extend s' to a map $A^{(1)}((t))$
$$\downarrow$$
$$Z(A_h) =: Z_h$$

and note that $Z_h \to Z$ = Poisson centre ? $\overset{?}{=} \mathcal{O}_X^p$ is surjective.

Let (X, Ω) be an affine. Then
reduced symplectic

Hamiltonian vector fields in T_X are closed under operation $\xi \longmapsto \xi^{(p)}$ [iterate vector field p times as a derivation on functions]

if and only if Ω is locally exact in the Zariski topology, and there is a bijection) v. strong condition.

$\left\{ \begin{array}{l} \text{restricted structure extensions} \\ \text{Poisson bundle with } h, \Omega \end{array} \right\} \cong \left\{ \eta \in H^0(X, \Omega^1_X / d\mathcal{O}_X) \text{ such that } d\eta = \Omega \right\}$

$H^1_{AR}(X) = \text{torsor}$

Let's fix a certain symplectic S-manifold $(X, \Omega, \mathcal{O}^{(0)})$.

Definition A Frobenius-constant quantization is a sheaf $\mathcal{O}_h$ of certain quantized algebras (flat over $k((h))$) such that

$$\mathcal{O}_h / h = \mathcal{O}_X \text{ as restricted Poisson algebras}$$

According what we saw before, this implies

$$Z_h = Z(\mathcal{O}_h) \cong \mathcal{O}_X^{p}((h))$$

$\mathcal{O}_X' \subseteq \mathcal{O}_X$ stays central in the quantization.

Theorem Let X be a symplectic restricted manifold as above, such that

$$F: H^0(X, \mathcal{O}_X^{(1)}) \longrightarrow H^0(X, \mathcal{O}_X) \text{ is bijective}$$

for $i = 1, 2, 3$. Then X admits a Frobenius-constant quantization, and set $Q(X, \Omega)$ of such objects with $H^1_{et}(X, \mathcal{H}_{log})$

where $\mathcal{H}_{log} \subset \Omega^1_X$ is the sheaf of logarithmic 1-forms, i.e. diagram

$$\begin{tikzcd}
0 \arrow[r] & (\mathcal{O}_X^p)^\times \arrow[r, hook] & \mathcal{O}_X^\times \arrow[r] \arrow[dr, "f \mapsto \frac{df}{f}"'] & \mathcal{H}_{log} \arrow[r] \arrow[d, phantom, "\cap"] & 0 \\
& & & \Omega^1_X &
\end{tikzcd}$$

— conditions on X are very restrictive.