Derived Algebraic Geometry

Lecture 11 of 14: Even more ∞-categories.
Beginnings of Derived Algebraic Geometry.

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References for this lecture:
J. Lurie, *'Stable Infinity Categories'*, math.CT/0608228,
B. Toën, *'Derived Algebraic Geometry'*, arXiv:1401.1044.

These slides available at
`http://people.maths.ox.ac.uk/~joyce/`

Plan of talk:

11.1. Stable ∞-categories

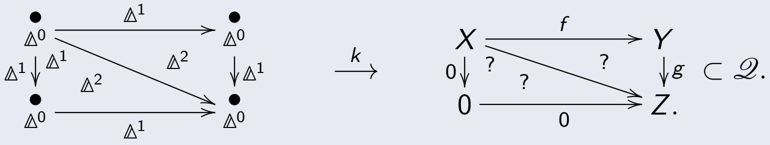
Definition

Let $\mathcal{Q}$ be a quasicategory. A *zero object* is an object $0 \in \mathcal{Q}$ which is both an initial object and a terminal object.

Suppose $\mathcal{Q}$ has a zero object, and consider commuting squares

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ \downarrow 0 & \Uparrow & g \downarrow \\ 0 & \xrightarrow{\quad 0 \quad} & Z \end{array} \tag{11.1}$$

in $\mathcal{Q}$. This really means we have a map of simplicial sets



So far (11.1) just means that $g \circ f = 0$, so that $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a complex. We call $X \xrightarrow{f} Y \xrightarrow{g} Z$ a *triangle* in $\mathcal{Q}$.

Definition (Continued.)

We say that $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a *fibre sequence*, or *exact triangle*, and write $X = \operatorname{Ker} g$, if (11.1) is a *homotopy Cartesian square* (∞-*Cartesian square*), i.e. if (11.1) is the limit in $\mathcal{Q}$ of the map of simplicial sets

$$
\begin{array}{ccc}
 & & \underset{\Delta^0}{\bullet} \\
 & & \big\downarrow \Delta^1 \\
\underset{\Delta^0}{\bullet} & \xrightarrow[\Delta^1]{\qquad\qquad} & \underset{\Delta^0}{\bullet}
\end{array}
\quad \xrightarrow{k} \quad
\begin{array}{ccc}
 & & Y \\
 & & \big\downarrow g \\
0 & \xrightarrow[0]{\qquad\qquad} & Z.
\end{array}
\subset \mathcal{Q}.
$$

Dually, we say that $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a *cofibre sequence*, or *coexact triangle*, and write $Z = \operatorname{Coker} f$, if (11.1) is a homotopy coCartesian square (∞-coCartesian square).

Definition

A quasicategory $\mathcal{Q}$ is a *stable ∞-category* if

- $\mathcal{Q}$ has a zero object 0.
- Every morphism $g : Y \to Z$ in $\mathcal{Q}$ has a *kernel* (that is, g extends to an exact triangle $X \xrightarrow{f} Y \xrightarrow{g} Z$).
- Every morphism $f : X \to Y$ in $\mathcal{Q}$ has a *cokernel* (that is, f extends to a coexact triangle $X \xrightarrow{f} Y \xrightarrow{g} Z$).
- Every exact triangle is coexact, and vice versa.

This is a simple definition with remarkable consequences. There are lots of stable ∞-categories in nature, and stable ∞-categories have very good properties, you can do lots of beautiful mathematics in them. For example:

Theorem 11.1

Let $\mathcal{Q}$ be a stable ∞-category. Then the homotopy category $\mathrm{Ho}(\mathcal{Q})$ is a triangulated category.

Theorem 11.1 is surprising as triangulated categories have lots of extra structure which apparently wasn't there in the definition of stable ∞-category. For example, triangulated categories are additive, so that $\mathrm{Hom}(X, Y)$ is an abelian group, but it is not obvious why $\mathrm{Ho}(\mathscr{Q})$ should be additive.

Every object X in $\mathscr{Q}$ has a *loop space* ΩX in an exact triangle $\Omega X \to 0 \to X$, and a *suspension* ΣX in a coexact triangle $X \to 0 \to \Sigma X$. These extend to ∞-functors $\Omega : \mathscr{Q} \to \mathscr{Q}$ and $\Sigma : \mathscr{Q} \to \mathscr{Q}$. As exact and coexact triangles are the same we see that $\Omega \circ \Sigma X \simeq X$ and $\Sigma \circ \Omega X \simeq X$, and Ω, Σ are inverse functors up to equivalence. On the homotopy triangulated category, $\mathrm{Ho}(\Sigma) : \mathrm{Ho}(\mathscr{Q}) \to \mathrm{Ho}(\mathscr{Q})$ is the shift functor [1].

If $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a (co)exact triangle then f extends to $W \xrightarrow{e} X \xrightarrow{f} Y$, and one can show that $W \cong \Omega Z$. We get a long exact sequence with each consecutive three terms a (co)exact triangle:

$$\cdots \to \Omega^2 Z \to \Omega X \to \Omega Y \to \Omega Z \to X \xrightarrow{f} Y \xrightarrow{g} Z \to \Sigma X \to \Sigma Y \to \Sigma Z \to \Sigma^2 X \to \cdots$$

For example: to define the loop space functor $\Omega : \mathscr{Q} \to \mathscr{Q}$ we have to make an infinite number of choices: on every object X (0-simplex in $\mathscr{Q}$) we choose a kernel ΩX for $0 \to X$, and similarly for all n-simplices in $\mathscr{Q}$. Each of these choices lies in a simplicial set of possible choices, and the definition of limits in quasicategories implies that this simplicial set is contractible.

General Principle

- When working in quasicategories, everything is a simplicial set.
- Something is 'unique' if it lies in a contractible simplicial set.

If $\mathscr{Q}$ is a quasicategory with a zero object, and all finite limits exist in $\mathscr{Q}$, then one can define the loop space ∞-functor $\Omega : \mathscr{Q} \to \mathscr{Q}$ as before. One can then define a stable ∞-category $\mathrm{Sp}(\mathscr{Q})$ of *spectra* in $\mathscr{Q}$, roughly by formally inverting Ω. Objects in $\mathrm{Sp}(\mathscr{Q})$ are sequences $(E_i)_{i\in\mathbb{Z}}$ of objects in $\mathscr{Q}$ with morphisms $E_i \to \Omega(E_{i+1})$. When $\mathscr{Q}$ is the quasicategory of topological spaces **Top**, this gives the usual notion of spectra in algebraic topology.

Derived categories and stable ∞-categories

Let $\mathcal{A}$ be an abelian category satisfying suitable conditions. Then (Lurie) the derived categories $D(\mathcal{A}), D^{\pm}(\mathcal{A}), D^b(\mathcal{A})$ can be written as (equivalent to) the homotopy categories of stable ∞-categories $\mathbb{D}(\mathcal{A}), \mathbb{D}^{\pm}(\mathcal{A}), \mathbb{D}^b(\mathcal{A})$. For example, if $\mathcal{A}$ has enough projectives, then we can define $\mathbb{D}^-(\mathcal{A})$ to be the nerve of the full simplicial subcategory of projective complexes in $\mathrm{Com}^-(\mathcal{A})$.

11.2. Dg-categories, Segal categories

Definition

Let $\mathbb{K}$ be a field. A *dg-category* over $\mathbb{K}$ is a category $\mathscr{C}$ enriched in $\mathbb{Z}$-graded chain complexes over $\mathbb{K}$. That is, $\mathscr{C}$ has objects $X, Y, \ldots$, and for all X, Y we have a chain complex $\mathrm{Hom}^\bullet(X, Y)$:

$$\cdots \xrightarrow{\mathrm{d}^{k-1}} \mathrm{Hom}^k(X, Y) \xrightarrow{\mathrm{d}^k} \mathrm{Hom}^{k+1}(X, Y) \xrightarrow{\mathrm{d}^{k+1}} \cdots$$

and for all X, Y, Z associative unital morphisms of complexes $\mu_{XYZ} : \mathrm{Hom}^\bullet(Y, Z) \otimes_{\mathbb{K}} \mathrm{Hom}^\bullet(X, Y) \to \mathrm{Hom}^\bullet(X, Z)$.

Given a dg-category $\mathscr{C}$, one can form an ordinary category $H^0(\mathscr{C})$ with the same objects and with morphisms $\mathrm{Hom}_{H^0(\mathscr{C})}(X, Y) = H^0(\mathrm{Hom}^\bullet(X, Y))$.

There is a notion of *pretriangulated dg-category* $\mathscr{C}$, which has a suspension functor Σ and a class of distinguished triangles (being pretriangulated is a property of $\mathscr{C}$, not extra structure). Then $H^0(\mathscr{C})$ is a triangulated category.

Dg-categories are a kind of $\mathbb{K}$-linear ∞-category, which predate the development of ∞-categories. Pretriangulated dg-categories were important as an enhanced notion of triangulated category, in which one could deal with problems such as the nonfunctoriality of the cone. Many derived categories such as $D^b \operatorname{coh}(X)$ are of the form $H^0(\mathscr{C})$ for a pretriangulated dg-category $\mathscr{C}$.

Definition

For dg-categories $\mathscr{C}, \mathscr{D}$, a *dg-functor* $F : \mathscr{C} \to \mathscr{D}$ is defined in the obvious way, mapping $X \mapsto F(X)$ on objects, and morphisms of complexes $F_{X,Y} : \operatorname{Hom}^\bullet_{\mathscr{C}}(X, Y) \to \operatorname{Hom}^\bullet_{\mathscr{D}}(F(X), F(Y))$ preserving all the structure. We call F a *quasi-equivalence* if $F_{X,Y}$ is a quasi-isomorphism for all X, Y and $H^0(F) : H^0(\mathscr{C}) \to H^0(\mathscr{D})$ is an equivalence of ordinary categories.
Dg-categories form a model category $\mathfrak{dg}\text{-}\mathfrak{Cat}$ in which the weak equivalences are quasi-equivalences.

From an ∞-category point of view, dg-categories are more-or-less equivalent to $\mathbb{K}$-linear stable ∞-categories.

Segal categories

Segal categories are yet another model for ∞-categories, a weak form of simplicial categories. Roughly, a Segal category has a set/class of *objects* $X, Y, Z, \ldots$, and for all objects X, Y a simplicial set $\mathrm{Hom}(X, Y)$, and for all X, Y, Z a diagram of simplicial sets

$$\begin{array}{ccc} \mathrm{Hom}(X,Y,Z) & \xrightarrow{\mu_{XYZ}} & \mathrm{Hom}(X,Z) \\ \simeq \big\downarrow \iota_{XYZ} & & \\ \mathrm{Hom}(Y,Z) \times \mathrm{Hom}(X,Y), & & \end{array}$$

with ι_{XYZ} a weak equivalence. The idea is that composition $\mathrm{Hom}(Y,Z) \times \mathrm{Hom}(X,Y) \to \mathrm{Hom}(X,Z)$ is $\mu_{XYZ} \circ \iota_{XYZ}^{-1}$, except that ι_{XYZ}^{-1} doesn't exist until we invert weak equivalences in **SSets**. So as in quasicategories, composition of morphisms is not unique, but involves a choice.

11.3. Derived Algebraic Geometry: foundations

As far as I am concerned, there are two main foundational theories of Derived Algebraic Geometry on the market:

- Bertrand Toën and Gabriele Vezzosi, a series of papers from 2001, an important one being 'HAG II', math.AG/0404373.
- Jacob Lurie's 2006 PhD thesis, which mutated into 'Derived Algebraic Geometry I–XIV', and then into $\geqslant 3$ enormous books 'Higher Topos Theory' math.CT/0608040, 'Higher Algebra', and 'Spectral Algebraic Geometry'.

There is also a series of books by Dennis Gaitsgory and Nick Rozenblyum called 'A study in Derived Algebraic Geometry', but I don't count these, as Gaitsgory–Rozenblyum seem to be obsessed by ind-coherent sheaves, which I don't see as the main point of DAG (the main point being extending schemes, stacks, Hartshorne, etc. to the 'derived' world?). I guess Gaitsgory–Rozenblyum mostly want to prove geometric Langlands.

Toën–Vezzosi (which I know better than Lurie) work with model categories, simplicial categories, and dg-categories — older technology now — and their focus is more clearly on doing Algebraic Geometry and dragging Hartshorne into the 21$^{\rm st}$ century (or at least into the 1990s). Lurie works in quasicategories — new and shiny — and developed much of the ∞-category theory himself. He cares about derived schemes and derived stacks, but also about a bunch of other things as well — ∞-categories have applications in Algebraic Topology, TQFTs, etc. Lurie is a towering figure in mathematics. You can find lots of his writings (1000s of pages) on his webpage. They are usually very long, but very clear and beautifully written.

It is generally believed that the Toën–Vezzosi and Lurie theories should be equivalent on their intersection. I cite references to both, and usually get away with it.

For short introductions I recommend two survey papers on DAG by Bertrand Toën, arXiv:1401.1044 and math.AG/0604504.

Higher stacks

Classical Artin $\mathbb{K}$-stacks are functors $F : \mathbf{Sch}_{\mathbb{K}}^{\mathrm{op}} \to \mathbf{Groupoids}$, or equivalently $F : \mathbf{Alg}_{\mathbb{K}} \to \mathbf{Groupoids}$. Morally, these have two essentially orthogonal directions of generalization:

- *Higher Artin stacks* $\mathbf{HArt}_{\mathbb{K}}$, in which we replace the target **Groupoids** by the ∞-category **SSets** (basically ∞-groupoids).
- *Derived Artin stacks* $\mathbf{DArt}_{\mathbb{K}}$, in which we replace the domain $\mathbf{Alg}_{\mathbb{K}}$ by an ∞-category of derived algebras, either *simplicial* $\mathbb{K}$*-algebras* $\mathbf{SAlg}_{\mathbb{K}}$, or (when $\mathrm{char}\,\mathbb{K} = 0$) cdgas $\mathbf{cdga}_{\mathbb{K}}$.

Actually it doesn't really make sense to talk about ∞-functors into an ordinary (2-)category **Groupoids**, so for derived stacks we consider ∞-functors $\mathbf{SAlg}_{\mathbb{K}} \to \mathbf{SSets}$, so one defines derived stacks as generalizations of higher stacks. But there is an ∞-subcategory $\mathbf{DArt}_{\mathbb{K}}^1 \subset \mathbf{DArt}_{\mathbb{K}}$ of *derived Artin* 1*-stacks*, which on the classical level are Artin rather than higher stacks.

I'll start with higher stacks, and go on to derived stacks next lecture.

What are higher stacks for? Moduli spaces for $D^b \operatorname{coh}(X)$

Let X be a smooth projective $\mathbb{K}$-scheme (this can be weakened a lot). Then people care about the categories $\operatorname{Vect}(X) \subset \operatorname{coh}(X) \subset D^b \operatorname{coh}(X)$. The moduli stacks $\mathcal{M}_{\operatorname{Vect}(X)} \subset \mathcal{M}_{\operatorname{coh}(X)}$ of objects in $\operatorname{Vect}(X)$ and $\operatorname{coh}(X)$ exist as Artin $\mathbb{K}$-stacks. However, no moduli stack of objects in $D^b \operatorname{coh}(X)$ exists, as an Artin $\mathbb{K}$-stack. We can define a moduli functor $F : \mathbf{Sch}_{\mathbb{K}}^{\mathrm{op}} \to \mathbf{Groupoids}$ parametrizing objects $E^\bullet$ in $D^b \operatorname{coh}(X)$, but it is *not a stack*, that is, it is not a 2-sheaf over (fppf, smooth etc.) open covers $\{U_i \hookrightarrow U,\ i \in I\}$ in $\mathbf{Sch}_{\mathbb{K}}$.
This is because (say perfect) complexes $E^\bullet \to U \times X$ do not form a 2-sheaf over U if $\operatorname{Ext}^k(E^\bullet, E^\bullet) \neq 0$ for $k < 0$, that is, if there are nontrivial morphisms $E^\bullet \to E^\bullet[k]$ for $k < 0$. This does not happen for sheaves $E \in \operatorname{coh}(X) \subset D^b \operatorname{coh}(X)$, which have $\operatorname{Ext}^k(E, E) = 0$ for $k < 0$. The problem is *negative Exts* $\operatorname{Ext}^{<0}(E^\bullet, E^\bullet)$ in $D^b \operatorname{coh}(X)$.

For the next bit of the lecture I am going to use '∞-category' to mean some suitable version of $(\infty, 1)$-categories (model categories, simplicial categories, Segal categories, quasicategories, dg-categories, ...) without caring exactly which kind.
The solution to defining moduli stacks in $D^b \operatorname{coh}(X)$ is to pass from the ordinary category $D^b \operatorname{coh}(X)$ to an ∞-category version $\mathbb{D}^b \operatorname{coh}(X)$, and define an ∞-categorical notion of ∞-sheaf (homotopy sheaf), which given an open cover $\{U_i \hookrightarrow U,\ i \in I\}$ in $\mathbf{Sch}_{\mathbb{K}}$, involves choosing $(n-1)$-morphisms in **SSets** on n-fold overlaps $U_1 \cap \cdots \cap U_n$ for all $n \geqslant 1$. Then complexes $E^\bullet \to U \times X$ in $\mathbb{D}^b \operatorname{coh}(U \times X)$ form an ∞-sheaf over U. So when we define a moduli ∞-functor $F : \mathbf{Sch}_{\mathbb{K}}^{\mathrm{op}} \to \mathbf{SSets}$, it has an ∞-sheaf property over open covers $\{U_i \hookrightarrow U,\ i \in I\}$ in $\mathbf{Sch}_{\mathbb{K}}$.
We define an ∞-category $\mathbf{HSta}_{\mathbb{K}}$ of *higher stacks*, the full ∞-subcategory of all F in $\mathrm{Fun}_\infty(\mathbf{Sch}_{\mathbb{K}}^{\mathrm{op}}, \mathbf{SSets})$ satisfying the ∞-sheaf condition. Then we define an ∞-subcategory $\mathbf{HArt}_{\mathbb{K}} \subset \mathbf{HSta}_{\mathbb{K}}$ of *higher Artin stacks*, which satisfy inductive conditions involving atlases.

What are higher stacks like?

Unfortunately I haven't got a very good answer to this.

Ordinary stacks X have *isotropy groups* $\mathrm{Iso}(x)$ at each point $x \in X$. For a moduli stack $\mathcal{M}$ of sheaves E, we have $\mathrm{Iso}([E]) = \mathrm{Aut}(E) \subset \mathrm{Ext}^0(E, E)$.

A higher stack X has *higher isotropy groups* $\mathrm{Iso}^i(x)$ for $i = 0, 1, \ldots$ at each point x, with $\mathrm{Iso}^i(x)$ abelian for $i > 0$ and $\mathrm{Iso}^i(x) = 0$ for $i \gg 0$, where for a moduli stack of complexes $E^\bullet$, we have $\mathrm{Iso}^0([E^\bullet]) \cong \mathrm{Aut}(E^\bullet) \subset \mathrm{Ext}^0(E^\bullet, E^\bullet)$ and $\mathrm{Iso}^i([E^\bullet]) \cong \mathrm{Ext}^{-i}(E^\bullet, E^\bullet)$ for $i > 0$.

If $\mathrm{Iso}^i(x) = 0$ for $i \geqslant n$ then $\mathrm{Iso}^0(x), \ldots, \mathrm{Iso}^{n-1}(x)$ should be the homology of an *n-group*. I'm not sure n-groups are properly defined yet, but 2-groups at least are well studied. A *2-group* is a quadruple (G_0, G_1, ρ, ω), where G_0 is a group, G_1 an abelian group, $\rho : G_0 \to \mathrm{Aut}(G_1)$ a morphism and $\omega \in H^3(BG_0, G_1)$.

So, our favourite local model for a higher Artin stack might be something like 'the quotient $[V/\boldsymbol{G}]$ of a $\mathbb{K}$-scheme V by an algebraic n-group $\boldsymbol{G}$.'

Simple examples of higher stacks

Let G be an abelian algebraic $\mathbb{K}$-group, e.g. $G = \mathbb{Z}_2$. People often write the quotient stack $[*/G]$ as BG. There is a higher Artin stack B^2G which classifies principal $[*/G]$-bundles. This is the functor $F : \mathbf{Sch}_{\mathbb{K}}^{\mathrm{op}} \to$ (2-groupoids) $\subset$ **SSets** such that $F(U)$ is the 2-groupoid (2-category with all 1-morphisms equivalences) of principal $[*/G]$-bundles $\pi : P \to U$, in the sense of Artin stacks. Here U is a $\mathbb{K}$-scheme, but P is an Artin stack which looks locally like $U \times [*/G]$.
Then B^2G is $[*/\boldsymbol{G}]$ for $\boldsymbol{G}$ the 2-group $(\{1\}, G, 0, 0)$.

Homotopy sheaves

Given a topological space X and ordinary open covers $\{U_i \hookrightarrow U, i \in I\}$ of open $U \subset X$, or more generally given a $\mathbb{K}$-scheme X and Grothendieck open covers (étale, smooth, ...) of $U \hookrightarrow X$, or on the site $\mathbf{Sch}_{\mathbb{K}}$, in §4.3, §5.1 and §5.2 we defined *sheaves* and 2-*sheaves* (*stacks*) on X. To define higher and derived stacks, we need an ∞-categorical notion of sheaf.

To do this, one starts with 'hypercovers': a simplicial notion of open cover, which is a simplicial object $U^\bullet : \Delta^{\rm op} \to \mathbf{Sch}_{\mathbb{K}}$ in $\mathbf{Sch}_{\mathbb{K}}$ satisfying conditions. For an open cover $\{U_i \hookrightarrow U, i \in I\}$ of U, the corresponding hypercover is

$$\coprod_{i_1 \in I} U_{i_1} \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \begin{array}{c}\displaystyle\coprod_{i_1,i_2 \in I} U_{i_1 i_2} = \\ \displaystyle\coprod_{i_1,i_2 \in I} U_{i_1} \times_U U_{i_2}\end{array} \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \begin{array}{c}\displaystyle\coprod_{i_1,i_2,i_3 \in I} U_{i_1 i_2 i_3} = \\ \displaystyle\coprod_{i_1,i_2,i_3 \in I} U_{i_1} \times_U U_{i_2} \times_U U_{i_3}\end{array} \cdots .$$

U is the colimit of this diagram in $\mathbf{Sch}_{\mathbb{K}}$, we glue the U_i to make U.

A *homotopy presheaf* is an (∞-)functor $F : \mathrm{Open}(X)^{\mathrm{op}} \to \mathbf{SSets}$. For each hypercover $U^\bullet : \Delta^{\mathrm{op}} \to \mathrm{Open}(X)$ of $U \in \mathrm{Open}(X)$ we get a corresponding cosimplicial object $F \circ U^\bullet : \Delta \to \mathbf{SSets}$. Pullbacks along $U_{i_1\cdots i_k} \hookrightarrow U$ induce a natural transformation $\underline{F(U)} \Rightarrow F \circ U^\bullet$, where $\underline{F(U)} : \Delta \to \mathbf{SSets}$ is the constant functor with value $F(U)$. We call F a *homotopy sheaf* (*∞-sheaf*) if for all such $U^\bullet \to U$, the induced morphism $F(U) \to \mathrm{holim}(F \circ U^\bullet)$ is a weak equivalence in $\mathbf{SSets}$, where $\mathrm{holim}(F \circ U^\bullet)$ is the homotopy limit of $\underline{F(U)} : \Delta \to \mathbf{SSets}$ in $\mathbf{SSets}$.

The homotopy sheaf condition involves data and conditions on all n-fold overlaps $U_{i_1} \cap \cdots \cap U_{i_n}$, packaged neatly using the language of simplicial sets.

We can then define the ∞-category $\mathbf{HSta}_{\mathbb{K}}$ of *higher* $\mathbb{K}$*-stacks* to be the full ∞-subcategory of F in $\mathrm{Fun}_\infty(\mathbf{Sch}_{\mathbb{K}}^{\mathrm{op}}, \mathbf{SSets})$ or $\mathrm{Fun}_\infty(\mathbf{Alg}_{\mathbb{K}}, \mathbf{SSets})$ which satisfy the homotopy sheaf condition for all hypercovers $U^\bullet \to U$, in some choice of Grothendieck topology.

Higher Artin stacks

Definition

Suppose we have defined the ∞-category $\mathbf{HSta}_{\mathbb{K}}$ of higher $\mathbb{K}$-stacks, with discrete ∞-subcategories $\mathbf{Aff}_{\mathbb{K}} \subset \mathbf{Sch}_{\mathbb{K}} \subset \mathbf{HSta}_{\mathbb{K}}$, and all (homotopy) fibre products exist in $\mathbf{HSta}_{\mathbb{K}}$.
We define a notion of *n-Artin stack* for $n \geqslant -1$, by induction on n:

- A -1-*Artin stack* is an affine scheme.
- A morphism $h : Y \to Z$ in $\mathbf{HSta}_{\mathbb{K}}$ is -1-*representable*, or *affine*, if $X \times_{g,Z,h} Y$ is an affine scheme for all $g : X \to Z$ with $X \in \mathbf{Aff}_{\mathbb{K}}$.
- Suppose by induction that $(n-1)$-Artin stacks and $(n-1)$-representable morphisms are defined. Then $X \in \mathbf{HSta}_{\mathbb{K}}$ is an *n-Artin stack* if there exists an $(n-1)$-representable, smooth, surjective morphism $\pi : U \to X$ with U a disjoint union of affine schemes. A morphism $h : Y \to Z$ in $\mathbf{HSta}_{\mathbb{K}}$ is *n-representable* if $X \times_{g,Z,h} Y$ is an n-Artin stack for all $g : X \to Z$ with $X \in \mathbf{Aff}_{\mathbb{K}}$.
- A stack X which is locally an n-Artin stack for some n is a *higher Artin stack*.

n-Artin stacks and Artin n-stacks

Write $\mathbf{HArt}_{\mathbb{K}} \subset \mathbf{HSta}_{\mathbb{K}}$ for the full ∞-subcategory of higher Artin stacks. Confusingly, there is a slightly different notion of *Artin n-stack*:

Definition

Let X be a higher Artin stack. We say that X is a (*higher*) *Artin n-stack* if for all $A \in \mathbf{Alg}_{\mathbb{K}}$, the simplicial set $X(A)$ is n-truncated, that is, $\pi_i(X(A)) = 0$ for all $i > n$. Then

$$n\text{-Artin stacks} \;\subset\; \text{Artin } n+1\text{-stacks.}$$

The notion of Artin n-stack is more useful than n-Artin stack. Artin 1-stacks are equivalent to ordinary Artin stacks. Artin 0-stacks are more-or-less the same thing as algebraic spaces. The moduli stack of complexes $E^\bullet$ in $D^b\,\mathrm{coh}(X)$ with $\mathrm{Ext}^{\leqslant -n}(E^\bullet, E^\bullet) = 0$ is an Artin n-stack.

Write $\mathbf{HArt}^n_{\mathbb{K}} \subset \mathbf{HArt}_{\mathbb{K}}$ for the full ∞-subcategory of higher Artin n-stacks.

One nice fact about higher moduli stacks

Toën–Vaquié prove that if X is a smooth projective $\mathbb{K}$-scheme then the moduli stack $\mathcal{M}_{D^b \operatorname{coh}(X)}$ of objects in $D^b \operatorname{coh}(X)$ exists as a higher Artin $\mathbb{K}$-stack.

In general I find higher stacks difficult to visualize and say anything meaningful about. But one thing does work nicely: over $\mathbb{K} = \mathbb{C}$ a higher $\mathbb{C}$-stack X has a 'topological realization' $X^{\rm top}$, a topological space natural up to homotopy equivalence, so we can define the (co)homology $H_*(X) := H_*(X^{\rm top})$, $H^*(X) := H^*(X^{\rm top})$.

It turns out that the (co)homology of the higher stack $\mathcal{M}_{D^b \operatorname{coh}(X)}$ is often computable, and is much nicer than the (co)homology of the Artin stack $\mathcal{M}_{\operatorname{coh}(X)}$, which is usually not computable.

Basically this is because $\mathcal{M}_{\operatorname{coh}(X)}$ is like an 'abelian monoid in stacks', with addition $\oplus$ in $\operatorname{coh}(X)$. But $\mathcal{M}_{D^b \operatorname{coh}(X)}$ is like an 'abelian group in stacks', as $[1] : D^b \operatorname{coh}(X) \to D^b \operatorname{coh}(X)$ acts like an (up to homotopy) inverse for addition $\oplus$ in $D^b \operatorname{coh}(X)$; and abelian groups are much simpler than monoids.

Derived Algebraic Geometry

Lecture 12 of 14: Derived schemes and derived stacks

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References for this lecture:
B. Toën, *'Derived Algebraic Geometry'*, arXiv:1401.1044,
B. Toën, *'Higher and derived stacks: a global overview'*, math.AG/0604504.

These slides available at
`http://people.maths.ox.ac.uk/∼joyce/`

Plan of talk:

12.1. Definitions of derived schemes and derived stacks

It turns out there is a good notion of derived scheme (Toën arXiv:1401.1044, Lurie), although the early Toën–Vezzosi theory went straight to derived stacks without discussing derived schemes.

Definition

Let $\mathbb{K}$ be a field. A *derived ringed* $\mathbb{K}$*-space* $\boldsymbol{X} = (X, \mathcal{O}_X)$ is a topological space X with a homotopy sheaf (∞-sheaf) $\mathcal{O}_X$ of simplicial commutative $\mathbb{K}$-algebras. Then $\pi_0(\mathcal{O}_X)$ is an ordinary sheaf of ordinary commutative $\mathbb{K}$-algebras, and $\pi_i(\mathcal{O}_X)$ is an ordinary sheaf of modules over $\pi_0(\mathcal{O}_X)$ for all $i > 0$.
We call $\boldsymbol{X} = (X, \mathcal{O}_X)$ (usually just abbreviated to $\boldsymbol{X}$) a *derived* $\mathbb{K}$*-scheme* if $(X, \pi_0(\mathcal{O}_X))$ is an ordinary $\mathbb{K}$-scheme, and $\pi_i(\mathcal{O}_X) \in \mathrm{qcoh}(X, \pi_0(\mathcal{O}_X))$ for all $i > 0$. We call $(X, \mathcal{O}_X)$ an *affine derived* $\mathbb{K}$*-scheme* if also $(X, \pi_0(\mathcal{O}_X))$ is an affine $\mathbb{K}$-scheme.
Write $\mathbf{DAff}_{\mathbb{K}} \subset \mathbf{DSch}_{\mathbb{K}}$ for the ∞-categories of derived (affine) $\mathbb{K}$-schemes.

There is a global sections ∞-functor $\Gamma : \mathbf{DAff}_{\mathbb{K}}^{\mathrm{op}} \to \mathbf{SAlg}_{\mathbb{K}}$ mapping $(X, \mathcal{O}_X) \mapsto \pi_*(\mathcal{O}_X)$, where $\pi : X \to *$ is the projection. It is an equivalence of ∞-categories, with a homotopy inverse $\mathrm{Spec} : \mathbf{SAlg}_{\mathbb{K}} \to \mathbf{DAff}_{\mathbb{K}}^{\mathrm{op}}$. Thus, we can alternatively define an affine derived $\mathbb{K}$-scheme as $\mathrm{Spec}\, A^\bullet$ for $A^\bullet$ a simplicial commutative $\mathbb{K}$-algebra, and a derived $\mathbb{K}$-scheme to be a derived ringed $\mathbb{K}$-space $(X, \mathcal{O}_X)$ which is locally equivalent to derived affine $\mathbb{K}$-schemes.

There is an inclusion $i : \mathbf{Alg}_{\mathbb{K}} \hookrightarrow \mathbf{SAlg}_{\mathbb{K}}$ which is right adjoint to $\pi_0 : \mathbf{SAlg}_{\mathbb{K}} \to \mathbf{Alg}_{\mathbb{K}}$. These extend to adjunctions

$$\mathbf{Aff}_{\mathbb{K}} \underset{t_0}{\overset{i}{\rightleftarrows}} \mathbf{DAff}_{\mathbb{K}}, \qquad \mathbf{Sch}_{\mathbb{K}} \underset{t_0}{\overset{i}{\rightleftarrows}} \mathbf{DSch}_{\mathbb{K}}.$$

That is, ordinary (affine) $\mathbb{K}$-schemes X include into derived (affine) $\mathbb{K}$-schemes $\boldsymbol{X} = i(X)$, and each derived (affine) $\mathbb{K}$-scheme $\boldsymbol{X} = (X, \mathcal{O}_X)$ has a *classical truncation* $t_0(\boldsymbol{X}) = (X, \pi_0(\mathcal{O}_X))$.

Derived stacks

Definition

In §11.3 we defined the ∞-category $\mathbf{HSta}_{\mathbb{K}}$ of *higher* $\mathbb{K}$*-stacks* to be the full ∞-subcategory of F in $\mathrm{Fun}_\infty(\mathbf{Alg}_{\mathbb{K}}, \mathbf{SSets})$ satisfying the homotopy sheaf condition for all hypercovers $U^\bullet \to U$, in some choice of Grothendieck topology (use fppf for the Artin case below).
Similarly, we define the ∞-category $\mathbf{DSta}_{\mathbb{K}}$ of *derived* $\mathbb{K}$*-stacks* to be the full ∞-subcategory of F in $\mathrm{Fun}_\infty(\mathbf{SAlg}_{\mathbb{K}}, \mathbf{SSets})$ satisfying the homotopy sheaf condition for all hypercovers $U^\bullet \to U$, in some choice of Grothendieck topology.
The Yoneda embedding gives an ∞-functor $Y : \mathbf{SAlg}_{\mathbb{K}}^{\mathrm{op}} \to \mathbf{DSta}_{\mathbb{K}}$.
We think of objects in the essential image of Y as derived affine $\mathbb{K}$-schemes, considered as derived stacks.
We then define *derived n-Artin stacks* for $n \geqslant -1$ by induction, exactly as for higher stacks, but starting with derived -1-Artin stacks being derived affine schemes. A derived stack $\boldsymbol{X}$ which is locally a derived n-Artin stack for some n is a *derived Artin stack*.

If $\boldsymbol{X}$ is a derived Artin stack then $t_0(\boldsymbol{X})$ is a higher Artin stack. (So 'derived' includes 'higher' for derived stacks.) We call $\boldsymbol{X}$ a *derived Artin n-stack* if $t_0(\boldsymbol{X})$ is a higher Artin n-stack. If $\boldsymbol{X}$ is a derived Artin 1-stack then $t_0(\boldsymbol{X})$ is an ordinary Artin stack. Write $\mathbf{DArt}_{\mathbb{K}} \subset \mathbf{DSta}_{\mathbb{K}}$ for the full ∞-subcategory of derived Artin stacks, and $\mathbf{DArt}_{\mathbb{K}}^n \subset \mathbf{DArt}_{\mathbb{K}}$ for the full ∞-subcategory of derived Artin n-stacks. Then

$$\mathbf{DSch}_{\mathbb{K}} \subset \mathbf{DArt}_{\mathbb{K}}^0 \subset \mathbf{DArt}_{\mathbb{K}}^1 \subset \cdots \subset \mathbf{DArt}_{\mathbb{K}}^n \subset \mathbf{DArt}_{\mathbb{K}}.$$

We have adjunctions

$$\mathbf{HArt}_{\mathbb{K}} \underset{t_0}{\overset{i}{\rightleftarrows}} \mathbf{DArt}_{\mathbb{K}}, \qquad \mathbf{HArt}_{\mathbb{K}}^n \underset{t_0}{\overset{i}{\rightleftarrows}} \mathbf{DArt}_{\mathbb{K}}^n.$$

The (co)units of the adjunctions give morphisms

$$t_0 \circ i(X) \xrightarrow{\cong} X, \qquad i \circ t_0(\boldsymbol{X}) \overset{\iota}{\hookrightarrow} \boldsymbol{X}.$$

If $\boldsymbol{X} \in \mathbf{DArt}_{\mathbb{K}}$ with classical truncation $X = t_0(\boldsymbol{X}) = i \circ t_0(\boldsymbol{X})$ there is an inclusion morphism $\iota : X \hookrightarrow \boldsymbol{X}$. Think of $\boldsymbol{X}$ as an *infinitesimal formal thickening* of X, like a scheme X is a formal thickening of its reduced subscheme $X^{\mathrm{red}} \hookrightarrow X$.

Locally finitely presented derived schemes and stacks

There is a notion of when a simplicial commutative $\mathbb{K}$-algebra $A^\bullet$ is *finitely presented*. Roughly, it means there are only finitely many generators and (higher) relations. When $\mathrm{char}\,\mathbb{K}=0$, the parallel notion for cdgas $A^\bullet=(A^*,\mathrm{d})$ is that A^* is a free graded polynomial $\mathbb{K}$-algebra with finitely many generators, all in degrees $\leqslant 0$.

A derived $\mathbb{K}$-scheme or $\mathbb{K}$-stack $\boldsymbol{X}$ is *locally finitely presented* if it is locally modelled on $\mathrm{Spec}\,A^\bullet$ for finitely presented $A^\bullet$. Locally finitely presented $\boldsymbol{X}$ are particularly nice. They have perfect (co)tangent complexes $\mathbb{L}_{\boldsymbol{X}},\mathbb{T}_{\boldsymbol{X}}$ (later).

If X is a smooth projective $\mathbb{K}$-scheme then the derived moduli stack $\boldsymbol{\mathcal{M}}$ of objects in $D^b\,\mathrm{coh}(X)$ is a locally finitely presented derived Artin stack (Toën–Vaquié).

Note that if X is a singular scheme or stack then $i(X)$ is generally not locally finitely presented, so $i(X)$ is not 'nice' as a derived stack. Often it is better to consider a derived version $\boldsymbol{X}$ of X with $\boldsymbol{X}\neq i(X)$.

12.2. Cotangent complexes of derived stacks

A smooth n-manifold X has a *cotangent bundle* T^*X, a rank n vector bundle $T^*X \to X$. Smooth maps $g : X \to Z$, $h : Y \to Z$ are called *transverse* if for all $x \in X$, $y \in Y$ with $g(x) = h(y) = z \in Z$, the linear map $D_xg \oplus D_yh : T_z^*Z \to T_x^*X \oplus T_y^*Y$ is injective. If g, h are transverse, a fibre product $W = X \times_{g,Z,h} Y$ exists in **Man**, with $\dim W = \dim X + \dim Y - \dim Z$, in a Cartesian square

$$\begin{array}{ccc} W & \xrightarrow{f} & Y \\ \downarrow e & & h \downarrow \\ X & \xrightarrow{g} & Z. \end{array} \tag{12.1}$$

There is an exact sequence of cotangent bundles

$$0 \to (g\circ e)^*(T^*Z) \xrightarrow{e^*(Dg)\oplus f^*(Dh)} \begin{array}{c} e^*(T^*X) \\ \oplus f^*(T^*Y) \end{array} \xrightarrow{De\oplus -Df} T^*W \to 0. \tag{12.2}$$

We would like to generalize all this to (derived) schemes and stacks.

If X is a classical $\mathbb{K}$-scheme, there is a notion of *cotangent sheaf* Ω_X, which lies in $\mathrm{coh}(X)$ if X is locally of finite type, and is a vector bundle if X is smooth. But given a Cartesian square (12.1) in $\mathbf{Sch}_{\mathbb{K}}$, the analogue of (12.2) for cotangent sheaves Ω_X is only left exact unless (12.1) satisfies strong transversality conditions.

There is a theory of (co)tangent complexes for derived (and classical) schemes and stacks. Here are some important features:

(i) When going from classical to derived we replace bundles and sheaves by complexes. So the *cotangent complex* $\mathbb{L}_{\boldsymbol{X}}$ of a derived stack $\boldsymbol{X}$ is a complex, in a derived category of quasicoherent sheaves.

(ii) For each $\boldsymbol{X} = (X, \mathcal{O}_X)$ there is a triangulated category $L_{\mathrm{qcoh}}(\boldsymbol{X})$ of sheaves of $\mathcal{O}_X$-modules with quasicoherent cohomology. It has a t-structure with heart $\mathrm{qcoh}(\boldsymbol{X}) \cong \mathrm{qcoh}(t_0(\boldsymbol{X}))$. But if $\boldsymbol{X} \not\cong i(X)$ then $L_{\mathrm{qcoh}}(\boldsymbol{X}) \not\cong D(\mathrm{qcoh}(X))$ in general.

(iii) Morphisms $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ give exact pullback functors $\boldsymbol{f}^* : L_{\mathrm{qcoh}}(\boldsymbol{Y}) \to L_{\mathrm{qcoh}}(\boldsymbol{X})$.

(iv) Under very mild conditions, each derived stack $\boldsymbol{X}$ has a *cotangent complex* $\mathbb{L}_{\boldsymbol{X}}$ in $L_{\rm qcoh}(\boldsymbol{X})$. Each morphism $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ induces a morphism $\mathbb{L}_{\boldsymbol{f}}:\boldsymbol{f}^*(\mathbb{L}_{\boldsymbol{Y}})\to\mathbb{L}_{\boldsymbol{X}}$ in $L_{\rm qcoh}(\boldsymbol{X})$.
(v) If $\boldsymbol{X}$ is a derived $\mathbb{K}$-scheme with classical truncation $X=t_0(\boldsymbol{X})$ then $\mathbb{L}_{\boldsymbol{X}}$ has cohomology in degrees $(-\infty,0]$ in the t-structure on $L_{\rm qcoh}(\boldsymbol{X})$, and $H^0(\mathbb{L}_{\boldsymbol{X}})\cong\Omega_X$ is the classical cotangent sheaf.
(vi) If $\boldsymbol{X}$ is *locally finitely presented* then $\mathbb{L}_{\boldsymbol{X}}$ is a *perfect complex* (locally modelled on a bounded complex of vector bundles $\mathcal{E}^\bullet=(0\to E^a\to E^{a+1}\to\cdots\to E^b\to 0)$ on $\boldsymbol{X}$).
A perfect complex $\mathcal{E}^\bullet$ has a *rank* $\mathrm{rank}\,\mathcal{E}^\bullet=\sum_k(-1)^k\,\mathrm{rank}\,E^k$ in $\mathbb{Z}$, locally constant on $\boldsymbol{X}$. We define the *virtual dimension* $\mathrm{vdim}\,\boldsymbol{X}=\mathrm{rank}\,\mathbb{L}_{\boldsymbol{X}}$.
Although duality $\mathbb{D}_{\boldsymbol{X}}=\mathcal{H}om(-,\mathcal{O}_{\boldsymbol{X}})$ may not be well behaved on $L_{\rm qcoh}(\boldsymbol{X})$ (does not square to the identity), it is well behaved on perfect complexes. So in this case we define the *tangent complex* $\mathbb{T}_{\boldsymbol{X}}=\mathbb{L}_{\boldsymbol{X}}^\vee$, and $\mathbb{L}_{\boldsymbol{X}}\cong\mathbb{T}_{\boldsymbol{X}}^\vee$.

(vii) Let X be a smooth projective $\mathbb{K}$-scheme. Then there exists a derived moduli stack $\boldsymbol{\mathcal{M}}$ of objects $E^\bullet$ in $D^b\operatorname{coh}(X)$, which is a locally finitely presented derived Artin $\mathbb{K}$-stack. At each point $[E^\bullet]\in\boldsymbol{\mathcal{M}}$ the (co)tangent complexes satisfy $H^k(\mathbb{T}_{\boldsymbol{\mathcal{M}}}|_{[E^\bullet]})\cong \operatorname{Ext}^{k-1}(E^\bullet,E^\bullet)$, $H^k(\mathbb{L}_{\boldsymbol{\mathcal{M}}}|_{[E^\bullet]})\cong \operatorname{Ext}^{1-k}(E^\bullet,E^\bullet)^*$. Thus the (co)tangent complexes $\mathbb{T}_{\boldsymbol{\mathcal{M}}},\mathbb{L}_{\boldsymbol{\mathcal{M}}}$ know about the full deformation theory $\operatorname{Ext}^*(E^\bullet,E^\bullet)$ of $\boldsymbol{\mathcal{M}}$ near $E^\bullet$.

(viii) Suppose we are given a homotopy Cartesian square in $\mathbf{DArt}_{\mathbb{K}}$:

$$\begin{array}{ccc} \boldsymbol{W} & \xrightarrow[\quad f \quad]{} & \boldsymbol{Y} \\ \Big\downarrow \boldsymbol{e} & \eta \Uparrow & \boldsymbol{h} \Big\downarrow \\ \boldsymbol{X} & \xrightarrow{\quad \boldsymbol{g} \quad} & \boldsymbol{Z}. \end{array} \tag{12.3}$$

Then there is a distinguished triangle of cotangent complexes

$$(\boldsymbol{g}\circ\boldsymbol{e})^*(\mathbb{L}_{\boldsymbol{Z}}) \xrightarrow{\boldsymbol{e}^*(\mathbb{L}_{\boldsymbol{g}})\oplus\boldsymbol{f}^*(\mathbb{L}_{\boldsymbol{h}})} \boldsymbol{e}^*(\mathbb{L}_{\boldsymbol{X}})\oplus\boldsymbol{f}^*(\mathbb{L}_{\boldsymbol{Y}}) \xrightarrow{\mathbb{L}_{\boldsymbol{e}}\oplus-\mathbb{L}_{\boldsymbol{f}}} \mathbb{L}_{\boldsymbol{W}} \xrightarrow{[+1]}. \tag{12.4}$$

It is an important and remarkable fact that this holds *without any transversality conditions on* $\boldsymbol{g},\boldsymbol{h}$. It is evidence that we have found the 'right' definition of derived schemes and stacks.

In fact cotangent complexes $\mathbb{L}_X^{\mathrm{Ill}}$ for classical schemes X were defined by Illusie in the '70s, and are compatible with the derived version by $\mathbb{L}_X^{\mathrm{Ill}} \cong \mathbb{L}_{i(X)}$, considering $i(X)$ as a derived scheme. However, Illusie's cotangent complexes do *not* satisfy the analogues of (vi)–(viii), they are not as well behaved.

For (viii), note that although $t_0 : \mathbf{DSch}_{\mathbb{K}} \to \mathbf{Sch}_{\mathbb{K}}$ preserves (homotopy) fibre products (i.e. $t_0(\boldsymbol{X} \times_{\boldsymbol{Z}} \boldsymbol{Y}) \cong t_0(\boldsymbol{X}) \times_{t_0(\boldsymbol{Z})} t_0(\boldsymbol{Y})$), $i : \mathbf{Sch}_{\mathbb{K}} \to \mathbf{DSch}_{\mathbb{K}}$ does not. Thus, given morphisms $g : X \to Z$, $h : Y \to Z$ of classical $\mathbb{K}$-schemes, they have fibre products $W = X \times_{h,Z,h} Y$ in $\mathbf{Sch}_{\mathbb{K}}$ and $\boldsymbol{W} = X \times_{h,Z,h} Y$ in $\mathbf{DSch}_{\mathbb{K}}$ with $W \cong t_0(\boldsymbol{W})$, but $\boldsymbol{W} \not\cong i(W)$ in general. The distinguished triangle (12.4) computes $\mathbb{L}_{\boldsymbol{W}}$, not $\mathbb{L}_W$.

General Principle

Results for classical schemes/stacks which hold under transversality/smoothness/flatness assumptions often hold for derived schemes/stacks without such assumptions.

12.3. Examples of derived schemes and stacks

Morally, *smooth* $\mathbb{K}$-schemes (and stacks) are the same in the classical and derived worlds, but singular derived schemes have more complicated structure at their singularities than singular classical schemes. 'Nice' derived schemes (e.g. locally finitely presented) may be built from smooth schemes by repeated fibre products. So my favourite example of a derived scheme $\boldsymbol{W}$ is a fibre product $\boldsymbol{W}=X\times_Z Y$ of smooth classical schemes X,Y,Z. For instance, X,Y could be smooth subschemes of a smooth scheme Z which intersect non-transversely, and $\boldsymbol{W}=X\cap Y$ could be their derived intersection. Equation (12.4) implies that $\boldsymbol{W}$ has cotangent complex perfect in degrees $[-1,0]$:

$$\mathbb{L}_{\boldsymbol{W}}\simeq\bigl[\underset{-1}{T^*Z}|_{\boldsymbol{W}}\xrightarrow{Dg\oplus Dh}\underset{0}{T^*X|_{\boldsymbol{W}}\oplus T^*Y|_{\boldsymbol{W}}}\bigr].$$

A derived scheme $\boldsymbol{W}$ with $\mathbb{L}_{\boldsymbol{W}}$ perfect in $[-1,0]$ is called *quasi-smooth*. (If $\boldsymbol{W}$ is smooth then $\mathbb{L}_{\boldsymbol{W}}$ is perfect in $[0,0]$.)

More generally, if V is a smooth $\mathbb{K}$-scheme, $E \to V$ is a vector bundle, and $s : V \to E$ a section then the derived zero locus $\boldsymbol{X} = s^{-1}(0)$ is quasi-smooth with cotangent complex

$$\mathbb{L}_{\boldsymbol{X}} \simeq \big[\underset{-1}{E^*|_{\boldsymbol{X}}} \xrightarrow{\;\;Ds\;\;} \underset{0}{T^*V|_{\boldsymbol{X}}}\big].$$

When $\mathrm{char}\,\mathbb{K} = 0$ we can write affine derived $\mathbb{K}$-schemes as $\mathrm{Spec}\,A^\bullet$ for $A^\bullet$ a cdga over $\mathbb{K}$ in degrees $\leqslant 0$. Take $A^\bullet = (A^*, \mathrm{d})$ to be *finitely presented*. That is, as a graded $\mathbb{K}$-algebra we have $A^* = \mathbb{K}[x^i_j : (i,j) \in I]$ the polynomial superalgebra generated by finitely many graded variables x^i_j for (i,j) in a finite indexing set I, with $\deg x^i_j = i \leqslant 0$, where x^i_j is an even (odd) variable if i is even (odd). Then $\mathrm{d} : A^* \to A^{*+1}$ is generated by choices of $\mathrm{d}(x^i_j)$ with $\deg(\mathrm{d}(x^i_j)) = i + 1$, and must satisfy $\mathrm{d}^2 = 0$.

For example, take $A^* = \mathbb{K}[x_1, \ldots, x_l, y_1, \ldots, y_m, z_1, \ldots, z_n]$ with $\deg x_i = -2$, $\deg y_j = -1$, $\deg z_k = 0$. We must have $\mathrm{d}z_k = 0$ as $\mathrm{d}z_k \in A^1 = 0$. Also $\mathrm{d}y_j \in A^0 = \mathbb{K}[z_1, \ldots, z_n]$, so $\mathrm{d}y_j = f_j(z_1, \ldots, z_n)$ for polynomials $f_1, \ldots, f_m$. And $\mathrm{d}x_i \in A^{-1} = \mathbb{K}[z_1, \ldots, z_n]\langle y_1, \ldots, y_m\rangle$, so $\mathrm{d}x_i = \sum_{j=1}^m g_{ij}(z_1, \ldots, z_n)y_j$. For $\mathrm{d}^2 = 0$ we must have

$$\textstyle\sum_{j=1}^m g_{ij}(z_1, \ldots, z_n) f_j(z_1, \ldots, z_n) = 0, \qquad i = 1, \ldots, l. \tag{12.5}$$

We have $H^0(A^\bullet) = \mathbb{K}[z_1, \ldots, z_n]/(f_1 = 0, \ldots, f_m = 0)$. Thus the degree -1 variables $y_1, \ldots, y_m$ correspond to *relations* $f_j = 0$ on the degree 0 variables $z_1, \ldots, z_n$. By (12.5), the degree -2 variables $x_1, \ldots, x_l$ correspond to *relations on the relations*, linear dependencies between the equations $f_j = 0$ for $j = 1, \ldots, m$.

The derived scheme $\boldsymbol{X} = \operatorname{Spec} A^\bullet$ has classical truncation $X = t_0(\boldsymbol{X})$ the subscheme $\{(z_1, \ldots, z_n) \in \mathbb{A}^n : f_j(z_1, \ldots, z_n) = 0, j = 1, \ldots, m\}$, independent of the x_i. The cotangent complex is

$$\mathbb{L}_{\boldsymbol{X}} \simeq \bigl[\underset{-2}{\mathcal{O}_{\boldsymbol{X}}^l} \xrightarrow{(g_{ij})_{i\leqslant l}^{j\leqslant m}} \underset{-1}{\mathcal{O}_{\boldsymbol{X}}^m} \xrightarrow{(\partial f_j/\partial z_k)_{j\leqslant m}^{k\leqslant n}} \underset{0}{\mathcal{O}_{\boldsymbol{X}}^n}\bigr].$$

Thus, a derived scheme remembers not just classical (degree 0) variables z_i, but also relations on the z_i (in degree -1), relations on the relations (in degree -2), relations on the relations on the relations (in degree -3), and so on.

Many moduli problems in Algebraic Geometry have a moduli stack $\mathcal{M}$, as an Artin (or higher) stack, obtained by defining a moduli functor $F : \mathbf{Alg}_{\mathbb{K}} \to \mathbf{Groupoids}$ (or $\mathbf{SSets}$), where $F(A)$ is the (∞-)groupoid of families of objects in the moduli problem over the base $\mathbb{K}$-scheme $U = \operatorname{Spec} A$, and proving F is an Artin stack.

Similarly, many moduli problems have a *derived moduli stack* $\boldsymbol{\mathcal{M}}$ with $\mathcal{M} = t_0(\boldsymbol{\mathcal{M}})$, obtained by defining a moduli ∞-functor $F : \mathbf{SAlg}_{\mathbb{K}} \to \mathbf{SSets}$, where $F(A^\bullet)$ is the ∞-groupoid of families of objects in the moduli problem over the base derived $\mathbb{K}$-scheme $\boldsymbol{U} = \operatorname{Spec} A^\bullet$. Then $\boldsymbol{\mathcal{M}}$ is an enhancement of $\mathcal{M}$ containing more information, and $\mathbb{L}_{\boldsymbol{\mathcal{M}}}$ encodes the deformation theory of objects in the moduli problem better than $\mathbb{L}_{\mathcal{M}}$. Results of Toën–Vaquié prove the existence of a derived moduli stack $\boldsymbol{\mathcal{M}}$ of objects in many dg-categories $\boldsymbol{\mathcal{T}}$, such as that with $\operatorname{Ho}(\boldsymbol{\mathcal{T}}) = D^b \operatorname{coh}(X)$.