Derived Algebraic Geometry

Lecture 13 of 14: Obstruction theories, virtual classes, and enumerative invariants

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Helpful reference for this lecture:
L. Battistella, F. Carocci and C. Manolache, *'Virtual classes for the working mathematician'*, arXiv:1804.06048.

These slides available at
`http://people.maths.ox.ac.uk/~joyce/`

Plan of talk:

13.1. Enumerative invariants in Algebraic Geometry
Introduction: Bézout's Theorem

Let C, D be curves in $\mathbb{CP}^2$ of degrees m, n. If C, D intersect transversely then $|C \cap D| = mn$. If C, D intersect non-transversely, but still in finitely many points, then considering $C \cap D = C \times_{\mathbb{CP}^2} D$ as a non-reduced scheme of dimension 0, then Bézout's Theorem says that length$(C \cap D) = mn$. However, if $C \cap D$ has dimension 1 then you cannot recover mn from $C \cap D$.

We can think of this in terms of *homology*: we have $[C] = m[\mathbb{CP}^1]$, $[D] = n[\mathbb{CP}^1]$ in $H_2(\mathbb{CP}^2, \mathbb{Z})$, and $[C] \bullet [D] = mn$ in $H_0(\mathbb{CP}^2, \mathbb{Z}) \cong \mathbb{Z}$, where $\bullet$ is the intersection product on $H_*(\mathbb{CP}^2, \mathbb{Z})$. So Bézout's Theorem concerns recovering the homology class $[C] \bullet [D] \in H_0(\mathbb{CP}^2, \mathbb{Z})$ from the scheme $C \cap D = C \times_{\mathbb{CP}^2} D$.

Suppose we take $\boldsymbol{X} = C \times_{\mathbb{CP}^2} D$ to be the fibre product in *derived* schemes $\mathbf{DSch}_{\mathbb{K}}$. Then using the derived scheme structure we can define a *virtual class* $[\boldsymbol{X}]_{\mathrm{virt}}$ in $H_0(\mathbb{CP}^2, \mathbb{Z})$ with $[\boldsymbol{X}]_{\mathrm{virt}} = mn$. This works even when $C \cap D$ has dimension 1.

Theorem (Virtual classes)

Suppose $\boldsymbol{X}$ *is a derived* $\mathbb{K}$*-scheme which is* **quasi-smooth** *(i.e.* $\mathbb{L}_{\boldsymbol{X}}$ *is perfect in the interval* $[-1,0]$*), and* **proper** *(i.e. the classical truncation* $X = t_0(\boldsymbol{X})$ *is proper; over* $\mathbb{C}$*, this means that the complex analytic topological space* $X^{\rm an}$ *is compact and Hausdorff). Then we can define a* **virtual class** $[\boldsymbol{X}]_{\rm virt}$ *in Chow homology* $A_{\mathrm{vdim}\,\boldsymbol{X}}(X,\mathbb{Z})$*, where the* **virtual dimension** *is* $\mathrm{vdim}\,\boldsymbol{X} = \mathrm{rank}\,\mathbb{L}_{\boldsymbol{X}}$ *in* $\mathbb{Z}$*. If* $\mathbb{K}=\mathbb{C}$ *we can take* $[\boldsymbol{X}]_{\rm virt} \in H_{2\,\mathrm{vdim}\,\boldsymbol{X}}(X^{\rm an},\mathbb{Z})$*.*

Virtual classes satisfy a number of properties, including **deformation invariance***: they are unchanged under continuous deformations of* $\boldsymbol{X}$*.*

If instead $\boldsymbol{X}$ *is a* **derived Deligne–Mumford** $\mathbb{K}$**-stack** *the above holds in* $\mathbb{Q}$*-homology* $A_*(X,\mathbb{Q})$*,* $H_*(X^{\rm an},\mathbb{Q})$*.*

We can also drop the condition that X *is proper, but then* $[\boldsymbol{X}]_{\rm virt}$ *lies in* **Borel–Moore homology** $H_*^{\rm BM}(X^{\rm an},\mathbb{Z})$*.*

As Chow homology is already of Borel–Moore type, properness is often omitted in theorems, but you usually need it in applications.

Actually, virtual classes $[\boldsymbol{X}]_{\rm virt}$ were invented circa 1994-6, about 10 years before (quasi-smooth) derived schemes. The standard reference in Algebraic Geometry is K. Behrend and B. Fantechi, *'The intrinsic normal cone'*, alg-geom/9601010.

Behrend and Fantechi define *obstruction theories* on a $\mathbb{K}$-scheme or Deligne–Mumford $\mathbb{K}$-stack X, and use them to define virtual classes.

Definition

Let X be a (classical) $\mathbb{K}$-scheme or Deligne–Mumford $\mathbb{K}$-stack, quasi-separated and locally of finite type. An *obstruction theory* $(\mathcal{E}^\bullet,\phi)$ on X is a perfect complex $\mathcal{E}^\bullet$ on X perfect in $[-1,0]$ and a morphism $\phi:\mathcal{E}^\bullet\rightarrow\mathbb{L}_X$ in $D_{\rm qcoh}(X)$, where $\mathbb{L}_X$ is the (Illusie) cotangent complex of X, such that $h^i(\phi):h^i(\mathcal{E}^\bullet)\rightarrow h^i(\mathbb{L}_X)$ is an isomorphism in $\mathrm{coh}(X)$ for $i=0$ and surjective for $i=-1$.
The *virtual dimension* of $(\mathcal{E}^\bullet,\phi)$ is $\mathrm{vdim}(\mathcal{E}^\bullet,\phi)=\mathrm{rank}\,\mathcal{E}^\bullet$ in $\mathbb{Z}$.

Given $X,(\mathcal{E}^\bullet,\phi)$, Behrend–Fantechi define $[X]_{\rm virt}\in A_{\mathrm{vdim}(\mathcal{E}^\bullet,\phi)}(X)$. The (confusing) definition involves intersection theory in Artin stacks.

Obstruction theories and quasi-smooth derived schemes

What do obstruction theories have to do with Derived Algebraic Geometry? Suppose $\boldsymbol{X}$ is a quasi-smooth derived scheme with classical truncation $X = t_0(\boldsymbol{X})$. Then there is an inclusion $\iota : X \hookrightarrow \boldsymbol{X}$. Properties of cotangent complexes give a morphism

$$\mathbb{L}_\iota : \iota^*(\mathbb{L}_{\boldsymbol{X}}) \longrightarrow \mathbb{L}_X. \tag{13.1}$$

As $\boldsymbol{X}$ is quasi-smooth, $\mathbb{L}_{\boldsymbol{X}}$ is perfect in $[-1,0]$ (that is, $\mathbb{L}_{\boldsymbol{X}}$ is locally modelled on a two-term complex $0 \to E^{-1} \to E^0 \to 0$ with E^{-1}, E^0 vector bundles), so $\iota^*(\mathbb{L}_{\boldsymbol{X}})$ is perfect in $[-1,0]$. Properties of classical truncation (that $\mathbb{L}_{X/\boldsymbol{X}}$ is concentrated in $(-\infty,-2]$) imply that $h^i(\mathbb{L}_\iota)$ is an isomorphism for $i \geqslant 0$ and surjective for $i=-1$. Therefore (13.1) is an obstruction theory.

So, the classical truncation X of any quasi-smooth derived scheme or D–M stack $\boldsymbol{X}$ has a natural obstruction theory – a semi-classical shadow of the derived $\boldsymbol{X}$, which knows about the virtual class.

Because Algebraic Geometers have got used to working with obstruction theories, and are frightened of DAG, mostly they still work with obstruction theories on classical schemes and D–M stacks, rather than in the derived world. So, they might form a classical moduli scheme $\mathcal{M}$, and then construct an obstruction theory $(\mathcal{E}^\bullet,\phi)$ on $\mathcal{M}$ by hand, using derived category techniques. However, almost invariably, what is secretly going on is that there is also a natural derived moduli scheme $\boldsymbol{\mathcal{M}}$, and for some geometric reason $\boldsymbol{\mathcal{M}}$ is quasi-smooth (often as $\mathrm{Ext}^{>2}(E,E)=0$ for $[E]\in\boldsymbol{\mathcal{M}}$), and $(\mathcal{E}^\bullet,\phi)=(\iota^*(\mathbb{L}_{\boldsymbol{\mathcal{M}}}),\mathbb{L}_\iota)$.

There are sometimes advantages to working with derived schemes instead of obstruction theories. For example, constructing obstruction theories on a fibre product $X\times_Z Y$ from obstruction theories on X,Y,Z is a problem because of non-functoriality. But forming the corresponding derived fibre product $\boldsymbol{X}\times_{\boldsymbol{Z}}\boldsymbol{Y}$ and checking it is quasi-smooth may be easy.

How to think about virtual classes

Our favourite example of a quasi-smooth derived $\mathbb{C}$-scheme $\boldsymbol{X}$ is a *derived zero locus*: let V be a smooth $\mathbb{C}$-scheme, and $E \to V$ a vector bundle, and $s : V \to E$ a section, and take $\boldsymbol{X} = s^{-1}(0)$, which we can think of as the (homotopy) fibre product $V \times_{0,E,s} V$ in $\mathbf{DSch}_{\mathbb{K}}$ of smooth $\mathbb{K}$-schemes V, V, E. The cotangent complex is

$$\mathbb{L}_{\boldsymbol{X}} \simeq \bigl[\underset{-1}{E^*|_{\boldsymbol{X}}} \xrightarrow{Ds} \underset{0}{T^*V|_{\boldsymbol{X}}}\bigr],$$

so $\boldsymbol{X}$ is quasi-smooth with $\mathrm{vdim}\,\boldsymbol{X} = \dim V - \mathrm{rank}\,E$. Let $\boldsymbol{X}$ be proper. If we could perturb s a little bit to $\tilde{s} : V \to E$ which was transverse (i.e. $0(V)$ and $\tilde{s}(V)$ intersect transversely in E) then $\tilde{X} = \tilde{s}^{-1}(0)$ would be proper and smooth of dimension $\mathrm{vdim}\,\boldsymbol{X}$, that is, a compact complex manifold, which has a fundamental class $[\tilde{X}]_{\mathrm{fund}}$ in $H_{2\,\mathrm{vdim}\,\boldsymbol{X}}(V, \mathbb{Z})$. If also V is proper then $[\tilde{X}]_{\mathrm{fund}} = [V] \cap c_{\mathrm{rank}\,E}(E)$. Although transverse perturbations $\tilde{s}$ may not exist, the formula $[\boldsymbol{X}]_{\mathrm{virt}} = [V] \cap c_{\mathrm{rank}\,E}(E)$ makes sense in $H_{2\,\mathrm{vdim}\,\boldsymbol{X}}(V, \mathbb{Z})$. In fact we can define it in $H_{2\,\mathrm{vdim}\,\boldsymbol{X}}(X, \mathbb{Z})$ without assuming V proper.

Thus, if a quasi-smooth derived $\mathbb{C}$-scheme $\boldsymbol{X}$ may be written as a derived zero locus $s^{-1}(0)$ of a section $s : V \to E$ of a vector bundle $E \to V$ over a smooth $\mathbb{C}$-scheme V, we can define the virtual class by $[\boldsymbol{X}]_{\rm virt} = [V] \cap c_{\mathrm{rank}\, E}(E)$.

In fact every quasi-smooth derived scheme $\boldsymbol{X}$ is (Zariski) locally of the form $s^{-1}(0)$, but we don't know how to build a global presentation of $\boldsymbol{X}$ as a derived zero locus. So the subtle thing about constructing virtual classes – which Behrend–Fantechi solved with their notion of 'intrinsic normal cone' – is to define a virtual class using only local presentations of $\boldsymbol{X}$.

There is a differential-geometric analogue of quasi-smooth derived Deligne–Mumford stacks: the *Kuranishi spaces* of Fukaya–Oh–Ohta–Ono. See my D. Joyce, *'Kuranishi spaces as a 2-category'*, arXiv:1510.07444 for a modern definition. Compact oriented Kuranishi spaces have virtual classes, which are used in symplectic geometry, as moduli spaces of J-holomorphic curves are Kuranishi spaces.

13.2. Examples of enumerative invariant theories
Maps from curves and Gromov–Witten theory

We discuss theories in which one can construct quasi-smooth moduli schemes and stacks, and so define enumerative invariants.

Example (Moduli spaces of maps.)

Let X, Y be smooth $\mathbb{K}$-schemes with X projective. Then one can form a derived moduli scheme $\mathbf{Map}(X, Y)$ of morphisms $f : X \to Y$. The Zariski tangent space $T_f\mathbf{Map}(X, Y)$ is the sheaf cohomology group $H^0(f^*(TY))$, and the cotangent complex $\mathbb{L}_{\mathbf{Map}(X,Y)}$ has $H^i\bigl(\mathbb{L}_{\mathbf{Map}(X,Y)}|_f\bigr) \cong H^{-i}(f^*(TY))^*$. Hence if $\dim X = m$ then $\mathbb{L}_{\mathbf{Map}(X,Y)}$ is perfect in $[-m, 0]$ as $H^i(f^*(TY)) = 0$ for $i > m$, so if $m = 1$, that is, X is a curve, then $\mathbf{Map}(X, Y)$ is quasi-smooth.

The *Gromov–Witten invariants* of a smooth $\mathbb{K}$-scheme Y 'count' quasi-smooth derived moduli stacks $\bar{\boldsymbol{\mathcal{M}}}_{g,k}(\alpha)$ with points $(\Sigma, f, \boldsymbol{p})$, where Σ is a genus g projective curve (varying, not fixed), $f : \Sigma \to Y$ a morphism with $f_*([\Sigma]) = \alpha \in H_2(Y, \mathbb{Z})$, and $\boldsymbol{p} = (p_1, \ldots, p_k)$ with $p_1, \ldots, p_k$ distinct points in Σ. To ensure $\bar{\boldsymbol{\mathcal{M}}}_{g,k}(\alpha)$ is a proper Deligne–Mumford stack we must allow Σ to have nodal singularities, and require $(\Sigma, f, \boldsymbol{p})$ to satisfy a 'stability condition'. Then there is a virtual class $[\bar{\boldsymbol{\mathcal{M}}}_{g,k}(\alpha)]_{\rm virt}$ in $A_*(\bar{\mathcal{M}}_{g,k}(\alpha), \mathbb{Q})$. We can push this forward along the 'evaluation map' $\bar{\boldsymbol{\mathcal{M}}}_{g,k}(\alpha) \to Y^k$ mapping $(\Sigma, f, \boldsymbol{p}) \mapsto \boldsymbol{p}$ to get $[\bar{\boldsymbol{\mathcal{M}}}_{g,k}(\alpha)]_{\rm virt} \in A_*(Y^k, \mathbb{Q})$, which is basically the Gromov–Witten invariant.

Gromov–Witten invariants are important in String Theory and Mirror Symmetry, and are used to define Quantum Cohomology. A good book is D.A. Cox and S. Katz, *'Mirror Symmetry and Algebraic Geometry'*, A.M.S., 1999.

Invariants counting vector bundles and coherent sheaves

Example (Moduli spaces of coherent sheaves.)

Let X be a smooth projective $\mathbb{K}$-scheme, and $\boldsymbol{\mathcal{M}}$ the derived moduli stack of coherent sheaves on X, a derived Artin 1-stack. At a point $[E]\in\boldsymbol{\mathcal{M}}$ we have $H^i(\mathbb{L}_{\boldsymbol{\mathcal{M}}}|_{[E]})\cong \mathrm{Ext}^{1-i}(E,E)^*$. Thus if $\dim X=m$ then $\mathbb{L}_{\boldsymbol{\mathcal{M}}}$ is perfect in the interval $[1-m,1]$. Hence if $m\leqslant 2$ then $\mathbb{L}_{\boldsymbol{\mathcal{M}}}$ is perfect in the interval $[-1,1]$, that is, $\boldsymbol{\mathcal{M}}$ is quasi-smooth in the sense of derived Artin stacks.

We cannot yet define a virtual class $[\boldsymbol{\mathcal{M}}]_{\rm virt}$, as $\boldsymbol{\mathcal{M}}$ is neither proper, nor a Deligne–Mumford stack. The usual solution is to fix a Chern character α and a 'stability condition' τ on $\mathrm{coh}(X)$ (explained in §13.3), and then we have open substacks $\boldsymbol{\mathcal{M}}^{\rm st}_\alpha(\tau)\subseteq\boldsymbol{\mathcal{M}}^{\rm ss}_\alpha(\tau)\subset\boldsymbol{\mathcal{M}}$ of (semi)stable sheaves in class α. Under good conditions $\boldsymbol{\mathcal{M}}^{\rm st}_\alpha(\tau)$ has a fine moduli scheme $\mathfrak{M}^{\rm st}_\alpha(\tau)$, and $\boldsymbol{\mathcal{M}}^{\rm ss}_\alpha(\tau)$ has a proper coarse moduli scheme $\mathfrak{M}^{\rm ss}_\alpha(\tau)$.

Thus if $\boldsymbol{\mathcal M}_\alpha^{\rm st}(\tau)=\boldsymbol{\mathcal M}_\alpha^{\rm ss}(\tau)$ then $\mathfrak M_\alpha^{\rm st}(\tau)=\mathfrak M_\alpha^{\rm ss}(\tau)$ is a proper fine moduli scheme. As stable sheaves E have $\mathrm{Aut}(E)=\mathbb G_m$, the projection $\mathcal M_\alpha^{\rm st}(\tau)\to\mathfrak M_\alpha^{\rm st}(\tau)$ is a $[*/\mathbb G_m]$-fibration. We can enhance $\mathfrak M_\alpha^{\rm st}(\tau)$ to a derived fine moduli scheme $\boldsymbol{\mathfrak M}_\alpha^{\rm st}(\tau)$ such that $\boldsymbol{\mathcal M}_\alpha^{\rm st}(\tau)\to\boldsymbol{\mathfrak M}_\alpha^{\rm st}(\tau)$ is a $[*/\mathbb G_m]$-fibration, and $\mathbb L_{\boldsymbol{\mathfrak M}_\alpha^{\rm st}(\tau)}$ is perfect in $[1-m,0]$, so $\boldsymbol{\mathfrak M}_\alpha^{\rm st}(\tau)$ is quasi-smooth if $m\leqslant 2$, and has a virtual class $[\boldsymbol{\mathfrak M}_\alpha^{\rm st}(\tau)]_{\rm virt}$. In particular, this can be used to define invariants counting stable=semistable vector bundles and coherent sheaves on projective surfaces X (algebraic Donaldson invariants). Defining invariants when $\boldsymbol{\mathcal M}_\alpha^{\rm st}(\tau)\neq\boldsymbol{\mathcal M}_\alpha^{\rm ss}(\tau)$ is complicated — see for example Mochizuki 2008, Joyce arXiv:2111.04694.

We cannot work with the coarse moduli scheme $\mathfrak M_\alpha^{\rm ss}(\tau)$ directly, as it is not known to have a quasi-smooth derived enhancement $\boldsymbol{\mathfrak M}_\alpha^{\rm ss}(\tau)$.

One can also define invariants counting stable=semistable coherent sheaves on Fano 3-folds by this method, as $\mathrm{Ext}^3(E,E)=0$ for E semistable with $\mathrm{rank}\,E>0$, and therefore $\mathbb L_{\boldsymbol{\mathcal M}}$ is perfect in $[-1,1]$ rather than $[-2,1]$ on $\boldsymbol{\mathcal M}_\alpha^{\rm ss}(\tau)$, so $\boldsymbol{\mathcal M}_\alpha^{\rm ss}(\tau)$ is quasi-smooth.

Virtual classes without quasi-smoothness

There are two cases when we can define virtual classes for derived moduli stacks $\boldsymbol{\mathcal{M}}$ which are not quasi-smooth: moduli stacks of coherent sheaves on Calabi–Yau 3- and 4-folds. Then, as in §14, $\boldsymbol{\mathcal{M}}$ has an additional structure, a *k-shifted symplectic structure* ω for $k=-1,-2$, and we can define virtual classes $[\boldsymbol{\mathcal{M}}]_{\rm virt}$ using ω by a different method. (The Calabi–Yau 3-fold case can also be reduced to Behrend–Fantechi virtual classes, but this is often not the best point of view.)

Invariants counting (semi)stable coherent sheaves on Calabi–Yau 3-folds are called *Donaldson–Thomas invariants*, see Thomas math.AG/9806111, Joyce–Song arXiv:0810.5645, Kontsevich–Soibelman arXiv:0811.2435. They satisfy *wall-crossing formulae* under change of stability condition.

Invariants counting (semi)stable coherent sheaves on Calabi–Yau 4-folds are still being developed — see Borisov–Joyce arXiv:1504.00690 and Oh–Thomas arXiv:2009.05542.

13.3. Stability and semistability, and GIT

For all the enumerative invariants we have discussed, we need to pick a 'stability condition' τ and a topological invariant α, and restrict to moduli stacks $\mathcal{M}_\alpha^{\rm st}(\tau) \subseteq \mathcal{M}_\alpha^{\rm ss}(\tau)$ of τ-(semi)stable objects in class α. There are several reasons for this:

- To form virtual classes we really want to be in the world of schemes (or at worst Deligne–Mumford stacks) rather than Artin stacks. But often no moduli scheme of all objects in the problem exists: you need to restrict to stable objects (for a fine moduli scheme) or semistable objects (for a coarse moduli scheme).
- We also need our moduli spaces to be proper ($\approx$ compact and Hausdorff). Often, coarse moduli schemes of semistable objects are automatically proper. Restricting to stables can kill compactness if stable$\neq$semistable. Allowing unstables can kill Hausdorffness.
- Often, a complex object (e.g. complex vector bundle) may be (poly)stable iff it admits a solution of an interesting p.d.e. (e.g. Hermitian–Einstein equations) – a reason to care about stability.

An example: semistability for vector bundles on curves

Let X be a smooth connected projective curve over $\mathbb{K}$. For each nonzero vector bundle $E \to X$ we have the *rank* $\mathrm{rank}\, E$ and the *degree* $\deg E = \int_X c_1(E)$. Define the *slope* $\mu(E) = \frac{\deg E}{\mathrm{rank}\, E}$. We call E *stable* (or *semistable*) if for all vector subbundles $F \subset E$ with $0 \neq F \neq E$ we have $\mu(F) < \mu(E)$ (or $\mu(F) \leqslant \mu(E)$). We call E *polystable* if $E = F_1 \oplus \cdots \oplus F_k$ with F_i stable and $\mu(F_1) = \cdots = \mu(F_k)$. Polystable implies semistable.

Theorem (Narasimhan–Seshadri 1965, . . .)

For all $r > 0$ and $d \in \mathbb{Z}$ there exist an open inclusion $\mathcal{M}^{\mathrm{st}}_{(r,d)} \subseteq \mathcal{M}^{\mathrm{ss}}_{(r,d)}$, where $\mathcal{M}^{\mathrm{st}}_{(r,d)}$ is a fine moduli scheme for stable vector bundles $E \to X$ and $\mathcal{M}^{\mathrm{ss}}_{(r,d)}$ is a coarse moduli scheme for semistable vector bundles $E \to X$, both of rank r and degree d. Also $\mathcal{M}^{\mathrm{st}}_{(r,d)}$ is smooth of dimension $r^2(g-1)+1$ and $\mathcal{M}^{\mathrm{ss}}_{(r,d)}$ is proper. If r, d are coprime then $\mathcal{M}^{\mathrm{st}}_{(r,d)} = \mathcal{M}^{\mathrm{ss}}_{(r,d)}$, so $\mathcal{M}^{\mathrm{st}}_{(r,d)}$ has a fundamental class $[\mathcal{M}^{\mathrm{st}}_{(r,d)}]_{\mathrm{fund}}$. Points of $\mathcal{M}^{\mathrm{st}}_{(r,d)}$ (or $\mathcal{M}^{\mathrm{ss}}_{(r,d)}$) correspond to isomorphism classes of stable (or polystable) bundles.

This is an example of a stability condition on an abelian category.

Definition

Let $\mathcal{A}$ be an abelian category. A *stability condition* $(\tau, T, \leqslant)$ on $\mathcal{A}$ is a total order $(T, \leqslant)$ and a map $\tau : \{\text{nonzero objects in } \mathcal{A}\} \to T$ such that if $0 \to E \to F \to G \to 0$ is a short exact sequence of nonzero objects in $\mathcal{A}$ then either $\tau(E) < \tau(F) < \tau(G)$, or $\tau(E) = \tau(F) = \tau(G)$, or $\tau(E) > \tau(F) > \tau(G)$.
Given such $(\tau, T, \leqslant)$, a nonzero object E in $\mathcal{A}$ is called *stable* (or *semistable*) if whenever $0 \neq F \subsetneq E$ in $\mathcal{A}$ then $\tau(F) < \tau(E)$ (or $\tau(F) \leqslant \tau(E)$, respectively).

Example

Let $\mathcal{A} = \mathrm{coh}(X)$ for X a curve. Set $(T, \leqslant) = (\mathbb{R} \amalg \{\infty\}, \leqslant)$ and for $0 \neq E \in \mathrm{coh}(X)$ define $\tau(E) = \frac{\deg E}{\mathrm{rank}\, E}$ if $\mathrm{rank}\, E > 0$ and $\tau(E) = \infty$ if $\mathrm{rank}\, E = 0$. Then $(\tau, T, \leqslant)$ is a stability condition, and (semi)stable objects $E \in \mathrm{coh}(X)$ with $\mathrm{rank}\, E > 0$ are semistable vector bundles in the previous sense.

Harder–Narasimhan filtrations

Theorem

Let $\mathcal{A}$ be an abelian category and $(\tau, T, \leqslant)$ a stability condition on $\mathcal{A}$, and suppose $\mathcal{A}, \tau$ satisfy some noetherian/artinian conditions. Then for every nonzero object $E \in \mathcal{A}$ there is a unique filtration $0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_n = E$ in $\mathcal{A}$ such that $F_i = E_i/E_{i-1}$ is τ-semistable for $i = 1, \ldots, n$ with $\tau(F_1) > \cdots > \tau(F_n)$. This is called the **Harder–Narasimhan filtration**.

We prove this by induction by taking $E_1 = F_1$ to be the subobject of E with $\tau(F_1)$ maximal, and then F_2 the subobject of E/E_1 with $\tau(F_2)$ maximal, and so on.

One moral is that as every object in $\mathcal{A}$ can be broken up uniquely into semistable objects, so to 'classify' all objects in $\mathcal{A}$ it is enough to classify semistable objects in $\mathcal{A}$ (e.g. to understand their moduli spaces). Compare classifying finite groups by classifying simple groups.

Geometric Invariant Theory and (semi)stability

Let V be a $\mathbb{K}$-scheme and G a (reductive) algebraic $\mathbb{K}$-group acting on V. Then we have an Artin $\mathbb{K}$-stack $[V/G]$. Suppose, instead, that we want to form a quotient V/G in the world of schemes. *Geometric Invariant Theory* (*GIT*) tells you how to do this. You should choose a 'linearization' $\mathcal{L}$, i.e. a G-equivariant ample line bundle $\mathcal{L}\to V$. Then GIT gives G-equivariant open subschemes $V^{\rm st}\subseteq V^{\rm ss}\subseteq V$ of $\mathcal{L}$-(semi)stable points such that a quotient scheme $V^{\rm st}/\!/_{\mathcal{L}}G$ exists in a strong sense (the Artin stack $[V^{\rm st}/G]$ is the scheme $V^{\rm st}/\!/_{\mathcal{L}}G$, at least if G acts effectively on V), and a quotient scheme $V^{\rm ss}/\!/_{\mathcal{L}}G$ exists in a weaker sense ($\pi:[V^{\rm ss}/G]\to V^{\rm ss}/\!/_{\mathcal{L}}G$ is a 'coarse moduli space'), and $V^{\rm st}/\!/_{\mathcal{L}}G\subseteq V^{\rm ss}/\!/_{\mathcal{L}}G$ is open. Furthermore, if V is projective then $V^{\rm ss}/\!/_{\mathcal{L}}G$ is projective. Thus, if $V^{\rm st}=V^{\rm ss}$ then the Artin stack $[V^{\rm st}/G]$ is a projective scheme.

Magic Fact

For many interesting classes of $\mathbb{K}$-linear abelian categories $\mathcal{A}$ (e.g. $\mathcal{A} = \mathrm{coh}(X)$ for smooth projective X, $\mathcal{A} = \text{mod-}\mathbb{C}Q, \ldots$), if $\mathcal{M}$ is the moduli stack of objects in $\mathcal{A}$, then we can cover $\mathcal{M}$ by larger and larger open substacks $U \subset \mathcal{M}$ with $U \cong [V/G]$ for V a (quasi)projective $\mathbb{K}$-scheme and G a reductive algebraic $\mathbb{K}$-group (usually $G = \mathrm{GL}(N, \mathbb{K})$ for $N \gg 0$).

Furthermore, there is a way to choose a GIT linearization $\mathcal{L}$ of the G-action on V such that $[V^{\mathrm{st}}/G], [V^{\mathrm{ss}}/G] \subseteq U \subset \mathcal{M}$ are the moduli stacks $\mathcal{M}_\alpha^{\mathrm{st}}(\tau), \mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$ of τ-(semi)stable objects in class α in $\mathcal{A}$ for some natural stability condition $(\tau, T, \leqslant)$ on $\mathcal{A}$.

Thus $\mathcal{M}_\alpha^{\mathrm{st}}(\tau)$ has a fine moduli scheme, and $\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$ an (often projective) coarse moduli scheme.

I don't know why GIT (semi)stability should coincide with abelian category (semi)stability, but in practice it always seems to work, e.g. for Gieseker stability on $\mathrm{coh}(X)$. Using this we can often construct *proper moduli schemes* of semistable objects, which is vital for any theory of enumerative invariants.

Derived Algebraic Geometry

Lecture 14 of 14: Pantev–Toën–Vaquié–Vezzosi's shifted symplectic Derived Algebraic Geometry

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References for this lecture:

T. Pantev, B. Toën, M. Vaquié and G. Vezzosi, *'Shifted symplectic structures'*, arXiv:1111.3209,

D. Calaque, T. Pantev, B. Toën, M. Vaquié and G. Vezzosi, *'Shifted Poisson structures and deformation quantization'*, arXiv:1506.03699.

These slides available at

http://people.maths.ox.ac.uk/∼joyce/

Plan of talk:

14. Shifted symplectic Derived Algebraic Geometry
Introduction

Definition

A *Calabi–Yau m-fold X* is a smooth projective $\mathbb{C}$-scheme of dimension m, such that $K_X \cong \mathcal{O}_X$.

Calabi–Yau manifolds, especially Calabi–Yau 3-folds, are very important in Geometry and String Theory. Suppose X is a Calabi–Yau 3-fold, and $\boldsymbol{\mathcal{M}}$ the derived moduli stack of coherent sheaves on X. As $\dim X = 3$, $\mathbb{L}_{\boldsymbol{\mathcal{M}}}$ is perfect in the interval $[-2, 1]$, not $[-1, 1]$, so $\boldsymbol{\mathcal{M}}$ is not quasi-smooth. Nonetheless, Thomas math.AG/9806111 defined enumerative invariants counting (semi)stable coherent sheaves on X.

Here is Thomas' idea. Suppose α is a Chern character on X and τ a Gieseker stability condition such that $\mathcal{M}_\alpha^{\rm st}(\tau)=\mathcal{M}_\alpha^{\rm ss}(\tau)$, so that $\mathcal{M}_\alpha^{\rm st}(\tau)$ has a proper fine moduli scheme. If $[E]\in\mathcal{M}_\alpha^{\rm st}(\tau)$ then $\mathrm{Hom}(E,E)=\mathbb{C}$. By Serre duality,
$\mathrm{Ext}^3(E,E)\cong\mathrm{Ext}^3(E,E\otimes K_X)\cong\mathrm{Hom}(E,E)^*\cong\mathbb{C}$ as $K_X\cong\mathcal{O}_X$.
As $\mathrm{Ext}^3(E,E)$ is independent of $[E]\in\mathcal{M}_\alpha^{\rm st}(\tau)$, it is possible to modify the natural obstruction theory on $\mathcal{M}_\alpha^{\rm st}(\tau)$, deleting the Ext^3 terms, to make it perfect in $[-1,1]$. So he could define a virtual class $[\mathcal{M}_\alpha^{\rm st}(\tau)]_{\rm virt}$ in $H_0(\mathcal{M}_\alpha^{\rm st}(\tau),\mathbb{Z})$, and then
$DT_\alpha(\tau)=\int_{[\mathcal{M}_\alpha^{\rm st}(\tau)]_{\rm virt}}1\in\mathbb{Z}$ is the *Donaldson–Thomas invariant*.
The theory was generalized by Joyce–Song arXiv:0810.5645, Kontsevich–Soibelman arXiv:0811.2435, and others.
These Donaldson–Thomas invariants are just the start of a long story: there is a 'categorification' in which $DT_\alpha(\tau)$ is the graded dimension of an exotic cohomology theory of $\mathcal{M}_\alpha^{\rm st}(\tau)$, a 'motivic' extension to invariants taking values in a larger ring, and so on.
This should be understood in the derived world, as a consequence of $\boldsymbol{\mathcal{M}}$ having a '-1-shifted symplectic structure'.

14.1. Classical and derived symplectic geometry
Classical symplectic geometry

Let M be a smooth manifold. Then M has a tangent bundle and cotangent bundle T^*M. We have k-forms $\omega \in C^\infty(\Lambda^k T^*M)$, and the de Rham differential $\mathrm{d}_{dR} : C^\infty(\Lambda^k T^*M) \to C^\infty(\Lambda^{k+1} T^*M)$. A k-form ω is *closed* if $\mathrm{d}_{dR}\omega = 0$.

A 2-form ω on M is *nondegenerate* if $\omega\cdot : TM \to T^*M$ is an isomorphism. This is possible only if $\dim M = 2n$ for $n \geqslant 0$. A *symplectic structure* is a closed, nondegenerate 2-form ω on M.

Symplectic geometry is the study of symplectic manifolds (M, ω).

A *Lagrangian* in (M, ω) is a submanifold $i : L \to M$ such that $\dim L = n$ and $i^*(\omega) = 0$.

Shifted symplectic Derived Algebraic Geometry

Pantev, Toën, Vaquié and Vezzosi (arXiv:1111.3209) defined a version of symplectic geometry in Derived Algebraic Geometry.

Let $\boldsymbol{X}$ be a derived $\mathbb{K}$-scheme. The cotangent complex $\mathbb{L}_{\boldsymbol{X}}$ has exterior powers $\Lambda^p\mathbb{L}_{\boldsymbol{X}}$. The *de Rham differential* $\mathrm{d}_{dR}:\Lambda^p\mathbb{L}_{\boldsymbol{X}}\to\Lambda^{p+1}\mathbb{L}_{\boldsymbol{X}}$ is a morphism of complexes. Each $\Lambda^p\mathbb{L}_{\boldsymbol{X}}$ is a complex, so has an internal differential $\mathrm{d}:(\Lambda^p\mathbb{L}_{\boldsymbol{X}})^k\to(\Lambda^p\mathbb{L}_{\boldsymbol{X}})^{k+1}$. We have $\mathrm{d}^2=\mathrm{d}_{dR}^2=\mathrm{d}\circ\mathrm{d}_{dR}+\mathrm{d}_{dR}\circ\mathrm{d}=0$.

A *p-form of degree k* on $\boldsymbol{X}$ for $k\in\mathbb{Z}$ is an element $[\omega^0]$ of $H^k(\Lambda^p\mathbb{L}_{\boldsymbol{X}},\mathrm{d})$. A *closed p-form of degree k* on $\boldsymbol{X}$ is an element

$$[(\omega^0,\omega^1,\ldots)]\in H^k\bigl(\textstyle\bigoplus_{i=0}^\infty\Lambda^{p+i}\mathbb{L}_{\boldsymbol{X}}[i],\mathrm{d}+\mathrm{d}_{dR}\bigr).$$

There is a projection $\pi:[(\omega^0,\omega^1,\ldots)]\mapsto[\omega^0]$ from closed p-forms $[(\omega^0,\omega^1,\ldots)]$ of degree k to p-forms $[\omega^0]$ of degree k.

Nondegenerate 2-forms and symplectic structures

Let $[\omega^0]$ be a 2-form of degree k on $\boldsymbol{X}$. Then $[\omega^0]$ induces a morphism $\omega^0 : \mathbb{T}_{\boldsymbol{X}} \to \mathbb{L}_{\boldsymbol{X}}[k]$, where $\mathbb{T}_{\boldsymbol{X}} = \mathbb{L}_{\boldsymbol{X}}^\vee$ is the tangent complex of $\boldsymbol{X}$. We call $[\omega^0]$ *nondegenerate* if $\omega^0 : \mathbb{T}_{\boldsymbol{X}} \to \mathbb{L}_{\boldsymbol{X}}[k]$ is a quasi-isomorphism.

If $\boldsymbol{X}$ is a derived scheme then the complex $\mathbb{L}_{\boldsymbol{X}}$ lives in degrees $(-\infty, 0]$ and $\mathbb{T}_{\boldsymbol{X}}$ in degrees $[0, \infty)$. So $\omega^0 : \mathbb{T}_{\boldsymbol{X}} \to \mathbb{L}_{\boldsymbol{X}}[k]$ can be a quasi-isomorphism only if $k \leqslant 0$, and then $\mathbb{L}_{\boldsymbol{X}}$ lives in degrees $[k, 0]$ and $\mathbb{T}_{\boldsymbol{X}}$ in degrees $[0, -k]$. If $k = 0$ then $\boldsymbol{X}$ is a smooth classical $\mathbb{K}$-scheme, and if $k = -1$ then $\boldsymbol{X}$ is quasi-smooth.

A closed 2-form $\omega = [(\omega^0, \omega^1, \ldots)]$ of degree k on $\boldsymbol{X}$ is called a *k-shifted symplectic structure* if $[\omega^0] = \pi(\omega)$ is nondegenerate.

Although the details are complex, PTVV are following a simple recipe for translating some piece of geometry from smooth manifolds/smooth classical schemes to derived schemes:

(i) replace manifolds/smooth schemes X by derived schemes $\boldsymbol{X}$.

(ii) replace vector bundles $TX, T^*X, \Lambda^p T^*X, \ldots$ by complexes $\mathbb{T}_{\boldsymbol{X}}, \mathbb{L}_{\boldsymbol{X}}, \Lambda^p \mathbb{L}_{\boldsymbol{X}}, \ldots$.

(iii) replace sections of $TX, T^*X, \Lambda^p T^*X, \ldots$ by cohomology classes of the complexes $\mathbb{T}_{\boldsymbol{X}}, \mathbb{L}_{\boldsymbol{X}}, \Lambda^p \mathbb{L}_{\boldsymbol{X}}, \ldots$, in degree $k \in \mathbb{Z}$.

(iv) replace isomorphisms of vector bundles by quasi-isomorphisms of complexes.

Note that in (iii), we can specify the degree $k \in \mathbb{Z}$ of the cohomology class (e.g. $[\omega] \in H^k(\Lambda^p \mathbb{L}_{\boldsymbol{X}})$), which doesn't happen at the classical level.

There is also a 'Poisson' version of the theory, due to Calaque–Pantev–Toën–Vaquié–Vezzosi 2015.

Calabi–Yau moduli schemes and moduli stacks

PTVV prove that if Y is a Calabi–Yau m-fold over $\mathbb{K}$ and $\boldsymbol{\mathcal{M}}$ is a derived moduli scheme or stack of (complexes of) coherent sheaves on Y, then $\boldsymbol{\mathcal{M}}$ has a $(2-m)$-shifted symplectic structure ω.

This suggests applications — lots of interesting geometry concerns Calabi–Yau moduli schemes, e.g. Donaldson–Thomas theory.

We can understand the associated nondegenerate 2-form $[\omega^0]$ in terms of *Serre duality*. At a point $[E]\in\boldsymbol{\mathcal{M}}$, we have $h^i(\mathbb{T}_{\boldsymbol{\mathcal{M}}}|_{[E]})\cong \mathrm{Ext}^{i+1}(E,E)$ and $h^i(\mathbb{L}_{\boldsymbol{\mathcal{M}}}|_{[E]})\cong \mathrm{Ext}^{1-i}(E,E)^*$. The Calabi–Yau condition gives $\mathrm{Ext}^i(E,E)\cong \mathrm{Ext}^{m-i}(E,E)^*$, which corresponds to $h^{i-1}(\mathbb{T}_{\boldsymbol{\mathcal{M}}}|_{[E]})\cong h^{i-1}(\mathbb{L}_{\boldsymbol{\mathcal{M}}}[2-m]|_{[E]})$. This is the cohomology at $[E]$ of the quasi-isomorphism $\omega^0:\mathbb{T}_{\boldsymbol{\mathcal{M}}}\to\mathbb{L}_{\boldsymbol{\mathcal{M}}}[2-m]$.

Lagrangians and Lagrangian intersections

Let $(\boldsymbol{X},\omega)$ be a k-shifted symplectic derived scheme or stack. Then Pantev et al. define a notion of *Lagrangian* $\boldsymbol{L}$ in $(\boldsymbol{X},\omega)$, which is a morphism $\boldsymbol{i}:\boldsymbol{L}\to\boldsymbol{X}$ of derived schemes or stacks together with a homotopy $i^*(\omega)\sim 0$ satisfying a nondegeneracy condition, implying that $\mathbb{T}_{\boldsymbol{L}}\simeq\mathbb{L}_{\boldsymbol{L}/\boldsymbol{X}}[k-1]$.

If $\boldsymbol{L},\boldsymbol{M}$ are Lagrangians in $(\boldsymbol{X},\omega)$, then the fibre product $\boldsymbol{L}\times_{\boldsymbol{X}}\boldsymbol{M}$ has a natural $(k-1)$-shifted symplectic structure.

If (S,ω) is a classical smooth symplectic scheme, then it is a 0-shifted symplectic derived scheme in the sense of PTVV, and if $L,M\subset S$ are classical smooth Lagrangian subschemes, then they are Lagrangians in the sense of PTVV. Therefore the (derived) Lagrangian intersection $L\cap M=L\times_S M$ is a -1-shifted symplectic derived scheme.

Examples of Lagrangians

Let $(\boldsymbol{X},\omega)$ be k-shifted symplectic, and $\boldsymbol{i}_a:\boldsymbol{L}_a\rightarrow\boldsymbol{X}$ be Lagrangian in $\boldsymbol{X}$ for $a=1,\ldots,d$. Then Ben-Bassat (arXiv:1309.0596) shows

$$\boldsymbol{L}_1\times_{\boldsymbol{X}}\boldsymbol{L}_2\times_{\boldsymbol{X}}\cdots\times_{\boldsymbol{X}}\boldsymbol{L}_d\longrightarrow(\boldsymbol{L}_1\times_{\boldsymbol{X}}\boldsymbol{L}_2)\times\cdots\times(\boldsymbol{L}_{d-1}\times_{\boldsymbol{X}}\boldsymbol{L}_d)\times(\boldsymbol{L}_d\times_{\boldsymbol{X}}\boldsymbol{L}_1)$$

is Lagrangian, where the r.h.s. is $(k-1)$-shifted symplectic by PTVV. This is relevant to defining 'Fukaya categories' of complex symplectic manifolds.

Let Y be a Calabi–Yau m-fold, so that the derived moduli stack $\boldsymbol{\mathcal{M}}$ of coherent sheaves (or complexes) on Y is $(2-m)$-shifted symplectic by PTVV, with symplectic form ω. Then

$$\boldsymbol{\mathcal{E}xact}\xrightarrow{\pi_1\times\pi_2\times\pi_3}(\boldsymbol{\mathcal{M}},\omega)\times(\boldsymbol{\mathcal{M}},-\omega)\times(\boldsymbol{\mathcal{M}},\omega)$$

is Lagrangian, where $\boldsymbol{\mathcal{E}xact}$ is the derived moduli stack of short exact sequences in $\mathrm{coh}(Y)$ (or distinguished triangles in $D^b\,\mathrm{coh}(Y)$). This is relevant to Cohomological Hall Algebras.

14.2. A 'Darboux theorem' for shifted symplectic schemes

Theorem 14.1 (Brav, Bussi and Joyce arXiv:1305.6302)

Suppose $(\boldsymbol{X},\omega)$ is a k-shifted symplectic derived $\mathbb{K}$-scheme for $k<0$. If $k\not\equiv 2 \mod 4$, then each $x\in\boldsymbol{X}$ admits a Zariski open neighbourhood $\boldsymbol{Y}\subseteq\boldsymbol{X}$ with $\boldsymbol{Y}\simeq\mathrm{Spec}(A,\mathrm{d})$ for (A,d) an explicit cdga generated by graded variables x_j^{-i},y_j^{k+i} for $0\leqslant i\leqslant -k/2$, and $\omega|_{\boldsymbol{Y}}=[(\omega^0,0,0,\ldots)]$ where x_j^l,y_j^l have degree l, and

$$\omega^0=\textstyle\sum_{i=0}^{[-k/2]}\sum_{j=1}^{m_i}\mathrm{d}_{dR}y_j^{k+i}\mathrm{d}_{dR}x_j^{-i}.$$

Also the differential d in (A,d) is given by Poisson bracket with a Hamiltonian H in A of degree $k+1$.

If $k\equiv 2 \mod 4$, we have two statements, one étale local with ω^0 standard, and one Zariski local with the components of ω^0 in the degree $k/2$ variables depending on some invertible functions.

This is extended to Artin stacks in Ben-Bassat–BBJ arXiv:1312.0090, and to Lagrangians in k-shifted symplectic derived schemes by Joyce–Safronov arXiv:1506.04024.

The case of -1-shifted symplectic derived schemes

When $k=-1$ the Hamiltonian H in the theorem has degree 0. Then Theorem 14.1 reduces to:

Corollary 14.2

Suppose $(\boldsymbol{X},\omega)$ is a -1-shifted symplectic derived $\mathbb{K}$-scheme. Then $(\boldsymbol{X},\omega)$ is Zariski locally equivalent to a derived critical locus $\mathbf{Crit}(H:U\to\mathbb{A}^1)$, for U a smooth classical $\mathbb{K}$-scheme and $H:U\to\mathbb{A}^1$ a regular function. Hence, the underlying classical $\mathbb{K}$-scheme $X=t_0(\boldsymbol{X})$ is Zariski locally isomorphic to a classical critical locus $\mathrm{Crit}(H:U\to\mathbb{A}^1)$.

Corollary 14.3

Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a classical moduli $\mathbb{K}$-scheme of coherent sheaves, or complexes of coherent sheaves, on Y. Then $\mathcal{M}$ is Zariski locally isomorphic to the critical locus $\mathrm{Crit}(H:U\to\mathbb{A}^1)$ of a regular function on a smooth $\mathbb{K}$-scheme.

14.3. Applications to Donaldson–Thomas theory

Let X be a Calabi–Yau 3-fold. The *Donaldson–Thomas invariants* $DT^\alpha(\tau)$ in $\mathbb{Z}$ or $\mathbb{Q}$ 'count' moduli schemes $\mathcal{M}_{\rm ss}^\alpha(\tau)$ of τ-semistable coherent sheaves on X in Chern class α. They were defined by Thomas 1998 when τ-stable$=\tau$-semistable using Behrend–Fantechi obstruction theories and virtual classes, and by Joyce–Song 2008 in general. In String Theory, I believe they are 'numbers of BPS states'.
In the τ-stable$=\tau$-semistable case, $\mathcal{M}_{\rm ss}^\alpha(\tau)$ is the classical truncation of a derived moduli scheme $\boldsymbol{\mathcal{M}}_{\rm ss}^\alpha(\tau)$, which is quasi-smooth. PTVV say that $\boldsymbol{\mathcal{M}}_{\rm ss}^\alpha(\tau)$ is -1-shifted symplectic, so BBJ say that $\boldsymbol{\mathcal{M}}_{\rm ss}^\alpha(\tau)$ is Zariski-locally a critical locus.
There is some interesting geometry associated with critical loci:

- Perverse sheaves of vanishing cycles.
- Motivic Milnor fibres.
- Categories of matrix factorizations.

We can use these to generalize Donaldson–Thomas theory.

Orientation data and perverse sheaves

Definition (based on Kontsevich and Soibelman 2008)

Let $(\boldsymbol{S},\omega)$ be a -1-shifted symplectic derived scheme or stack. Then the cotangent complex $\mathbb{L}_{\boldsymbol{S}}$ is a perfect complex on $\boldsymbol{S}$, and $K_{\boldsymbol{S}}=\det\mathbb{L}_{\boldsymbol{S}}$ is a line bundle on $\boldsymbol{S}$. *Orientation data* for $(\boldsymbol{S},\omega)$ is a choice of square root line bundle $K_{\boldsymbol{S}}^{1/2}$.

Actually, orientation data is more like a *spin structure* than an orientation.

Some notion of orientation data is needed for most generalizations of Donaldson–Thomas theory of Calabi–Yau 3-folds.

Theorem (Joyce–Upmeier arXiv:2001.00113)

Let X be a compact Calabi–Yau 3-fold and $(\boldsymbol{\mathcal{M}},\omega)$ the -1-shifted symplectic derived moduli stack of coherent sheaves (or complexes) on X. Then canonical orientation data exists for $(\boldsymbol{\mathcal{M}},\omega)$.

Theorem (Ben-Bassat-Bussi–Brav–Dupont–Joyce–Szendrői 2012)

Let $(\boldsymbol{S},\omega)$ be a -1-shifted symplectic derived scheme or stack with orientation data $K_{\boldsymbol{S}}^{1/2}$. Then we can construct a natural perverse sheaf $P^\bullet_{\boldsymbol{S}}$ on $\boldsymbol{S}$. The hypercohomology $\mathbb{H}^*(P^\bullet_{\boldsymbol{S}})$ is a graded vector space. When $\boldsymbol{S}$ is a Calabi–Yau 3-fold derived moduli scheme $\boldsymbol{\mathcal{M}}^\alpha_{\rm ss}(\tau)$ with τ-stable=τ-semistable, we have

$$DT^\alpha(\tau)=\sum_{i\in\mathbb{Z}}(-1)^i\dim\mathbb{H}^i(P^\bullet_{\boldsymbol{\mathcal{M}}^\alpha_{\rm ss}(\tau)}).$$

Thus the graded vector spaces $\mathbb{H}^*(P^\bullet_{\boldsymbol{\mathcal{M}}^\alpha_{\rm ss}(\tau)})$ categorify Donaldson–Thomas invariants. They are interpreted in String Theory as *vector spaces of BPS states*.

It is expected that for Calabi–Yau 3-fold moduli stacks $\boldsymbol{\mathcal{M}}$, one should make $\mathbb{H}^*(P^\bullet_{\boldsymbol{\mathcal{M}}})$ into an associative algebra, a *Cohomological Hall Algebra* as in Kontsevich–Soibelman 2010. The (conjectural) product should be defined using the shifted Lagrangian

$$\boldsymbol{\mathcal{E}xact}\xrightarrow{\pi_1\times\pi_2\times\pi_3}(\boldsymbol{\mathcal{M}},\omega)\times(\boldsymbol{\mathcal{M}},-\omega)\times(\boldsymbol{\mathcal{M}},\omega).$$

Donaldson–Thomas style invariants of C–Y 4-folds

There is also an interesting story for Calabi–Yau 4-folds, which uses PTVV −2-shifted symplectic structures on C–Y 4 moduli spaces.

Definition (Borisov–Joyce 2015)

Let $(\boldsymbol{S},\omega)$ be a -2-shifted symplectic derived scheme or stack (for example, a Calabi–Yau 4-fold moduli space), and set $K_{\boldsymbol{S}}=\det\mathbb{L}_{\boldsymbol{S}}$. Then ω induces a natural isomorphism $\iota:K_{\boldsymbol{S}}^{\otimes^2}\rightarrow\mathcal{O}_{\boldsymbol{S}}^{\otimes^2}$. An *orientation* of $(\boldsymbol{S},\omega)$ is an isomorphism $\jmath:K_{\boldsymbol{S}}\rightarrow\mathcal{O}_{\boldsymbol{S}}$ with $\jmath^{\otimes^2}=\iota$.

Theorem (Cao–Gross–Joyce 2019)

Let X be a compact Calabi–Yau 4-fold and $(\boldsymbol{\mathcal{M}},\omega)$ the -2-shifted symplectic derived moduli stack of coherent sheaves (or complexes) on X. Then an orientation exists for $(\boldsymbol{\mathcal{M}},\omega)$.

Using the BBJ Darboux Theorem, Theorem 14.1, we prove:

Theorem (Borisov–Joyce 2015, Oh–Thomas 2020.)

Let $(\boldsymbol{S},\omega)$ be a proper, oriented -2-shifted symplectic derived scheme over $\mathbb{C}$. Then we can construct a natural virtual class $[\boldsymbol{S}]_{\rm virt}$ in $H_*(S,\mathbb{Z})$.
When $\boldsymbol{S}$ is a Calabi–Yau 4-fold derived moduli scheme $\boldsymbol{\mathcal{M}}_{\rm ss}^{\alpha}(\tau)$ with τ-stable$=\tau$-semistable, this allows us to define Donaldson–Thomas style 'DT4 invariants' of Calabi–Yau 4-folds.

CY4 virtual classes are different to Behrend–Fantechi virtual classes, they have half the expected dimension.
PTVV's shifted symplectic theory is a (rare) example in which Derived Algebraic Geometry has powerful applications in classical Algebraic Geometry, and you really need the derived point of view.
Shifted symplectic structures would not work on classical stacks, it is essential to work with derived stacks.