Derived Algebraic Geometry

Lecture 5 of 14: Classical stacks

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Helpful reference for this lecture:
T.L. Gomez, *'Algebraic stacks'*, Proc. Indian Acad. Sci. Math. Sci. 111 (2001), 1–31. math.AG/9911199.

These slides available at
`http://people.maths.ox.ac.uk/∼joyce/`

Plan of talk:

Introduction

Stacks are geometric spaces in algebraic geometry which generalize schemes. Over a field $\mathbb{K}$ we have $\mathbb{K}$-schemes $\subset$ algebraic $\mathbb{K}$-spaces $\subset$ Deligne–Mumford $\mathbb{K}$-stacks $\subset$ Artin $\mathbb{K}$-stacks, in (2-)categories

$$\mathbf{Sch}_{\mathbb{K}} \subset \mathbf{AlgSp}_{\mathbb{K}} \subset \mathbf{DMSta}_{\mathbb{K}} \subset \mathbf{Art}_{\mathbb{K}}\,.$$

Here $\mathbf{Sch}_{\mathbb{K}}, \mathbf{AlgSp}_{\mathbb{K}}$ are ordinary categories, or *discrete* 2-categories (i.e. a 2-category $\mathfrak{C}$ in which any 2-morphism $\eta : f \Rightarrow f$ is the identity id_f, so that $\mathfrak{C}$ is essentially the same as $\mathrm{Ho}(\mathfrak{C})$), and $\mathbf{DMSta}_{\mathbb{K}}, \mathbf{Art}_{\mathbb{K}}$ are genuine 2-categories, with nontrivial 2-morphisms.

Each Artin stack X has isotropy groups $\mathrm{Iso}_X(x)$ for each $\mathbb{K}$-point x in X, which are algebraic $\mathbb{K}$-groups, such as $\mathrm{GL}(n, \mathbb{K})$. Essentially, X is Deligne–Mumford if $\mathrm{Iso}_X(x)$ is finite for all x, and an algebraic space if $\mathrm{Iso}_X(x) = \{1\}$ for all x. If an algebraic $\mathbb{K}$-group Γ acts on a $\mathbb{K}$-scheme U then $[U/\Gamma]$ is an Artin stack. It is Deligne–Mumford if the stabilizer groups $\mathrm{Stab}_\Gamma(u)$ are finite for $u \in U$, and an algebraic space if the action is free.

If X is an Artin $\mathbb{K}$-stack it has a smooth surjective morphism $\pi : U \to X$ with $U \in \mathbf{Sch}_{\mathbb{K}}$, called an atlas. Here 'smooth' is in the algebraic geometry sense of having smooth fibres (in differential geometry it would be a submersion). If X is Deligne–Mumford or an algebraic space we can take π to be étale (a covering map) instead of smooth. Algebraic spaces are basically the quotients of schemes by étale equivalence relations. It is difficult to write down examples of algebraic spaces which are not schemes.

We do not really know what a stack X is — there is no beautiful definition like that of a topological space, which would make a geometer happy. However, we *do* know what a (1-)morphism $f : U \to X$ is, when U is a (possibly affine) scheme. So the idea is to concoct a definition of X solely in terms of (1-)morphisms $f : U \to X$ for schemes U, and their relations. Then we pretend that this is the real definition.

There are a lot of technicalities in the theory of stacks. I may lie a little bit, e.g. I'm not sure (and don't care) if a Deligne–Mumford stack is exactly the same as an Artin stack with finite isotropy groups.

Stacks were invented to handle moduli problems in which a moduli scheme does not exist. Nearly all interesting moduli problems in Algebraic Geometry have a moduli stack. In fact, this is almost a tautology: we can write a moduli problem as a moduli functor $F:\mathbf{Sch}_{\mathbb{K}}^{\mathrm{op}}\to\mathbf{Groupoids}$, where $\mathbf{Sch}_{\mathbb{K}}^{\mathrm{op}}$ is the opposite category of $\mathbf{Sch}_{\mathbb{K}}$. We *define* Artin $\mathbb{K}$-stacks to be a class of functors $\mathbf{Sch}_{\mathbb{K}}^{\mathrm{op}}\to\mathbf{Groupoids}$ which have a bunch of properties we expect of well behaved moduli functors. Then the moduli functor F *is* the moduli stack.

If X is an Artin stack corresponding to a functor $F:\mathbf{Sch}_{\mathbb{K}}^{\mathrm{op}}\to\mathbf{Groupoids}$, then for each $U\in\mathbf{Sch}_{\mathbb{K}}$, $F(U)$ is a groupoid, where objects $f,g\in F(U)$ correspond to 1-morphisms $f,g:U\to X$ in $\mathbf{Art}_{\mathbb{K}}$, and morphisms $\eta:f\to g$ in $F(U)$ correspond to 2-morphisms $\eta:f\Rightarrow g$ in $\mathbf{Art}_{\mathbb{K}}$.

This is the sense in which although we don't know what X is, we do know what a (1-)morphism $f:U\to X$ is, for U an (affine) scheme.

5.1. Moduli functors and moduli schemes

As an example, let X be a smooth projective $\mathbb{K}$-scheme. We wish to define a moduli space $\mathcal{M}$ of algebraic vector bundles (locally free coherent sheaves) on X, possibly Gieseker or slope (semi)stable.

We could try this either in the world of schemes, or stacks.

For schemes, define a moduli functor $F_{\mathrm{Vect}} : \mathbf{Sch}_{\mathbb{K}}^{\mathrm{op}} \rightarrow \mathbf{Sets}$ by:

- For each $\mathbb{K}$-scheme U, define $F_{\mathrm{Vect}}(U)$ to be the set of isomorphism classes $[E]$ of vector bundles $E \rightarrow X \times U$, where we may require $E|_{X\times\{u\}}$ to be semistable for each $u \in U$.
- For each scheme morphism $\phi : U \rightarrow V$, define $F_{\mathrm{Vect}}(\phi) : F_{\mathrm{Vect}}(V) \rightarrow F_{\mathrm{Vect}}(U)$ to map $[E] \mapsto [(\mathrm{id}_X \times \phi)^*(E)]$, where $\mathrm{id}_X \times \phi : X \times U \rightarrow X \times V$.

There are variations on this: we could replace $\mathbf{Sch}_{\mathbb{K}}^{\mathrm{op}}$ by $\mathbf{AlgSp}_{\mathbb{K}}^{\mathrm{op}}$ as in Laumon–Moret-Bailly, or by the category of $\mathbb{K}$-algebras $\mathbf{Alg}_{\mathbb{K}}$. The difference is not important.

Now let $\mathcal{M}$ be any $\mathbb{K}$-scheme. We define the functor $F_{\mathcal{M}} = \mathrm{Hom}(-,\mathcal{M}) : \mathbf{Sch}_{\mathbb{K}}^{\mathrm{op}} \to \mathbf{Sets}$ by $F_{\mathcal{M}}(U) = \mathrm{Hom}_{\mathbf{Sch}_{\mathbb{K}}}(U,\mathcal{M})$ and $F_{\mathcal{M}}(\phi) : \alpha \mapsto \alpha \circ \phi$ for $\phi : U \to V$ a morphism in $\mathbf{Sch}_{\mathbb{K}}$ and $\alpha : V \to \mathcal{M}$. The Yoneda Lemma implies that $F_{\mathcal{M}}$ determines $\mathcal{M}$ up to canonical isomorphism.

We say that $\mathcal{M}$ is a *fine moduli scheme* for vector bundles on X if there exists a natural isomorphism of functors $\eta : F_{\mathcal{M}} \Rightarrow F_{\mathrm{Vect}}$. A fine moduli scheme need not exist, but if it does it is unique up to canonical isomorphism.

Now $F_{\mathcal{M}}(\mathcal{M}) = \mathrm{Hom}(\mathcal{M},\mathcal{M}) \ni \mathrm{id}_{\mathcal{M}}$, so $\eta(\mathrm{id}_{\mathcal{M}}) \in F_{\mathrm{Vect}}(\mathcal{M})$. Write $\eta(\mathrm{id}_{\mathcal{M}}) = [E_{\mathrm{univ}}]$. Then $E_{\mathrm{univ}} \to X \times \mathcal{M}$ is a vector bundle. It has the *universal property* that whenever U is a $\mathbb{K}$-scheme and $E \to X \times U$ a vector bundle (possibly (semi)stable over U) then there is a unique morphism $\phi : U \to \mathcal{M}$ such that there exists an isomorphism $\alpha : E \xrightarrow{\cong} (\mathrm{id}_X \times \phi)^*(E_{\mathrm{univ}})$. (Note that α need not be unique.) This property of $\mathcal{M}, E_{\mathrm{univ}}$ characterizes $\mathcal{M}$ uniquely. Taking $U = \mathrm{Spec}\,\mathbb{K}$ shows $\mathbb{K}$-points of $\mathcal{M}$ are in 1-1 correspondence with isomorphism classes of (semistable) vector bundles $E \to X$.

This notion of fine moduli schemes is good when it works, but often it doesn't. There are far more functors $F : \mathbf{Sch}_{\mathbb{K}}^{\mathrm{op}} \to \mathbf{Sets}$ up to natural isomorphism than there are schemes, so for a random F it is very unlikely that there exists a scheme $\mathcal{M}$ with $F \cong F_{\mathcal{M}}$.

As a general rule, if the objects in your problem (e.g. vector bundles) have nontrivial automorphisms, fine moduli schemes may not exist. This is because the moduli space $\mathcal{M}$ should have a geometric structure encoding automorphism groups – as in isotropy groups of stacks.

If a fine moduli scheme does not exist, one may be able to define a *coarse moduli scheme*, which is a scheme $\mathcal{M}$ with a natural transformation $\eta : F_{\mathrm{Vect}} \Rightarrow F_{\mathcal{M}}$ (*not* a natural isomorphism) which is 'as close to being a natural isomorphism as possible'.

It turns out that (at least to a first approximation):

- Fine and coarse moduli schemes of all vector bundles don't exist.
- Fine moduli schemes of stable bundles exist.
- Coarse moduli schemes of semistable bundles exist.

Making nice moduli schemes was why (semi)stability was invented.

Sheaf property of the functors $F_{\mathcal{M}}$

Let $\mathcal{M}$ be a scheme, so we have $F_{\mathcal{M}} = \mathrm{Hom}(-,\mathcal{M}) : \mathbf{Sch}_{\mathbb{K}}^{\mathrm{op}} \to \mathbf{Sets}$. Let $U \in \mathbf{Sch}_{\mathbb{K}}$, and let $\{U_i : i \in I\}$ be an open cover of U (Zariski or étale) with inclusions $\iota_i : U_i \hookrightarrow U$. Then we have sets $F_{\mathcal{M}}(U), F_{\mathcal{M}}(U_i)$ and morphisms $F_{\mathcal{M}}(\iota_i) : F_{\mathcal{M}}(U_i) \to F_{\mathcal{M}}(U)$. Now morphisms $f : U \to \mathcal{M}$ form a *sheaf* on U. That is, if we are given morphisms $f_i : U_i \to \mathcal{M}$ for $i \in I$ with $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for $i,j \in I$, then there is a unique $f : U \to \mathcal{M}$ with $f|_{U_i} = f_i$ for $i \in I$. We can express this sheaf property solely in terms of the functor $F_{\mathcal{M}}$ and open covers in $\mathbf{Sch}_{\mathbb{K}}$. So it makes sense to say that a functor $F : \mathbf{Sch}_{\mathbb{K}}^{\mathrm{op}} \to \mathbf{Sets}$ *is a sheaf on* $\mathbf{Sch}_{\mathbb{K}}^{\mathrm{op}}$.
However, the functor $F_{\mathrm{Vect}} : \mathbf{Sch}_{\mathbb{K}}^{\mathrm{op}} \to \mathbf{Sets}$ *need not* be a sheaf on $\mathbf{Sch}_{\mathbb{K}}^{\mathrm{op}}$, and if it is not then no fine moduli space exists. The reason is that we defined F_{Vect} by taking sets of *isomorphism classes* of vector bundles E, F, but if there are more than one isomorphism $E \cong F$ this forgets information. For example, isomorphism classes of line bundles on $\mathbb{CP}^1$ cannot be glued from the open cover $\mathbb{CP}^1 \setminus \{[1,0]\}$, $\mathbb{CP}^1 \setminus \{[0,1]\}$.

Functors to groupoids and the stack property

We can fix this problem by not taking isomorphism classes. Recall that *groupoids* are categories in which every morphism is an isomorphism. They form a category (in fact, a 2-category) **Groupoids**. Let us define a functor $\bar F_{\rm Vect}:\mathbf{Sch}_{\mathbb K}^{\rm op}\to\mathbf{Groupoids}$ as before, but instead of taking isomorphism classes, we just define $F(U)$ to be the groupoid with objects vector bundles $E\to X\times U$, and morphisms isomorphisms of such bundles. Our previous functor $F_{\rm Vect}$ is the composition of $\bar F_{\rm Vect}$ with the functor $\mathbf{Groupoids}\to\mathbf{Sets}$ taking isomorphism classes of objects.

Now vector bundles on $X\times U$ can be glued over an open cover $\{U_i:i\in I\}$ of U, in a 2-categorical sense: given vector bundles $E_i\to X\times U_i$ for $i\in I$, and isomorphisms $\eta_{ij}:E_i|_{X\times U_{ij}}\to E_j|_{X\times U_{ij}}$ with $\eta_{jk}\circ\eta_{ij}|_{U_{ijk}}=\eta_{ik}|_{U_{ijk}}$, etc., there is a vector bundle $E\to X$ with isomorphisms $E|_{U_i}=E_i$. That is, vector bundles over $X\times U$ are a 2-*sheaf* or *stack* on U. We can express this 2-sheaf property solely in terms of the functor $\bar F_{\rm Vect}$ and open covers in $\mathbf{Sch}_{\mathbb K}$. So it makes sense to say that $\bar F:\mathbf{Sch}_{\mathbb K}^{\rm op}\to\mathbf{Groupoids}$ *is a stack on* $\mathbf{Sch}_{\mathbb K}^{\rm op}$.

5.2. The definition of stacks (first version)

We will define a notion of when a functor $F : \mathbf{Sch}_{\mathbb{K}}^{\mathrm{op}} \to \mathbf{Groupoids}$ is a *stack*. This is just a categorical/sheafy concept; to get stacks which act like nice geometric spaces we need to impose extra conditions, giving Deligne–Mumford or Artin stacks, say.

To define stacks we need a notion of *open cover* in $\mathbf{Sch}_{\mathbb{K}}$ (Zariski, étale, fppf, smooth, ...), that is, when a collection of morphisms $\{\iota_i : U_i \to U,\ i \in I\}$ in $\mathbf{Sch}_{\mathbb{K}}$ is an open cover of U. For now, we assume we know what open covers are, and explain more later.

Here $\mathbf{Sch}_{\mathbb{K}}^{\mathrm{op}}$ is an ordinary category, but **Groupoids** is a 2-category. We can regard $\mathbf{Sch}_{\mathbb{K}}^{\mathrm{op}}$ as a 2-category with only identity 2-morphisms (a discrete 2-category), and then functors $\mathbf{Sch}_{\mathbb{K}}^{\mathrm{op}} \to \mathbf{Groupoids}$ are the same as 2-functors. This is helpful as then $\mathrm{Fun}_2(\mathbf{Sch}_{\mathbb{K}}^{\mathrm{op}}, \mathbf{Groupoids})$ is a 2-category, with objects 2-functors, 1-morphisms 2-natural transformations, and 2-morphisms modifications. As all morphisms in groupoids are isomorphisms, it is a (2,1)-category.

Definition 5.1

A (2-)functor $F : \mathbf{Sch}_{\mathbb{K}}^{\mathrm{op}} \to \mathbf{Groupoids}$ is a *stack* if whenever $\{\iota_i : U_i \to U,\ i \in I\}$ is an open cover of U in $\mathbf{Sch}_{\mathbb{K}}$, then:

(i) If $X, Y \in F(U)$ and $\varphi_i : X|_{U_i} \to Y|_{U_i}$ are morphisms in $F(U_i)$ for $i \in I$ with $\varphi_i|_{U_{ij}} = \varphi_j|_{U_{ij}}$ for $i, j \in I$ then there exists $\eta : X \to Y$ with $\eta|_{U_i} = \varphi_i$ for $i \in I$.

(ii) If $X, Y \in F(U)$ and $\varphi, \psi : X \to Y$ are morphisms in $F(U)$ with $\varphi|_{U_i} = \psi|_{U_i}$ for $i \in I$ then $\varphi = \psi$.

(iii) If $X_i \in F(U_i)$ and $\varphi_{ij} : X_i|_{U_{ij}} \to X_j|_{U_{ij}}$ are morphisms in $F(U_{ij})$ with $\varphi_{jk}|_{U_{ijk}} \circ \varphi_{ij}|_{U_{ijk}} = \varphi_{ik}|_{U_{ijk}}$ for all $i, j, k \in I$ then there exists $X \in F(U)$ and $\varphi_i : X|_{U_i} \to X_i$ for $i \in I$ such that $\varphi_{ij} \circ \varphi_i|_{U_{ij}} = \varphi_j|_{U_{ij}}$ for $i, j \in I$.

Here we use the notation that if $X \in F(U)$ then $X|_{U_i} = F(\iota_i)(X) \in F(U_i)$. Also $U_{ij} = U_i \times_{\iota_i, U, \iota_j} U_j$ (think of as $U_i \cap U_j$) and $U_{ijk} = U_i \times_U U_j \times_U U_k$ (think of as $U_i \cap U_j \cap U_k$).

All of (i)–(iii) are properties you would expect of something which is a sheaf in a reasonable sense, e.g. if $F(U)$ is the groupoid of some kind of sheaves on $X \times U$.

Write $\mathbf{Sta}_{\mathbb{K}} \subset \mathrm{Fun}_2(\mathbf{Sch}_{\mathbb{K}}^{\mathrm{op}}, \mathbf{Groupoids})$ for the full 2-subcategory of stacks. Functors $\mathbf{Sch}_{\mathbb{K}}^{\mathrm{op}} \to \mathbf{Groupoids}$ are sometimes called *prestacks*. We can regard prestacks as a class of (generally very horrible) spaces, and stacks as (slightly less horrible) spaces. Both are still too horrible to do geometry on in a meaningful way.

We can include the discrete 2-category $\mathbf{Sch}_{\mathbb{K}}$ of schemes, and the 2-category $\mathbf{GQuot}_{\mathbb{K}}$ of global quotients $[V/G]$ in §4, as full 2-subcategories $\mathbf{Sch}_{\mathbb{K}} \subset \mathbf{GQuot}_{\mathbb{K}} \subset \mathbf{Sta}_{\mathbb{K}} \subset \mathrm{Fun}_2(\mathbf{Sch}_{\mathbb{K}}^{\mathrm{op}}, \mathbf{Groupoids})$.

To do this for $\mathbf{Sch}_{\mathbb{K}}$, we include $\mathbf{Sch}_{\mathbb{K}} \to \mathrm{Fun}(\mathbf{Sch}_{\mathbb{K}}^{\mathrm{op}}, \mathbf{Sets})$ by mapping $\mathcal{M} \mapsto \mathrm{Hom}(-, \mathcal{M})$ as before, and compose with the functor $\mathbf{Sets} \hookrightarrow \mathbf{Groupoids}$ taking a set S to the groupoid with objects S and only identity morphisms.

Defining stacks using ‘categories fibred in groupoids’

There is an alternative version of stacks which defines a stack not as a functor $F:\mathbf{Sch}_{\mathbb{K}}^{\mathrm{op}}\rightarrow\mathbf{Groupoids}$, but as a category $\mathscr{C}$ with a functor $\Pi:\mathscr{C}\rightarrow\mathbf{Sch}_{\mathbb{K}}$. For the moduli stack of vector bundles on X, we define $\mathscr{C}_{\mathrm{Vect}}$ to be the category with objects (U,E) where $U\in\mathbf{Sch}_{\mathbb{K}}$ and $E\rightarrow X\times U$ is a vector bundle, and morphisms $(\phi,\eta):(U,E)\rightarrow(V,F)$, where $\phi:U\rightarrow V$ is a morphism in $\mathbf{Sch}_{\mathbb{K}}$ and $\eta:E\rightarrow(\mathrm{id}_X\times\phi)^*(F)$ is an isomorphism of vector bundles. The functor $\Pi_{\mathrm{Vect}}:\mathscr{C}_{\mathrm{Vect}}\rightarrow\mathbf{Sch}_{\mathbb{K}}$ maps $(U,E)\mapsto U$ and $(\phi,\eta)\mapsto\phi$.
To model functors to **Groupoids** rather than $\mathfrak{Cat}$, we require $\Pi:\mathscr{C}\rightarrow\mathbf{Sch}_{\mathbb{K}}$ to be a *category fibred in groupoids*, which roughly means that for each $U\in\mathbf{Sch}_{\mathbb{K}}$ the full subcategory $\mathscr{C}_U\subset\mathscr{C}$ with oibjkects $\{X\in\mathscr{C}:\Pi(X)=U\}$ is a groupoid.
Definition 5.1 of stacks has an analogue for categories fibred in groupoids. It is easy to translate between the two pictures, e.g. given $F:\mathbf{Sch}_{\mathbb{K}}^{\mathrm{op}}\rightarrow\mathbf{Groupoids}$ define $\mathscr{C}$ to have objects (U,E) for $U\in\mathbf{Sch}_{\mathbb{K}}$ and $E\in F(U)$, and morphisms $(\phi,\eta):(U,E)\rightarrow(V,E')$ for $\phi:U\rightarrow V$ in $\mathbf{Sch}_{\mathbb{K}}$ and $\eta:E\rightarrow F(\phi)(E')$ in $F(U)$.

What we mean by ‘open cover’ in $\mathbf{Sch}_{\mathbb{K}}$

The most obvious notion of open cover of schemes is an open cover in the Zariski topology, in the usual sense of topology. However, this turns out to be too coarse to be useful (for example, a principal $\mathbb{Z}_2$-bundle $P \to U$ in the Zariski topology is automatically trivial). Grothendieck introduced a more general notion of open cover (Grothendieck topology) in Algebraic Geometry. An *étale open cover* $\{\iota_i : U_i \to U,\ i \in I\}$ is a collection of schemes U_i and étale morphisms (covering maps) $\iota_i : U_i \to U$ such that $\coprod_{i\in I} \iota_i : \coprod_{i\in I} U_i \to U$ is surjective. For example, the map $\mathbb{C}^* \to \mathbb{C}^*$ mapping $z \mapsto z^k$ for $0 \neq k \in \mathbb{Z}$ is an étale open cover. A *smooth open cover* is similar, but the morphisms $\iota_i : U_i \to U$ must be smooth (basically the fibres are smooth, so morally ι_i looks locally like a projection $U \times \mathbb{A}^k \to U$).

5.3. Deligne–Mumford and Artin stacks

Deligne–Mumford (and Artin stacks) are nice classes of stacks in the étale (or smooth) topologies. Before we define them, we need some properties of objects and 1-morphisms in the 2-category $\mathbf{Sta}_{\mathbb{K}}$.

Definition

An object X in $\mathbf{Sta}_{\mathbb{K}}$ is *representable* if it is equivalent in the 2-category $\mathbf{Sta}_{\mathbb{K}}$ to a scheme (i.e. an object of the discrete 2-subcategory $\mathbf{Sch}_{\mathbb{K}} \subset \mathbf{Sta}_{\mathbb{K}}$).

It is a fact that all fibre products exist in the 2-category $\mathbf{Sta}_{\mathbb{K}}$ (we can write down an explicit construction). Hence products $X \times Y = X \times_{\mathrm{Spec}\,\mathbb{K}} Y$ exist.

A 1-morphism $f : X \to Y$ in $\mathbf{Sta}_{\mathbb{K}}$ is called *representable* if whenever $g : U \to Y$ is a 1-morphism in $\mathbf{Sta}_{\mathbb{K}}$ with U representable, then $X \times_{f,Y,g} U$ is representable. Roughly, this means that all the fibres of $f : X \to Y$ are schemes.

Definition

Every stack X in $\mathbf{Sta}_{\mathbb{K}}$ has a natural *diagonal* 1*-morphism* $\Delta_X : X \to X \times X$. We often require Δ_X to be a representable 1-morphism. Suppose U, V are $\mathbb{K}$-schemes and $f : U \to X$, $g : V \to X$ are 1-morphisms. Then $U \times_{f,X,g} V \cong (U \times V) \times_{f\times g, X\times X, \Delta_X} X$, so if Δ_X is representable then $U \times_{f,X,g} V$ is a $\mathbb{K}$-scheme. As this holds for all $g : V \to X$, this implies that $f : U \to X$ is representable if U is a $\mathbb{K}$-scheme.

Definition

Fix a Grothendieck topology (étale, smooth) used to define stacks. Let **P** be a property of morphisms of schemes $f : U \to V$ which is local on the target in the chosen topology, and stable under base change. Then we say that a representable 1-morphism $f : X \to Y$ of stacks has property **P** if for all 2-Cartesian squares

$$\begin{array}{ccc} U & \xrightarrow{\quad g \quad} & V \\ \downarrow & \Uparrow & \downarrow \\ X & \xrightarrow{\quad f \quad} & Y \end{array}$$

with V a scheme (hence U is a scheme as f is representable), then the $\mathbb{K}$-scheme morphism g has property $\boldsymbol{P}$.

In this way we can define when a representable 1-morphism of stacks is quasi-compact, separated, étale, smooth, ...

Think of V as an 'open set' in Y, and U as the 'pullback open set' in X. When f is representable, if V is a scheme then U is too. Thus representable 1-morphisms are locally modelled on scheme morphisms, and we can transfer property **P** to stack 1-morphisms.

Definition

Define $\mathbf{Sta}_{\mathbb{K}}$ using the étale topology. We call $X \in \mathbf{Sta}_{\mathbb{K}}$ a *Deligne–Mumford* $\mathbb{K}$*-stack* if:

(i) The diagonal morphism $\Delta_X : X \to X \times X$ is representable.

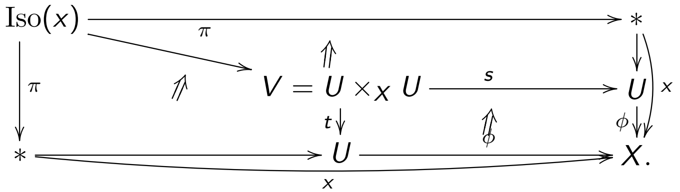
(ii) Δ_X is quasi-compact and separated.

(iii) There exists a $\mathbb{K}$-scheme $U \in \mathbf{Sch}_{\mathbb{K}} \subset \mathbf{Sta}_{\mathbb{K}}$ (called an *atlas*) and a morphism $\phi : U \to X$ which is étale and surjective.

We write $\mathbf{DMSta}_{\mathbb{K}} \subset \mathbf{Sta}_{\mathbb{K}}$ for the full 2-subcategory of Deligne–Mumford stacks.

Here Δ_X and ϕ are representable by (i), so the conditions on Δ_X, ϕ in (ii)–(iii) make sense. Think of $U = \coprod_{i \in I} U_i$ where $U_i \to X$ is an 'étale open set' (actually, $U \to X$ is one big 'étale open set'), and could take $U_i = \operatorname{Spec} A_i$ to be affine. So X admits an étale open cover by affine $\mathbb{K}$-schemes.

Let $x : * \to X$ be a $\mathbb{K}$-point, where $* = \operatorname{Spec} \mathbb{K}$. Write $\mathrm{Iso}_X(x)$ for the group of 2-morphisms $\eta : x \Rightarrow x$. As $\phi : U \to X$ is surjective, x factors via ϕ. There is a diagram of 2-Cartesian squares



(5.1)

As ϕ is representable, étale and quasicompact, so are s, t and π. As $\pi : \mathrm{Iso}(x) \to *$ is étale and quasicompact, we see that $\mathrm{Iso}(x)$ is a $\mathbb{K}$-scheme, discrete (as étale) and compact in the Zariski topology (as quasicompact), so $\mathrm{Iso}(x)$ is a finite group.
Thus, isotropy groups of Deligne–Mumford stacks are finite groups. Quotients $[U/\Gamma]$ for U a $\mathbb{K}$-scheme and Γ a finite group are Deligne–Mumford stacks.

Definition

Define $\mathbf{Sta}_{\mathbb{K}}$ using the smooth topology (technically should use the 'fppf topology' with an 'fppf open set' being a finitely presented flat morphism $U \to X$, because of better descent properties).
We call $X \in \mathbf{Sta}_{\mathbb{K}}$ an *Artin* $\mathbb{K}$*-stack* if:

(i) The diagonal morphism $\Delta_X : X \to X \times X$ is representable.

(ii) Δ_X is quasi-compact and separated.

(iii) There exists a $\mathbb{K}$-scheme $U \in \mathbf{Sch}_{\mathbb{K}} \subset \mathbf{Sta}_{\mathbb{K}}$ (called an *atlas*) and a morphism $\phi : U \to X$ which is smooth and surjective.

We write $\mathbf{Art}_{\mathbb{K}} \subset \mathbf{Sta}_{\mathbb{K}}$ for the full 2-subcategory of Artin stacks.

The argument of the previous slide using (5.1) shows that for each $\mathbb{K}$-point $x : * \to X$, $\mathrm{Iso}(x)$ is a smooth $\mathbb{K}$-scheme, and thus an algebraic $\mathbb{K}$-group, such as $\mathrm{GL}(n, \mathbb{K})$. Quotients $[U/\Gamma]$ for U a $\mathbb{K}$-scheme and Γ an algebraic $\mathbb{K}$-group are Artin $\mathbb{K}$-stacks.

Definition

A *groupoid object* (U, V, s, t, u, i, m) in a category $\mathscr{C}$, or simply *groupoid* in $\mathscr{C}$, consists of objects U, V in $\mathscr{C}$ and morphisms $s, t : V \to U$, $u : U \to V$, $i : V \to V$ and $m : V \times_{s,U,t} V \to V$ satisfying the identities

$$
\begin{gathered}
s \circ u = t \circ u = \mathrm{id}_U, \ \ s \circ i = t, \ \ t \circ i = s, \ \ s \circ m = s \circ \pi_2, \\
t \circ m = t \circ \pi_1, \ \ m \circ (i \times \mathrm{id}_V) = u \circ s, \ \ m \circ (\mathrm{id}_V \times i) = u \circ t, \\
m \circ (m \times \mathrm{id}_V) = m \circ (\mathrm{id}_V \times m) : V \times_U V \times_U V \longrightarrow V, \\
m \circ (\mathrm{id}_V \times u) = m \circ (u \times \mathrm{id}_V) : V = V \times_U U \longrightarrow V,
\end{gathered}
\tag{5.2}
$$

where we suppose all the fibre products exist.

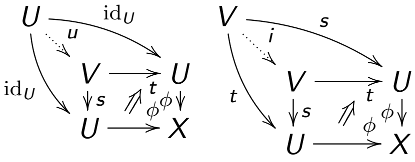
A groupoid in **Sets** is a groupoid in the usual sense, where U is the set of *objects*, V the set of *morphisms*, $s : V \to U$ the *source* of a morphism, $t : V \to U$ the *target* of a morphism, $u : U \to V$ the *unit* taking $X \mapsto \mathrm{id}_X$, i the *inverse* taking $f \mapsto f^{-1}$, and m the *multiplication* taking $(f, g) \mapsto f \circ g$ when $s(f) = t(g)$. Then (5.2) reduces to the usual axioms for a groupoid.

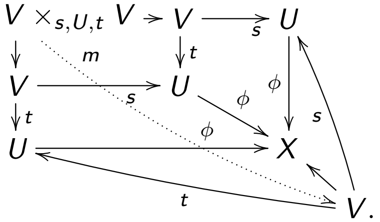
Stacks in terms of groupoids in schemes

Let X be a Deligne–Mumford or Artin stack, and $\phi : U \to X$ a choice of atlas. Form the 2-Cartesian square

$$\begin{array}{ccc} V = U \times_X U & \xrightarrow{\quad t \quad} & U \\ \downarrow s & \Uparrow & \phi \downarrow \\ U & \xrightarrow{\quad \phi \quad} & X \end{array} \tag{5.3}$$

Then V is a $\mathbb{K}$-scheme, and $s, t : V \to U$ are $\mathbb{K}$-scheme morphisms which are étale (for Deligne–Mumford) or smooth (for Artin). Define morphisms $u : U \to V$, $i : V \to U$, $m : V \times_U V \to V$ by the diagrams





Then (U, V, s, t, u, i, m) is a *groupoid in* $\mathbf{Sch}_{\mathbb{K}}$.

Thus, given a Deligne–Mumford or Artin $\mathbb{K}$-stack X, we can construct a (nonunique) groupoid (U, V, s, t, u, i, m) in $\mathbf{Sch}_{\mathbb{K}}$, where s, t are étale in the Deligne–Mumford case, and smooth in the Artin case. We can reconstruct X from this groupoid by taking (5.3) to be a 2-coCartesian square in $\mathbf{Sta}_{\mathbb{K}}$, so that X is a pushout $U \amalg_V U$. It is possible to develop the theory of stacks so that stacks are *defined* to be suitable groupoids in $\mathbf{Sch}_{\mathbb{K}}$, rather than functors. See for example Moerdijk, *Orbifolds as groupoids: an introduction*, math.DG/0203100 on this approach for orbifolds.

Here are two examples: suppose that $(s, t) : V \to U \times U$ is an embedding, making V into a subscheme of $U \times U$. Then we can regard V as the graph of an *equivalence relation* $\sim$ on U, where (5.2) is the equivalence relation axioms, and $X = U/\sim$ as the quotient of U by an equivalence relation. It is an algebraic space.

Suppose that a finite/algebraic group G acts on U by $\rho : G \times U \to U$. Take $V = G \times U$, $s = \pi_U$, $t = \rho$, $u = (1_G, \mathrm{id}_U)$, and i, m to come from inverses and multiplication in G. This defines a groupoid in $\mathbf{Sch}_{\mathbb{K}}$, the corresponding stack is $X = [U/G]$.

Derived Algebraic Geometry

Lecture 6 of 14: Abelian categories, coherent and quasicoherent sheaves

Dominic Joyce, Oxford University
Summer Term 2022

Good reference for this lecture:
Hartshorne, *Algebraic Geometry*.

These slides available at
`http://people.maths.ox.ac.uk/∼joyce/`

Plan of talk:

Introduction

Let X be a $\mathbb{K}$-scheme. Then we can define *algebraic vector bundles* $E \to X$. One way to define them is that E is a $\mathbb{K}$-scheme, with a smooth morphism $\pi : E \to X$, and the fibres $E_x = \pi^{-1}(x)$ of E are given a $\mathbb{K}$-vector space structure, and X may be covered by Zariski open $U \subseteq X$ such that $\pi^{-1}(U) \cong U \times \mathbb{A}^r$ for $r = \mathrm{rank}\, E$, such that $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is identified with the projection $U \times \mathbb{A}^r \to U$, and the vector space structure on the fibres E_u for $u \in U$ is identified with that on $\{u\} \times \mathbb{A}^r \cong \mathbb{A}^r$.
A second way to define vector bundles is as a special kind of coherent sheaf, see below.
If $E, F \to X$ are vector bundles, we can also define *morphisms* $\phi : E \to F$, which are scheme morphisms $\phi : E \to F$ such that $\pi \circ \phi = \pi$ and $\phi|_{E_x} : E_x \to F_x$ is linear for each $x \in X$. Write $\mathrm{Vect}(X)$ for the category of algebraic vector bundles on X.

Then $\mathrm{Vect}(X)$ is a $\mathbb{K}$-linear category (i.e. $\mathrm{Hom}(E,F)$ is a $\mathbb{K}$-vector space for all $E,F \in \mathrm{Vect}(X)$, and composition is bilinear), with a zero object $0 \to X$, direct sums $\oplus$, and tensor products $\otimes$. There is also a notion of *exact sequence*

$$0 \longrightarrow E \xrightarrow{\ \phi\ } F \xrightarrow{\ \psi\ } G \longrightarrow 0, \qquad (6.1)$$

in $\mathrm{Vect}(X)$, such that $\psi \circ \phi = 0$, and for each $x \in X$, the restriction of (6.1) to x is an exact sequence of $\mathbb{K}$-vector spaces. This makes $\mathrm{Vect}(X)$ into an *exact category*. However, an arbitrary morphism $\phi : E \to F$ may not have kernel or cokernel in $\mathrm{Vect}(X)$ — that is, ϕ may not extend to an exact sequence in $\mathrm{Vect}(X)$

$$0 \longrightarrow \mathrm{Ker}\,\phi \longrightarrow E \xrightarrow{\ \phi\ } F \longrightarrow \mathrm{Coker}\,\phi \longrightarrow 0,$$

as the dimensions of the kernel and cokernel of $\phi_x : E_x \to F_x$ may vary discontinuously with $x \in X$.

General Principle (Grothendieck)

It is better to work with a category that has nice properties, but which has nasty objects, than with a category that has nasty properties, but which has nice objects.

The category of *coherent sheaves* $\mathrm{coh}(X)$ is an enlargement of $\mathrm{Vect}(X)$ which is not just a $\mathbb{K}$-linear exact category, but an *abelian category* (it has a good notion of exact sequence, and every morphism has a kernel and a cokernel). It is in some sense the (or better, a) smallest abelian category containing $\mathrm{Vect}(X)$, and can be regarded as a 'completion' of $\mathrm{Vect}(X)$ by adding kernels and cokernels. Objects of $\mathrm{coh}(X)$ can be like singular vector bundles, or vector bundles supported on subschemes of X.

6.1 Abelian categories

Definition

Let $\mathscr{C}$ be a category, and $\mathbb{K}$ a field. We say that:

- $\mathscr{C}$ is *preadditive* if $\mathrm{Hom}(X, Y)$ has the structure of an abelian group for all $X, Y \in \mathscr{C}$, and composition is biadditive. It is $\mathbb{K}$-*linear* if instead $\mathrm{Hom}(X, Y)$ is a $\mathbb{K}$-vector space and composition is bilinear.
- $\mathscr{C}$ is *additive* if it is preadditive, and it has a *zero object* $0 \in \mathscr{C}$ which is both an initial and a terminal object, and for all $X, Y \in \mathscr{C}$ there is a *direct sum* $X \oplus Y$ which is both a product (fibre product $X \times_0 Y$ over 0) and a coproduct (pushout $X \amalg_0 Y$ over 0).
- $\mathscr{C}$ is *preabelian* if it is additive and every morphism $f : X \to Y$ has both a kernel and a cokernel.
- $\mathscr{C}$ is *abelian* if it is preabelian, and every injective morphism is a kernel, and every surjective morphism is a cokernel.

Note here that several concepts which are familiar from linear algebra actually have purely category-theoretic definitions, and so make sense in every (additive etc.) category.

For example, let $\mathscr{C}$ be a preadditive category. Then $\mathrm{Hom}(X, Y)$ is an abelian group for $X, Y \in \mathscr{C}$, so there is a unique *zero morphism* $0 : X \to Y$. Then for any morphism $f : X \to Y$, a *kernel* $\kappa : K \to X$ for f is an *equalizer* for $f : X \to Y$ and $0 : X \to Y$. That is, $f \circ \kappa = 0 \circ \kappa$, and κ has the universal property that if $\kappa' : K' \to X$ has $f \circ \kappa' = 0 \circ \kappa'$ then there is a unique morphism $\iota : K' \to K$ with $\kappa' = \kappa \circ \iota$. Similarly, a *cokernel* $\gamma : Y \to C$ for f is a coequalizer for $f : X \to Y$ and $0 : X \to Y$, satisfying the dual universal property. Kernels and cokernels need not exist, but if they do they are unique up to canonical isomorphism.

If $\mathscr{C}$ is preadditive with a zero object, a morphism $f : X \to Y$ is *injective* if its kernel is the unique morphism $0 : 0 \to X$, and *surjective* if its cokernel is $0 : Y \to 0$. This has nothing to do with injectivity of maps of sets, they are purely category-theoretic.

Example 6.1

- The category **Ab** of abelian groups is an abelian category.
- The category $\mathbf{Vect}_{\mathbb{K}}$ of $\mathbb{K}$-vector spaces over a field $\mathbb{K}$ is $\mathbb{K}$-linear abelian.
- If R is a ring, the category R-**mod** of (left) R-modules is abelian.
- If A is a $\mathbb{K}$-algebra, the category A-**mod** of A-modules is $\mathbb{K}$-linear abelian.
- There are lots of abelian full subcategories of the above with objects satisfying suitable conditions. For example, the subcategories of finite abelian groups, and of finitely generated abelian groups in **Ab**, and of finite-dimensional $\mathbb{K}$-vector spaces in $\mathbf{Vect}_{\mathbb{K}}$, and if R is a noetherian commutative ring, the subcategory of finitely generated R-modules in R-**mod**, are all abelian.

In general, categories of (sheaves of) modules tend to be abelian. An advantage of abelian categories is that lots of homological algebra (exact sequences etc.) works nicely in them.

6.2. Coherent and quasicoherent sheaves

Let X be a $\mathbb{K}$-scheme. Then we have a category $\mathrm{Vect}(X)$ of algebraic vector bundles. It is a $\mathbb{K}$-linear additive category which is an *exact category* (it has a distinguished class of 'short exact sequences' $0 \to E \to F \to G \to 0$ satisfying some axioms), but it is not abelian if $\dim X > 0$. We would like to embed $\mathrm{Vect}(X)$ into an abelian category $\mathrm{coh}(X)$, where $\mathrm{coh}(X)$ is 'as small as possible'. To do this we first rewrite $\mathrm{Vect}(X)$ as a category of modules.

Recall that X has a *structure sheaf* $\mathcal{O}_X$, a sheaf of commutative $\mathbb{K}$-algebras. Thus we can consider $\mathcal{O}_X$*-modules* $\mathcal{E} \to X$, which are sheaves (in the Zariski topology) in which for each open $U \subseteq X$, $\mathcal{E}(U)$ has the structure of a module over the $\mathbb{K}$-algebra $\mathcal{O}_X(U)$, and for $V \subseteq U \subseteq X$ the restriction morphism $\rho^{\mathcal{E}}_{UV} : \mathcal{E}(U) \to \mathcal{E}(V)$ is a module morphism over $\rho^{\mathcal{O}_X}_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$.

Such $\mathcal{O}_X$-modules form an abelian category $\mathcal{O}_X$-**mod**. This is not trivial: for an exact sequence $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ in $\mathcal{O}_X$-**mod**, if $U \subseteq X$ is open, it is generally *not* true that $0 \to \mathcal{E}(U) \to \mathcal{F}(U) \to \mathcal{G}(U) \to 0$ is exact in the abelian category $\mathcal{O}_X(U)$-**mod** ($\mathcal{F}(U) \to \mathcal{G}(U)$ may not be surjective), so we can't just deduce the abelian category axioms from the abelian category property of $\mathcal{O}_X(U)$-**mod** for each open $U \subseteq X$. However, for each (scheme-theoretic) point $x \in X$, the sequence of stalks $0 \to \mathcal{E}_x \to \mathcal{F}_x \to \mathcal{G}_x \to 0$ is exact in the abelian category $\mathcal{O}_{X,x}$-**mod**, and we prove $\mathcal{O}_X$-**mod** abelian using properties of stalks.

The action of $\mathcal{O}_X$ on itself makes $\mathcal{O}_X$ into an object in $\mathcal{O}_X$-**mod**.

If $\pi : E \to X$ is an algebraic vector bundle (defined as a scheme morphism with extra structure), we can define the *sheaf of sections* $\mathcal{E}$ of X, such that $\mathcal{E}(U)$ is the $\mathcal{O}_X(U)$-module of sections $s : U \to E|_U$ with $\pi|_U \circ s = \mathrm{id}_U$ of $E|_U$. Then $\mathcal{E} \in \mathcal{O}_X$-**mod**. An object of $\mathcal{O}_X$-**mod** comes from an algebraic vector bundle iff it is a *locally free sheaf*, i.e. it is locally isomorphic to $\mathcal{O}_X^{\oplus^r} = \mathcal{O}_X \oplus \cdots \oplus \mathcal{O}_X$.

The category $\mathcal{O}_X$-**mod** is too large, and its objects too horrible (infinite-dimensional, etc.) to work with. We will define better behaved abelian subcategories $\mathrm{coh}(X) \subset \mathrm{qcoh}(X) \subset \mathcal{O}_X$-**mod**, such that $\mathrm{Vect}(X)$ (considered as a subcategory of $\mathcal{O}_X$-**mod**) is a subcategory of $\mathrm{coh}(X)$.

We define $\mathbb{K}$-schemes using the spectrum functor $\mathrm{Spec} : \mathbf{Alg}_{\mathbb{K}}^{\mathrm{op}} \to \mathbf{RingedSpaces}_{\mathbb{K}}$. For each $A \in \mathbf{Alg}_{\mathbb{K}}$ with $\mathrm{Spec}\, A = U$, an affine scheme, there is a corresponding *module spectrum functor* $\mathrm{MSpec} : A$-**mod**$\to \mathcal{O}_U$-**mod**. For a $\mathbb{K}$-scheme X, we call $\mathcal{E}$ in $\mathcal{O}_X$-**mod** *quasi-coherent* if we can cover X by open affine $U \subseteq X$ with $\mathcal{E}|_U \cong \mathrm{MSpec}\, M$ for some $\mathcal{O}_X(U)$-module M. When X is *locally noetherian* (assume this from now on, e.g. it holds for quasiprojective $\mathbb{K}$-schemes), we call $\mathcal{E}$ *coherent* if the same is true with M a finitely generated $\mathcal{O}_X(U)$-module.

Locally on X, any quasicoherent sheaf $\mathcal{E}$ may be written as a cokernel $\mathcal{O}_X \otimes_{\mathbb{K}} V \to \mathcal{O}_X \otimes_{\mathbb{K}} W \to \mathcal{E} \to 0$, where V, W are possibly infinite-dimensional $\mathbb{K}$-vector spaces. A coherent sheaf is the same, but with V, W finite-dimensional. Thus, a coherent sheaf looks locally like the cokernel of a morphism of vector bundles. Think of vector bundles of infinite rank as typical objects in $\mathrm{qcoh}(X)$. Although the objects of $\mathrm{qcoh}(X)$ are not nice, the category $\mathrm{qcoh}(X)$ has some good categorical properties (for example, arbitrary infinite direct sums exist) which make it useful for some constructions.

If X is a projective $\mathbb{K}$-scheme, we can characterize $\mathrm{coh}(X)$ as the full subcategory of compact objects in $\mathrm{qcoh}(X)$ (an object $\mathcal{E} \in \mathrm{qcoh}(X)$ is *compact* if $\mathrm{Hom}(E, -)$ commutes with infinite filtered colimits in $\mathrm{qcoh}(X), \textbf{Sets}$).

6.3. Examples and properties of coherent sheaves

Example

Let X be a $\mathbb{K}$-scheme and $x \in X$ be a $\mathbb{K}$-point. We define the *skyscraper sheaf* $\mathcal{O}_x$, a coherent sheaf, by, for all open $U \subseteq X$

$$\mathcal{O}_x(U) = \begin{cases} \mathbb{K}, & x \in U, \\ 0, & x \notin U. \end{cases}$$

Define the action of $\mathcal{O}_X(U)$ on $\mathcal{O}_x(U)$ by $f \in \mathcal{O}_X(U)$ acts by multiplication by $f(x) \in \mathbb{K}$ if $x \in U$, and f acts trivially if $x \notin U$.

Every $\mathcal{E} \in \mathrm{coh}(X)$ has a *support* $\mathrm{supp}\,\mathcal{E} \subseteq X$, the smallest closed subset in the Zariski topology such that $\mathcal{E}|_{X \setminus \mathrm{supp}\,\mathcal{E}} \cong 0$. Then $\mathrm{supp}\,\mathcal{O}_x = \{x\}$. Take X to be smooth projective. The *dimension* of $\mathcal{E}$ is $\dim \mathrm{supp}\,\mathcal{E}$. Then $\dim \mathcal{O}_x = 0$. We call $\mathcal{E}$ a *torsion sheaf* if $\dim \mathcal{E} < \dim X$. We call $\mathcal{E}$ *torsion-free* if $\mathcal{E}$ has no nonzero subobject $0 \neq \mathcal{F} \subseteq \mathcal{E}$ with $\mathcal{F}$ a torsion sheaf.

Example

Let $[x, y] \in \mathbb{CP}^1$. There is a nonzero section $s \in H^0(\mathcal{O}_{\mathbb{CP}^1}(1))$, unique up to scale, such that $s|_{[x,y]} = 0$. Then we have an exact sequence in $\mathrm{coh}(X)$

$$0 \longrightarrow \mathcal{O}_{\mathbb{CP}^1} \xrightarrow{s} \mathcal{O}_{\mathbb{CP}^1}(1) \longrightarrow \mathcal{O}_{[x,y]} \longrightarrow 0.$$

Torsion sheaves often appear as cokernels of morphisms of vector bundles like this. But a torsion sheaf cannot be the kernel of a morphism of vector bundles; only torsion-free sheaves can. We can think of $\mathrm{coh}(X)$ (say for projective X) as the abelian category made by starting with $\mathrm{Vect}(X)$ and adding cokernels of all morphisms. We could instead have added all kernels; the result would have been the opposite category $\mathrm{coh}(X)^{\mathrm{op}}$, with the inclusion $\mathrm{Vect}(X) \hookrightarrow \mathrm{coh}(X)^{\mathrm{op}}$ mapping $\mathcal{E}$ to its dual $\mathcal{E}^*$.

If $k \geqslant 1$ there is a coherent sheaf $\mathcal{O}_{[x,y]}^{(k)}$ in an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{CP}^1} \xrightarrow{s^{\otimes^k}} \mathcal{O}_{\mathbb{CP}^1}(k) = \mathcal{O}_{\mathbb{CP}^1}(1)^{\otimes^k} \longrightarrow \mathcal{O}_{[x,y]}^{(k)} \longrightarrow 0,$$

with $\mathcal{O}_{[x,y]}^{(k)}(U) \cong \mathbb{C}^k$ if $x \in U$ and $\mathcal{O}_{[x,y]}^{(k)}(U) = 0$ otherwise.

Proposition (Classification of coherent sheaves on $\mathbb{CP}^1$.)

Every $\mathcal{E} \in \mathrm{coh}(\mathbb{CP}^1)$ is a finite direct sum of objects $\mathcal{O}_{\mathbb{CP}^1}(n)$ for $n \in \mathbb{Z}$ and $\mathcal{O}_{[x,y]}^{(k)}$ for $[x, y] \in \mathbb{CP}^1$ and $k \geqslant 1$. The summands are unique up to order. Also $\mathcal{E}$ is a vector bundle, and torsion-free, iff there are no summands $\mathcal{O}_{[x,y]}^{(k)}$, and $\mathcal{E}$ is torsion iff there are no summands $\mathcal{O}_{\mathbb{CP}^1}(n)$.

Example (Ideal sheaf of a point.)

Let X be a $\mathbb{K}$-scheme and $x \in X$ be a $\mathbb{K}$-point. Define the *ideal sheaf* I_x to be the subsheaf of sections s of $\mathcal{O}_X$ such that $s|_x = 0$. Then I_x is torsion-free and there is an exact sequence

$$0 \longrightarrow I_x \xrightarrow{\mathrm{inc}} \mathcal{O}_X \longrightarrow \mathcal{O}_x \longrightarrow 0.$$

If $\dim X = 1$ then I_x is a vector bundle, e.g. if $X = \mathbb{CP}^1$ then $I_{[x,y]} \cong \mathcal{O}_{\mathbb{CP}^1}(-1)$. But if $\dim X \geqslant 2$ then I_x is not a vector bundle, but a torsion-free sheaf with a singularity at x.

Example (Torsion-free sheaves as limits of vector bundles.)

Let $\mathbb{CP}^2$ have homogeneous coordinates $[x, y, z]$, so that $H^0(\mathcal{O}_{\mathbb{CP}^2}(1)) = \langle x, y, z\rangle_{\mathbb{C}}$. Let $\delta \in \mathbb{C}$, and consider the sheaf $\mathcal{E}_\delta$ defined by the exact sequence in $\mathrm{coh}(\mathbb{CP}^2)$

$$0 \longrightarrow \mathcal{O}_{\mathbb{CP}^2} \xrightarrow{(y,z,\delta\,\mathrm{id})} \mathcal{O}_{\mathbb{CP}^2}(1) \oplus \mathcal{O}_{\mathbb{CP}^2}(1) \oplus \mathcal{O}_{\mathbb{CP}^2} \longrightarrow \mathcal{E}_\delta \longrightarrow 0.$$

Then $\mathcal{E}_\delta$ is a vector bundle isomorphic to $\mathcal{O}_{\mathbb{CP}^2}(1) \oplus \mathcal{O}_{\mathbb{CP}^2}(1)$ if $\delta \neq 0$, and a torsion-free sheaf, non-vector bundle isomorphic to $(I_{[1,0,0]} \otimes \mathcal{O}_{\mathbb{CP}^2}(2)) \oplus \mathcal{O}_{\mathbb{CP}^2}$ if $\delta = 0$. This shows that torsion-free sheaves which are not locally free can be limits of vector bundles.

One reason to be interested in coherent sheaves is that, say when X is a smooth projective $\mathbb{K}$-scheme of dimension $\geqslant 2$, moduli spaces of vector bundles are generally noncompact, but (coarse) moduli schemes of (Gieseker semistable) coherent sheaves with fixed Chern character are proper (compact and Hausdorff).
So, we can compactify moduli spaces of (semistable) algebraic vector bundles by extending them to moduli spaces of (semistable) torsion-free coherent sheaves, which may be thought of algebraic vector bundles with mild singularities. Being able to form compact (proper) moduli schemes is essential in defining invariants.