# Derived Algebraic Geometry

Lecture 9 of 14: $\infty$-categories

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Helpful references for these two lectures:

E. Riehl, *'Homotopical categories: from model categories to $(\infty,1)$-categories'*, arXiv/1904.00886.

M. Groth, *'A short course on $\infty$-categories'*, arXiv/1007.2925.

These slides available at
`http://people.maths.ox.ac.uk/∼joyce/`

Plan of talk:

# Introduction

In this lecture '$\infty$-category' always means '$(\infty, 1)$-category', that is, all $n$-morphisms are invertible for $n \geqslant 2$. (Although '$n$-morphism' may not make sense, depending on your model for $\infty$-categories.)

There are a bunch of different but related structures which are more-or-less kinds of $\infty$-category:

- Model categories.
- Categories enriched in topological spaces.
- Simplicial categories; simplicial model categories.
- Quasicategories.
- . . . .

Of these, model categories are the oldest (Quillen 1967), and look least like an $\infty$-category (they have no visible higher morphisms). But most of the other kinds of $\infty$-category use model categories under the hood. Toën–Vezzosi's DAG is written in terms of model categories and simplicial categories. Lurie works with quasicategories, which may be the best/coolest version.

To see why model categories belong in the list, consider:

**General Principle**

If you start with an ordinary category $\mathscr{C}$ and invert some class of morphisms $\mathscr{W}$ in $\mathscr{C}$ ('weak equivalences'), the result $\mathscr{C}[\mathscr{W}^{-1}]$ should really be an $\infty$-**category** ($(\infty,1)$-category), with homotopy category $\mathrm{Ho}(\mathscr{C}[\mathscr{W}^{-1}])$ an ordinary category.

I'll return to this in 'Dwyer–Kan localization'. Model categories give you $\mathscr{C},\mathscr{W}$ and some other data, and give you techniques for computing things in $\mathrm{Ho}(\mathscr{C}[\mathscr{W}^{-1}])$, plus some things which secretly come from the (not explicitly defined) $\infty$-category $\mathscr{C}[\mathscr{W}^{-1}]$, e.g. you can define 'homotopy fibre products', which are really fibre products in the $\infty$-category $\mathscr{C}[\mathscr{W}^{-1}]$, pushed down to $\mathrm{Ho}(\mathscr{C}[\mathscr{W}^{-1}])$. As an example, consider derived categories $D(\mathcal{A})$ constructed from $\mathrm{Ho}(\mathrm{Com}(\mathcal{A}))$ by inverting the class $\mathscr{W}$ of quasi-isomorphisms. Really $D(\mathcal{A})=\mathrm{Ho}(\mathbb{D}(\mathcal{A}))$ for a stable $\infty$-category $\mathbb{D}(\mathcal{A})$.

Recall that a (2,1)-category $\mathfrak{C}$ is basically a 'category enriched in groupoids', that is, $\mathrm{Hom}(X, Y)$ is a groupoid rather than a set for objects $X, Y \in \mathfrak{C}$, where objects and morphisms in $\mathrm{Hom}(X, Y)$ are 1- and 2-morphisms in $\mathfrak{C}$.

Similarly, an $(\infty, 1)$-category is really a 'category enriched in $\infty$-groupoids'. But what is an $\infty$-groupoid? Two models for the (model/$\infty$-)category of $\infty$-groupoids are topological spaces **Top** (up to homotopy), and simplicial sets **SSets**. This is why categories enriched in topological spaces, and simplicial categories, are possible definitions of $\infty$-category.

As **Top** and **SSets** are Quillen equivalent as model categories, theories of $\infty$-categories based on **Top** and **SSets** are essentially equivalent. But noone uses categories enriched in **Top** except as motivation, as far as I know.

# 9.1. Model categories

Model categories were invented by Quillen to abstract methods of homotopy theory into category theory; a model category is one in which one can 'do homotopy theory'.

## Definition

A *model category* is a complete and cocomplete category $\mathscr{M}$ (i.e. all small limits and colimits exist) equipped with three distinguished classes of morphisms, the *weak equivalences* $\mathscr{W}$, the *fibrations* $\mathscr{F}$, and the *cofibrations* $\mathscr{C}$. These must satisfy:

(i) $\mathscr{W}, \mathscr{F}, \mathscr{C}$ are closed under composition and include identities.

(ii) $\mathscr{W}, \mathscr{F}, \mathscr{C}$ are closed under retracts. Here $f$ is a *retract* of $g$ if there exist $i, j, r, s$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & \mathrm{id}_X & & \\ X & \xrightarrow{\;i\;} & Y & \xrightarrow{\;r\;} & X \\ f\downarrow & & \downarrow g & & \downarrow f \\ X' & \xrightarrow{\;j\;} & Y' & \xrightarrow{\;s\;} & X'. \\ & & \mathrm{id}_{X'} & & \end{array}$$

## Definition (Continued.)

(iii) For $f : X \to Y$, $g : Y \to Z$ in $\mathscr{M}$, if two of $f, g, g \circ f$ are in $\mathscr{W}$ then so is the third.

(iv) A (co)fibration which is also a weak equivalence is called *acyclic*. Acyclic cofibrations have the *left lifting property* with respect to fibrations, and cofibrations have the left lifting property with respect to acyclic fibrations. Explicitly, if the square below commutes, where $i$ is a cofibration, $p$ is a fibration, and $i$ or $p$ is acyclic, then there exists $h$ as shown:

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & X \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow{\quad g \quad} & Y. \end{array}$$

(v) Every morphism $f$ in $\mathscr{M}$ can be written as $f = p \circ i$ for a fibration $p$ and an acyclic cofibration $i$.

(vi) Every morphism $f$ in $\mathscr{M}$ can be written as $f = p \circ i$ for an acyclic fibration $p$ and a cofibration $i$.

Two of $\mathscr{W}, \mathscr{C}, \mathscr{F}$ determine the third, so it is enough to define two.

## Example 9.1

(a) The category **Top** of topological spaces has a model structure with $\mathscr{W}$ the weak homotopy equivalences, and $\mathscr{F}$ the Serre fibrations (maps with the homotopy lifting property for CW complexes).

(b) If $R$ is a commutative ring then $\mathrm{Com}(R\text{-mod})$ has a model structure with weak equivalences quasi-isomorphisms and fibrations morphisms $\phi : E^\bullet \to F^\bullet$ with $\phi^k : E^k \to F^k$ surjective for all $k \in \mathbb{Z}$.

(c) Let $\mathcal{A}$ be a *Grothendieck abelian category* (an abelian category with extra conditions on coproducts and limits; includes $\mathrm{qcoh}(X)$ for any scheme $X$). Then $\mathrm{Com}(\mathcal{A})$ has a model structure with weak equivalences quasi-isomorphisms and cofibrations morphisms $\phi : E^\bullet \to F^\bullet$ with $\phi^k : E^k \to F^k$ injective for all $k \in \mathbb{Z}$.

## Definition

Any model category $\mathscr{M}$ has an *initial object* $\emptyset$ and a *terminal object* $*$. An object $X$ in $\mathscr{M}$ is called *fibrant* if $X \to *$ lies in $\mathscr{F}$, and *cofibrant* if $\emptyset \to X$ lies in $\mathscr{C}$.

If $X \in \mathscr{M}$ and there is a weak equivalence $w : C \to X$ with $C$ cofibrant then $C$ is a *cofibrant replacement* for $X$. If there is a weak equivalence $w : X \to F$ with $F$ fibrant then $F$ is a *fibrant replacement* for $X$. Such replacements always exist.

If $X \in \mathscr{M}$, a *cylinder object* is an object $X \times [0,1]$ in $\mathscr{M}$ with a factorization $X \amalg X \xrightarrow{c} X \times [0,1] \xrightarrow{w} X$ of the codiagonal $X \amalg X \to X$, with $c$ a cofibration and $w$ a weak equivalence. Cylinder objects exist by (vi).

A *path object* is an object $\mathrm{Map}([0,1], X)$ in $\mathscr{M}$ with a factorization $X \xrightarrow{w} \mathrm{Map}([0,1], X) \xrightarrow{f} X \times X$ of the diagonal $X \to X \times X$, with $w$ a weak equivalence and $f$ a fibration. Path objects exist by (v).

Morphisms $f, g : X \to Y$ are called *(left) homotopy equivalent* if there exists $h : X \times [0,1] \to Y$ with $h \circ c = f \amalg g$.

## Definition

The *homotopy category* is $\mathrm{Ho}(\mathscr{M}) = \mathscr{M}[\mathscr{W}^{-1}]$, the category obtained from $\mathscr{M}$ by formally inverting all weak equivalences. Note that this is independent of $\mathscr{C}, \mathscr{F}$.

The problem with $\mathscr{M}[\mathscr{W}^{-1}]$ is that it is difficult to say what the morphisms are. However, we have:

## Theorem (Fundamental Theorem of Model Categories.)

$\mathrm{Ho}(\mathscr{M})$ *is equivalent to the category whose objects are fibrant-cofibrant objects in* $\mathscr{M}$*, and whose morphisms are homotopy classes of morphisms in* $\mathscr{M}$*.*

That is, if we restrict to fibrant-cofibrant objects $X, Y$ then morphisms $\mathrm{Hom}(X, Y)$ in $\mathscr{M}[\mathscr{W}^{-1}]$ are easy to compute, and every object in $\mathscr{M}[\mathscr{W}^{-1}]$ is isomorphic to a fibrant-cofibrant object. In Example 9.1, the homotopy category $\mathrm{Ho}(\mathscr{M})$ is (a) the *homotopy category* **hTop** of homotopy types and (b),(c) the derived categories $D(R\text{-mod})$ and $D(\mathcal{A})$.

# Homotopy fibre products and homotopy pushouts

In **Top**, fibre products $X \times_{f,Z,g} Y$ and pushouts $X \amalg_{f,Z,g} Y$ exist, but if we change $f, g$ by homotopies, then $X \times_{f,Z,g} Y$, $X \amalg_{f,Z,g} Y$ need not stay homotopy equivalent. So fibre products and pushouts in **Top** are the wrong idea for homotopy theory. Instead one defines the *homotopy fibre product*

$$X \times^h_{f,Z,g} Y = X \times_{f,Z,\rho_0} \mathrm{Map}([0,1], Z) \times_{\rho_1,Z,g} Y,$$

where $\rho_i : \mathrm{Map}([0,1], Z) \to Z$ are evaluation at $i = 0, 1$ in $[0,1]$, and fibre products on the right are in **Top**. Then $X \times^h_{f,Z,g} Y$ is homotopy invariant, so the construction descends to $\mathrm{Ho}(\textbf{Top})$. Similarly one defines the *homotopy pushout*

$$X \amalg^h_{f,Z,g} Y = X \amalg_{f,Z,\iota_0} ([0,1] \times Z) \amalg_{\iota_1,Z,g} Y,$$

where $\iota_i : Z \to [0,1] \times Z$ maps $\iota_i : z \mapsto (i, z)$.

Since path objects $\mathrm{Map}([0,1],Z)$ and cylinder objects $[0,1]\times Z$ make sense in any model category $\mathscr{M}$, we can make the same definitions of homotopy fibre product and homotopy pushout in $\mathscr{M}$. What do these mean? In both $\mathrm{Ho}(\mathbf{Top})$ and $\mathrm{Ho}(\mathscr{M})$, in general homotopy fibre products and pushouts are *not* fibre products and pushouts (do not satisfy universal properties) in either the ordinary categories $\mathbf{Top},\mathscr{M}$, or the ordinary categories $\mathrm{Ho}(\mathbf{Top}),\mathrm{Ho}(\mathscr{M})$. Instead, the correct interpretation is that there are secretly $(\infty,1)$-categories $\mathbf{Top}^\infty$, $\mathscr{M}^\infty$ with homotopy categories $\mathrm{Ho}(\mathbf{Top}),\mathrm{Ho}(\mathscr{M})$, and homotopy fibre products and pushouts are actually $\infty$-*category fibre products and pushouts* in $\mathbf{Top}^\infty$, $\mathscr{M}^\infty$. While the input data $X,Y,Z,f,g$ for an $\infty$-category fibre product makes sense in the homotopy category $\mathrm{Ho}(\mathbf{Top}^\infty),\mathrm{Ho}(\mathscr{M}^\infty)$, the fibre product is characterized by a universal property involving $n$-morphisms in $\mathbf{Top}^\infty,\mathscr{M}^\infty$ for all $n$.
Thus model category techniques effectively give ways to do constructions in an $\infty$-category, without defining $\infty$-categories.

## General Principle

Model categories are strict forms of $(\infty, 1)$-categories.

That is, composition of morphisms in a model category is strictly associative (other notions of $(\infty, 1)$-category have composition non-associative, or even not uniquely defined).

There are 'strictification theorems' which allow you to pass from looser forms of $(\infty, 1)$-categories (e.g. Segal categories) to model categories. Typically, general constructions are done in the looser kinds of $(\infty, 1)$-category, and explicit computations are done in model categories.

# 9.2. Simplicial sets

## Definition

The *simplex category* $\Delta$ has objects set $[n]=\{0,1,\ldots,n\}$ for $n\geqslant 0$ and morphisms $f:[n]\rightarrow[m]$ the order preserving functions, that is, if $0\leqslant i\leqslant j\leqslant n$ then $f(i)\leqslant f(j)$.
A *simplicial set* is a functor $F:\Delta^{\mathrm{op}}\rightarrow$ **Sets**. A *morphism of simplicial sets* $\eta:F\rightarrow G$ is a natural transformation of functors $\eta:F\Rightarrow G$. This makes simplicial sets into a category **SSets**.

## Definition

The *topological n-simplex* $\Delta^n_{\mathrm{top}}$ is

$$\Delta^n_{\mathrm{top}}=\bigl\{(x_0,\ldots,x_n)\in\mathbb{R}^{n+1}:x_i\geqslant 0,\;\; x_0+\cdots+x_n=1\bigr\}.$$

If $f:[n]\rightarrow[m]$ is order-preserving we define $f_{\mathrm{top}}:\Delta^m_{\mathrm{top}}\rightarrow\Delta^n_{\mathrm{top}}$ by

$$f_{\mathrm{top}}(x_0,\ldots,x_m)=(y_0,\ldots,y_n),\quad y_i=\textstyle\sum_{j\in\{0,\ldots,m\}:f(j)=i}x_j.$$

This defines a functor $G:\Delta^{\mathrm{op}}\rightarrow$ **Top** mapping $[n]\mapsto\Delta^n_{\mathrm{top}}$, $f\mapsto f_{\mathrm{top}}$.

### Definition

Let $X$ be a (compactly-generated, Hausdorff) topological space. A *triangulation* of $X$ is data $(f_i^n:\Delta^n_{\rm top}\rightarrow X)_{i\in I^n}$ for $n\geqslant 0$, where $I^n$ is an indexing set, such that $f_i^n$ is a homeomorphism with a closed subset $\mathrm{Im}\, f_i^n\subseteq X$, and for each $j=0,\ldots,n$ the restriction of $f_i^n$ to the face $x_j=0$ of $\Delta^n_{\rm top}$ is $f_{i'}^{n-1}$ for some $i'\in I^{n-1}$, that is,

$$f_i^n(x_0,\ldots,x_{j-1},0,x_{j+1},\ldots,x_n)=f_{i'}^{n-1}(x_0,\ldots,x_{j-1},x_{j+1},\ldots,x_n),$$

and $X=\coprod_{n\geqslant 0}\coprod_{i\in I^n}f_i^n((\Delta^n_{\rm top})^\circ)$ is a stratification of $X$ into locally closed sets, satisfying some topological conditions.

Think of a surface divided into triangles.

## Definition

Let $F : \Delta^{\rm op} \to \mathbf{Sets}$ be a simplicial set. We define a topological space $X_F$ with a triangulation, the *topological realization* of $F$, by

$$X_F = \left(\textstyle\coprod_{n\geqslant 0} F([n]) \times \Delta^n_{\rm top}\right) / \sim,$$

where the equivalence relation $\sim$ is generated by

$$(s, f_{\rm top}(x_0, \ldots, x_m)) \simeq (F(f)s, (x_0, \ldots, x_m)),$$

for all $f : [n] \to [m]$ in $\Delta$, $s \in F([n])$ and $(x_0, \ldots, x_m) \in \Delta^m_{\rm top}$. The maps $f_i^n : \Delta^n_{\rm top} \to X$ in the triangulation of $X$ come from the inclusions $\Delta^n_{\rm top} \to \{s\} \times \Delta^n_{\rm top}$ for $s \in F([n])$ which are *nondegenerate*, that is, $s$ is not in the image of $F(g) : F([m]) \to F([n])$ for some $m < n$ and $g : [n] \to [m]$.

In this way we can define a *topological realization functor* $\mathbf{TR} : \mathbf{SSets} \to \mathbf{CGHaus} \subset \mathbf{Top}$, for $\mathbf{CGHaus}$ the subcategory of compactly generated Hausdorff spaces. It is a right Kan extension

$$\begin{array}{ccc} \Delta^{\rm op} & \xrightarrow{\text{Yoneda embedding}} & \mathbf{SSets} \\ & \searrow^{G} & \downarrow{\scriptstyle \mathbf{TR}} \\ & & \mathbf{CGHaus}. \end{array}$$

We can think of a simplicial set $F:\Delta^{\rm op}\to$ **Sets** as basically a topological space $X_F$ with a triangulation. However, $F([n])$ is not the set $I^n$ of $n$-simplices of the triangulation of $X_F$, because of *degenerate simplices*: $F([n])$ must also contain points coming from $m$-simplices of $X_F$ for $m=0,1\ldots,n-1$, corresponding to all surjective simplicial maps $\Delta^n_{\rm top}\to\Delta^m_{\rm top}$. For example, when $m<n$ the maps $f:[m]\to[n]$, $g:[n]\to[m]$ given by $f(i)=i$ and $g(j)=\max(i,m)$ have $g\circ f=\mathrm{id}_{[m]}$. Hence $F(g):F([m])\to F([n])$ has a left inverse $F(f)$, and is injective, so $F([n])$ is at least as big as $F([m])$ for all $n\geqslant m$.

So even a simplicial set $F:\Delta^{\rm op}\to$ **Sets** corresponding to a topological space with finitely simplices is quite a lot of data: we have $|F([n])|\to\infty$ as $n\to\infty$.

There is a good notion of *products* of simplicial sets (as fibre products over the terminal object in **SSets**). In topological realizations, this involves dividing $\Delta^m_{\rm top}\times\Delta^n_{\rm top}$ into $m+n$-simplices.

# The singular functor

There is also a functor **Sing** : **Top** $\to$ **SSets** which maps a space $X$ to the functor $F : \Delta^{\mathrm{op}} \to$ **Sets** with $F([n]) = \mathrm{Map}_{C^0}(\Delta^n_{\mathrm{top}}, X)$ for objects $[n]$ and $F(f) = - \circ f_{\mathrm{top}} : \mathrm{Map}_{C^0}(\Delta^n_{\mathrm{top}}, X) \to \mathrm{Map}_{C^0}(\Delta^m_{\mathrm{top}}, X)$ for morphisms $f : [m] \to [n]$, so that $f_{\mathrm{top}} : \Delta^m_{\mathrm{top}} \to \Delta^n_{\mathrm{top}}$.

There are model category structures on **Top** and **SSets** such that **TR** : **SSets** $\to$ **Top** and **Sing** : **Top** $\to$ **SSets** are homotopy inverses (**Top** and **SSets** are Quillen equivalent model categories), and the homotopy categories $\mathrm{Ho}($**Top**$)$ and $\mathrm{Ho}($**SSets**$)$ are equivalent categories.

The weak equivalences in the model category **SSets** are morphisms $\eta$ for which **TR**$(\eta)$ is a weak homotopy equivalence of topological spaces. The fibrations are 'Kan fibrations', and the cofibrations morphisms $\eta : F \Rightarrow G$ such that $\eta([n]) : F([n]) \to G([n])$ is injective for all $n$. All simplicial sets are cofibrant. The fibrant objects are called 'Kan complexes'.

The category **SSets** is used as a model for ∞-groupoids.

# 9.3. Simplicial categories

## Definition

A *simplicial category* $\mathscr{S}$ is a category enriched in simplicial sets. That is, $\mathscr{S}$ is a 'category' in which for all objects $X, Y$ in $\mathscr{S}$, the morphisms $\mathrm{Hom}(X, Y)$ is a simplicial set, and composition $\mu_{X,Y,Z} : \mathrm{Hom}(Y, Z) \times \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z)$ is a morphism of simplicial sets. Composition is strictly associative, that is, $\mu_{W,Y,Z} \circ (\mathrm{id}_{\mathrm{Hom}(Y,Z)} \times \mu_{W,X,Y}) = \mu_{W,X,Z} \circ (\mu_{X,Y,Z} \times \mathrm{id}_{\mathrm{Hom}(W,X)})$, rather than 'associative up to homotopy'.

Before we have met things like additive categories, in which $\mathrm{Hom}(X, Y)$ is an abelian group. This is still an ordinary category, with $\mathrm{Hom}(X, Y)$ a set, but $\mathrm{Hom}(X, Y)$ also has the additional structure of an abelian group.

For simplicial categories, it is probably best *not* to think of $\mathrm{Hom}(X, Y)$ as a set with extra structure (though you could think of the underlying set as $\mathbf{TR}(\mathrm{Hom}(X, Y)))$, so a simplicial category is not an ordinary category with extra structure.

Here is a related notion:

**Definition**

A *simplicial object in* $\mathfrak{Cat}$ is a functor $F : \Delta^{\mathrm{op}} \to \mathfrak{Cat}$.

Given a simplicial category $\mathscr{S}$, we can define $F_{\mathscr{S}} : \Delta^{\mathrm{op}} \to \mathfrak{Cat}$, a simplicial object in $\mathfrak{Cat}$, by taking $F_{\mathscr{S}}([n])$ to be the category with the same objects as $\mathscr{S}$ and with morphisms $X \to Y$ to be the set $\mathrm{Hom}_{\mathscr{S}}(X, Y)([n])$, and for a morphism $f : [m] \to [n]$ in $\Delta$ the functor $F_{\mathscr{S}} : F_{\mathscr{S}}([n]) \to F_{\mathscr{S}}([m])$ acts as the identity on objects and as $\mathrm{Hom}_{\mathscr{S}}(X, Y)(f) : \mathrm{Hom}_{\mathscr{S}}(X, Y)([n]) \to \mathrm{Hom}_{\mathscr{S}}(X, Y)([m])$ on morphisms.

Then simplicial categories correspond to special simplicial objects $F : \Delta^{\mathrm{op}} \to \mathfrak{Cat}$, those in which the set of objects of the category $F([n])$ is independent of $[n]$, and $F(f)$ acts as identity on sets of objects for all $f : [m] \to [n]$ in $\Delta$.

Any ordinary category $\mathscr{C}$ can be made into a simplicial category $\mathscr{S}$ by defining $\mathrm{Hom}_{\mathscr{S}}(X,Y)$ to map $[n] \mapsto \mathrm{Hom}_{\mathscr{C}}(X,Y)$ and $f \mapsto \mathrm{id}_{\mathscr{C}(X,Y)}$ and for all $[n]$ and $f : [m] \to [n]$.
Simplicial categories can be used as models for $(\infty,1)$-categories. To work in $\infty$-categories one does homotopy theory in the simplicial sets $\mathrm{Hom}(X,Y)$, using the model structure on **SSets**. Because of the Fundamental Theorem of Model Categories it is helpful to restrict to simplicial categories $\mathscr{S}$ such that $\mathrm{Hom}(X,Y)$ is a fibrant-cofibrant object in **SSets** (a 'Kan complex') for all $X, Y \in$ **SSets**. So we could *define* an $(\infty,1)$-category to be a simplicial category in which all Hom sets are Kan complexes. But we won't do this; instead, our preferred definition of $\infty$-category is *quasicategories* (next lecture).

# Derived Algebraic Geometry

Lecture 10 of 14: More about $\infty$-categories

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Summer Term 2022

These slides available at
`http://people.maths.ox.ac.uk/∼joyce/`

Plan of talk:

# 10.1. Simplicial sets and simplicial objects
## Face and degeneracy maps of simplicial sets

Return to the simplicial category $\Delta$. Define morphisms in $\Delta$, the *face maps* $\delta^{n,i}:[n-1]\to[n]$ and *degeneracy maps* $\sigma^{n,i}:[n+1]\to[n]$ for $i=0,\ldots,n$ by

$$\delta^{n,i}(j)=\begin{cases} j, & j<i, \\ j+1, & j\geqslant i, \end{cases} \quad \sigma^{n,i}(j)=\begin{cases} j, & j\leqslant i, \\ j-1, & j>i. \end{cases}$$

That is, $\delta^{n,i}$ misses $i$, and $\sigma^{n,i}$ repeats $i$. These satisfy the identities

$$\begin{aligned} \delta^{n,j}\circ\delta^{n-1,i} &= \delta^{n,i}\circ\delta^{n-1,j-1}, & 0\leqslant i<j\leqslant n, \\ \sigma^{n,j}\circ\sigma^{n+1,i} &= \sigma^{n,i}\circ\sigma^{n+1,j+1}, & 0\leqslant i\leqslant j\leqslant n, \\ \sigma^{n,j}\circ\delta^{n+1,i} &= \delta^{n,i}\circ\sigma^{n-1,j-1}, & 0\leqslant i<j\leqslant n, \\ \sigma^{n-1,j}\circ\delta^{n,i} &= \delta^{n-1,i-1}\circ\sigma^{n-2,j}, & 0\leqslant j<j+1<i\leqslant n, \\ \sigma^{n,j}\circ\delta^{n+1,i} &= \mathrm{id}, & i=j \text{ or } j+1. \end{aligned} \tag{10.1}$$

The category $\Delta$ is generated by the $\delta^{n,i},\sigma^{n,i}$ subject only to the relations $(10.1)$.

Now let $S:\Delta^{\rm op}\to\textbf{Sets}$ be a simplicial set. Write $S_n=S([n])$ and define *face maps* $d_{n,i}=S(\delta^{n,i}):S_n\to S_{n-1}$ and *degeneracy maps* $s_{n,i}=S(\sigma^{n,i}):S_n\to S_{n+1}$. Reversing directions of morphisms, these satisfy

$$\begin{aligned}
d_{n-1,i}\circ d_{n,j}&=d_{n-1,j-1}\circ d_{n,i}, && 0\leqslant i<j\leqslant n,\\
s_{n+1,i}\circ s_{n,j}&=s_{n+1,j+1}\circ s_{n,i}, && 0\leqslant i\leqslant j\leqslant n,\\
d_{n+1,i}\circ s_{n,j}&=s_{n-1,j-1}\circ d_{n,i}, && 0\leqslant i<j\leqslant n,\\
d_{n,i}\circ s_{n-1,j}&=s_{n-2,j}\circ d_{n-1,i-1}, && 0\leqslant j<j+1<i\leqslant n,\\
d_{n+1,i}\circ s_{n,j}&=\mathrm{id}, && i=j \text{ or } j+1.
\end{aligned}\tag{10.2}$$

As $\Delta$ is generated by the $\delta^{n,i},\sigma^{n,i}$ subject to (10.1), to define a simplicial set $S$ it is enough to give sets $S_n$ for $n\geqslant 0$ and maps $d_{n,i}:S_n\to S_{n-1}$ and $s_{n,i}:S_n\to S_{n+1}$ for $0\leqslant i\leqslant n$ satisfying (10.2), and all the other morphisms $S(f)$ in $S$ for $f:[k]\to[l]$ can be written as compositions of the $d_{n,i},s_{n,i}$.

This gives us a way to draw a picture of a simplicial set:

$$S_0 \begin{array}{c} \xleftarrow{\;d_{1,0}\;} \\ \xrightarrow{\;s_{0,0}\;} \\ \xleftarrow{\;d_{1,1}\;} \end{array} S_1 \begin{array}{c} \xleftarrow{\;d_{2,0}\;} \\ \xrightarrow{\;s_{1,1}\;} \\ \xleftarrow{\;d_{2,1}\;} \\ \xrightarrow{\;s_{1,0}\;} \\ \xleftarrow{\;d_{2,2}\;} \end{array} S_2 \begin{array}{c} \xleftarrow{\;d_{3,0}\;} \\ \xrightarrow{\;s_{2,2}\;} \\ \xleftarrow{\;d_{3,1}\;} \\ \xrightarrow{\;s_{2,1}\;} \\ \xleftarrow{\;d_{3,2}\;} \\ \xrightarrow{\;s_{2,0}\;} \\ \xleftarrow{\;d_{3,3}\;} \end{array} S_3 \quad \cdots . \tag{10.3}$$

# Simplicial objects in categories

Given a category $\mathscr{C}$, a *simplicial object in* $\mathscr{C}$ is a functor $F:\Delta^{\mathrm{op}}\to\mathscr{C}$. Thus a simplicial set is a simplicial object in **Sets**. One often abbreviates 'simplicial object in things' to 'simplicial thing'. For example, a *simplicial commutative* $\mathbb{K}$*-algebra* (simplicial object in $\mathbf{Alg}_{\mathbb{K}}$) is a functor $A:\Delta^{\mathrm{op}}\to\mathbf{Alg}_{\mathbb{K}}$. We can draw a picture of a simplicial commutative $\mathbb{K}$-algebra as a diagram (10.3), where the $S_i$ are commutative $\mathbb{K}$-algebras and the $d_{n,i}, s_{n,i}$ are algebra morphisms.

This will be important for Derived Algebraic Geometry as one model for 'derived commutative $\mathbb{K}$-algebras' are simplicial commutative $\mathbb{K}$-algebras. (If $\operatorname{char}\mathbb{K}=0$, another model is cdgas in degrees $\leqslant 0$.) So roughly speaking, a derived scheme should be a topological space with a homotopy sheaf (∞-sheaf) of simplicial commutative $\mathbb{K}$-algebras.

A *cosimplicial object in* $\mathscr{C}$ is a functor $F:\Delta\to\mathscr{C}$. Thus a simplicial commutative $\mathbb{K}$-algebra is a cosimplicial affine $\mathbb{K}$-scheme.

# (Higher) stacks as simplicial schemes

In Lecture 5 I explained that, given a (Deligne–Mumford or Artin) $\mathbb{K}$-stack $X$ and an atlas $u : U \to X$, one can construct a groupoid object $(U, V, s, t, u, i, m)$ in $\mathbf{Sch}_{\mathbb{K}}$, where $V = U \times_X U$, and $s, t : V \to U$, $u : U \to V$, $m : V \times_U V \to V$. In fact this is just part of a *simplicial* $\mathbb{K}$*-scheme* (simplicial object in $\mathbf{Sch}_{\mathbb{K}}$):

$$U \;\underset{\underset{t}{\longleftarrow}}{\overset{\overset{s}{\longleftarrow}}{\overset{u}{\longrightarrow}}}\; \begin{matrix} V = \\ U \times_X U \end{matrix} \;\underset{\underset{m}{\longleftarrow}}{\overset{\overset{\Pi_1}{\longleftarrow}}{\overset{\Delta_V}{\longrightarrow}\atop\overset{\Pi_2}{\longleftarrow}\atop\longrightarrow}}\; \begin{matrix} V \times_U V = \\ U \times_X U \times_X U \end{matrix} \;\begin{matrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\end{matrix}\; U \times_X U \times_X U \times_X U \;\cdots .$$

Pridham arXiv:0905.4044 explains that one can model *higher* $\mathbb{K}$*-stacks* — a generalization of stack, where for example moduli stacks of objects in $D^b \operatorname{coh}(X)$ should be higher stacks rather than Artin stacks – as a special kind of simplicial $\mathbb{K}$-scheme.

Ordinary stacks are functors $\mathbf{Alg}_{\mathbb{K}} \to \mathbf{Groupoids}$. To generalize them in the 'higher' direction, we need to replace **Groupoids** by **SSets** ($\infty$-groupoids), which corresponds to replacing $\mathbb{K}$-schemes by *simplicial* $\mathbb{K}$*-schemes*. To generalize them in the 'derived' direction, we need to replace $\mathbf{Alg}_{\mathbb{K}}$ by simplicial algebras, or equivalently, replace affine $\mathbb{K}$-schemes by *cosimplicial affine* $\mathbb{K}$*-schemes*. So the most general kind of derived higher $\mathbb{K}$-stack is like a 'simplicial cosimplicial $\mathbb{K}$-scheme'.

# Dwyer–Kan localization

Let $\mathscr{M}$ be a model category. Then we can define an explicit simplicial category $\mathscr{M}_{\rm DK}$ called the *Dwyer–Kan localization* or *hammock localization* of $\mathscr{M}$, where objects of $\mathscr{M}_{\rm DK}$ are objects of $\mathscr{M}$, and in the simplicial set $\mathrm{Hom}_{\rm DK}(X,Y)$, $n$-simplices are equivalence classes of commutative diagrams for $m \geqslant 0$

$$
\begin{array}{ccccccccccccccc}
 & & K_{10} & \xleftarrow{\sim} & L_{10} & \to & K_{20} & \xleftarrow{\sim} & L_{20} & \to & K_{30} & \to \cdots \xleftarrow{\sim} & L_{m0} & & \\
 & \nearrow & \downarrow\sim & & \downarrow\sim & & \downarrow\sim & & \downarrow\sim & & \downarrow\sim & & \sim\downarrow & \searrow & \\
 & & K_{11} & \xleftarrow{\sim} & L_{11} & \to & K_{21} & \xleftarrow{\sim} & L_{21} & \to & K_{31} & \to \cdots \xleftarrow{\sim} & L_{m1} & & \\
X & \nearrow & \downarrow\sim & & \downarrow\sim & & \downarrow\sim & & \downarrow\sim & & \downarrow\sim & & \sim\downarrow & \searrow & Y, \\
 & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & \cdots & \cdots & & \\
 & \searrow & \downarrow\sim & & \downarrow\sim & & \downarrow\sim & & \downarrow\sim & & \downarrow\sim & & \sim\downarrow & \nearrow & \\
 & & K_{1n} & \xleftarrow{\sim} & L_{1n} & \to & K_{2n} & \xleftarrow{\sim} & L_{2n} & \to & K_{3n} & \to \cdots \xleftarrow{\sim} & L_{mn} & &
\end{array}
$$

with $n+1$ rows, where morphisms '$\sim$' are weak equivalences, and the equivalence relation omits identities and composes composable morphisms, changing $m$, and for $m=0$ we take morphisms $X \to Y$.

Then regarding the ordinary category $\mathscr{M}$ as a simplicial category, there is a simplicial functor $\mathscr{M} \to \mathscr{M}_{\rm DK}$ mapping morphisms $X \to Y$ to the same morphisms with $m = 0$. The homotopy category $\mathrm{Ho}(\mathscr{M}_{\rm DK})$ is $\mathrm{Ho}(\mathscr{M}) = \mathscr{M}[\mathscr{W}^{-1}]$, where by definition the morphisms $X \to Y$ in $\mathrm{Ho}(\mathscr{M}_{\rm DK})$ are the connected components of the simplicial set $\mathrm{Hom}_{\rm DK}(X, Y)$.
We regard $\mathscr{M}_{\rm DK}$ as the ∞-category associated to the model category $\mathscr{M}$. In the last lecture I claimed:

## General Principle

If you start with an ordinary category $\mathscr{C}$ and invert some class of morphisms $\mathscr{W}$ in $\mathscr{C}$ ('weak equivalences'), the result $\mathscr{C}[\mathscr{W}^{-1}]$ should really be an ∞-**category** ($(\infty, 1)$-category), with homotopy category $\mathrm{Ho}(\mathscr{C}[\mathscr{W}^{-1}])$ an ordinary category.

Some justification is that it is natural to define $\mathrm{Hom}_{\mathscr{C}[\mathscr{W}^{-1}]}(X, Y)$ as the connected components of a simplicial set, so $\mathscr{C}[\mathscr{W}^{-1}]$ is naturally the homotopy category of a simplicial category.

# 10.2. Kan complexes and weak Kan complexes

## Definition

For $n \geqslant 0$, the *standard n-simplex* $\mathbb{\Delta}^n$ is the simplicial set $\mathrm{Hom}(-,[n]) : \Delta^{\mathrm{op}} \to$ **Sets**. Define morphisms of simplicial sets $\mathbb{\delta}^{n,i} : \mathbb{\Delta}^{n-1} \to \mathbb{\Delta}^n$ for $i = 0, 1 \ldots, n$ by $\mathbb{\delta}^{n,i} = - \circ \delta^{n,i}$, where $\delta^{n,i} : [n-1] \to [n]$ is the face map. Then $\mathbb{\delta}^{n,i}$ is an injective morphism in **SSets**, and $\mathbb{\delta}_{n,i}(\mathbb{\Delta}^{n-1})$ is a *simplicial subset* of $\mathbb{\Delta}^n$, that is, $\mathbb{\delta}^{n,i}(\mathbb{\Delta}^{n-1})([k]) \subseteq \mathbb{\Delta}^n([k])$ in **Sets** for each $k \geqslant 0$.

Define the $n-1$-*sphere* $\partial\mathbb{\Delta}^n$, as a simplicial subset of $\mathbb{\Delta}^n$, by

$$\partial\mathbb{\Delta}^n = \textstyle\bigcup_{i=0,\ldots,n} \mathbb{\delta}^{n,i}(\mathbb{\Delta}^{n-1}),$$

where for each $k \geqslant 0$ we take the union in subsets of $\mathbb{\Delta}^n([k])$. It is a simplicial set with an inclusion $\partial\mathbb{\Delta}^n \hookrightarrow \mathbb{\Delta}^n$. For $k = 0, \ldots, n$, define the *k-horn* $\mathbb{\Delta}^n_k$, as a simplicial subset of $\mathbb{\Delta}^n$, by

$$\mathbb{\Delta}^n_k = \textstyle\bigcup_{i=0,\ldots,n:\ i\neq k} \mathbb{\delta}^{n,i}(\mathbb{\Delta}^{n-1}).$$

It is a simplicial set with an inclusion $\mathbb{\Delta}^n_k \hookrightarrow \mathbb{\Delta}^n$. It is an *inner horn* if $0 < k < n$.

## Definition

A simplicial set $S$ is a *Kan complex* if for all $0 \leqslant k \leqslant n$ and all $f : \Delta^n_k \to S$ in **SSets**, there exists a (not necessarily unique) morphism $g : \Delta^n \to S$ making the following diagram commute:

$$\begin{array}{ccc} \Delta^n_k & \xrightarrow{\qquad\qquad f \qquad\qquad} & S. \\ & \searrow \quad \Delta^n \quad \overset{g}{\cdots\cdots\nearrow} & \end{array}$$

Then we say that *all horns in $S$ have fillers*.
We call $S$ a *weak Kan complex* if the above holds for all $0 < k < n$. Then we say that *all inner horns in $S$ have fillers*.

## Lemma

*If $X$ is a topological space then* $\mathbf{Sing}(X)$ *is a Kan complex.*

## Proof.

We must fill in the diagram in **Top**:

$$\begin{array}{ccc} \mathbf{TR}(\Delta^n_k) & \xrightarrow{\qquad\qquad f \qquad\qquad} & X. \\ & \searrow \quad \mathbf{TR}(\Delta^n) \quad \overset{g}{\cdots\cdots\nearrow} & \end{array}$$

This is possible as $\mathbf{TR}(\Delta^n)$ retracts onto $\mathbf{TR}(\Delta^n_k)$. □

# The nerve of a category

Kan complexes are the fibrant-cofibrant objects in the model category **SSets**. Here is a construction which yields weak Kan complexes:

## Definition

Let $\mathscr{C}$ be a small category. (*Small* means the objects form a set.) Define a simplicial set $\mathscr{N}(\mathscr{C})$ called the *nerve* of $\mathscr{C}$ as follows:

- 0-*simplices* (elements of $\mathscr{N}(\mathscr{C})([0])$) are objects $X \in \mathscr{C}$.
- 1-*simplices* (in $\mathscr{N}(\mathscr{C})([1])$) are morphisms $X_0 \xrightarrow{f_1} X_1$ in $\mathscr{C}$.
- *n-simplices* are sequences $X_0 \xrightarrow{f_1} X_1 \cdots X_{n-1} \xrightarrow{f_n} X_n$ in $\mathscr{C}$.
- *Face maps* $d_{n,i} : \mathscr{N}(\mathscr{C})([n]) \to \mathscr{N}(\mathscr{C})([n-1])$ omit $X_0, f_1$ for $i=0$, omit $X_n, f_n$ for $i = n$, and compose $f_i, f_{i+1}$ for $0<i<n$.
- *Degeneracy maps* $s_{n,i} : \mathscr{N}(\mathscr{C})([n]) \to \mathscr{N}(\mathscr{C})([n+1])$ insert $\mathrm{id}_{X_i} : X_i \to X_i$ into the sequence.

Functors $F : \mathscr{C} \to \mathscr{D}$ induce morphisms $\mathscr{N}(\mathscr{C}) \to \mathscr{N}(\mathscr{D})$.

We can characterize when a simplicial set is a nerve via horn-filling.

## Proposition 10.1

(a) *A simplicial set $S$ is isomorphic to the nerve $\mathcal{N}(\mathscr{C})$ of a small category $\mathscr{C}$ iff all inner horns $f : \Lambda^n_k \to S$ have unique fillers. This implies that $\mathcal{N}(\mathscr{C})$ is a* **weak Kan complex** *for all $\mathscr{C}$.*
(b) *A simplicial set $S$ is isomorphic to the nerve $\mathcal{N}(\mathscr{C})$ of a small groupoid $\mathscr{C}$ iff all horns $f : \Lambda^n_k \to S$ for $n > 1$ have unique fillers.*
(c) *The nerve $\mathcal{N}(\mathscr{C})$ of a small category $\mathscr{C}$ is a Kan complex iff $\mathscr{C}$ is a groupoid.*

Proposition 10.1 suggests that we can see *weak Kan complexes as generalizations of categories*, and *Kan complexes as generalizations of groupoids*. We already know that Kan complexes are the fibrant-cofibrant objects in **SSets**, so by the fundamental theorem of model theory, the category with objects Kan complexes and homotopy classes of morphisms between them is equivalent to $\mathrm{Ho}(\textbf{SSets})$, so a model for ∞-groupoids.

# 10.3. Quasicategories

Quasicategories are a model (arguably the best) for $(\infty, 1)$-categories, developed by Joyal and Lurie. Lurie went on to use them as the foundation for his theory of Derived Algebraic Geometry.

**Definition**

A *quasicategory* is a weak Kan complex.

Based on our definition of the nerve $\mathscr{N}(\mathscr{C})$ of a category $\mathscr{C}$, we will explain how to treat a quasicategory like an ($\infty$-)category.

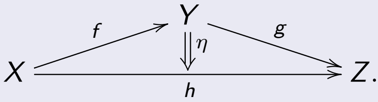
**Definition**

Let $\mathscr{Q}$ be a quasicategory.

- An *object* $X$ of $\mathscr{Q}$ is a 0-simplex (element $X \in \mathscr{Q}([0])$).
- A 1-*morphism* $f : X \to Y$ of $\mathscr{Q}$ is a 1-simplex (element $f \in \mathscr{Q}([1])$) with face maps $d_{1,1}(f) = X$ and $d_{1,0}(f) = Y$.
- The *identity* 1-*morphism* is $\mathrm{id}_X = s_{0,0}(X)$, from the degeneracy map $s_{0,0} : \mathscr{Q}([0]) \to \mathscr{Q}([1])$.

## Definition (Continued.)

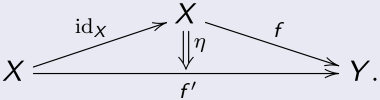
- Let $f : X \to Y$ and $g : Y \to Z$ be 1-morphisms in $\mathcal{Q}$. We say that $h : X \to Z$ is a choice of *composition* in $\mathcal{Q}$ if there exists a 2-simplex $\eta \in \mathcal{Q}([2])$ with $d_{2,2}(\eta) = f$, $d_{2,0}(\eta) = g$ and $d_{2,1}(\eta) = h$. We think of $\eta : g \circ f \Rightarrow h$ as a 2-morphism in $\mathcal{Q}$, and draw it as a picture of a 2-simplex, with $X, Y, Z$ as the vertices, $f, g, h$ as the edges, and $\eta$ as the 2-simplex:




  Note that *compositions are nonunique*. But as $\mathcal{Q}$ is a weak Kan complex, *compositions always exist*, as $X, Y, Z, f, g$ define a morphism $\Lambda^2_1 \to \mathcal{Q}$, and $h, \eta$ fill the horn to a morphism $\Delta^2 \to \mathcal{Q}$.

## Definition (Continued.)

- Let $f, f' : X \to Y$ be 1-morphisms. We say that $f, f'$ are 2-*isomorphic*, or *homotopic*, written $f \sim f'$, if there exists




  This is an equivalence relation on $f, f'$.
  Using the horn-filling condition for 3-horns we can show that if $h_1, h_2$ are possible compositions $g \circ f$ then $h_1 \sim h_2$.
- The *homotopy category* $\mathrm{Ho}(\mathscr{Q})$ is the category with *objects* $X$ the objects $X$ of $\mathscr{Q}$, and *morphisms* $[f] : X \to Y$ the $\sim$-equivalence classes of 1-morphisms $f : X \to Y$. *Identity morphisms* are $[\mathrm{id}_X] : X \to X$. The *composition* of $[f] : X \to Y$ and $[g] : Y \to Z$ is $[g] \circ [f] = [h] : X \to Z$, where $h : X \to Z$ is a choice of composition of 1-morphisms in $\mathscr{Q}$, and $[h]$ is independent of choices.
- If $\mathscr{Q} = \mathscr{N}(\mathscr{C})$ then $\mathrm{Ho}(\mathscr{Q}) \simeq \mathscr{C}$.

Many definitions in category theory have well-behaved analogues for quasicategories. Here are some examples:

**Definition**

- Let $\mathscr{Q}, \mathscr{R}$ be quasicategories. A *functor* $F : \mathscr{Q} \to \mathscr{R}$ is a morphism of simplicial sets.
- If $F, G : \mathscr{Q} \to \mathscr{R}$ are functors, a *natural transformation* $\eta : F \Rightarrow G$ is a morphism of simplicial sets $\eta : \mathbb{A}^1 \times \mathscr{Q} \to \mathscr{R}$ which restricts to $F$ on $0 \times \mathscr{Q}$ and to $G$ on $1 \times \mathscr{Q}$.
- Let $\mathscr{Q}$ be a quasi-category and $X, Y$ be objects in $\mathscr{Q}$. Define the *right Hom object* $\mathrm{Hom}^R_{\mathscr{Q}}(X, Y)$ to be the simplicial set whose $n$-simplices are morphisms $\mathbb{A}^{n+1} \to \mathscr{Q}$ which restrict to the constant map to $X$ on $\delta^{n+1,n+1}(\mathbb{A}^n) \subset \mathbb{A}^{n+1}$, and restrict to $Y$ on vertex $n+1$ of $\mathbb{A}^{n+1}$.
- An object $Y$ in $\mathscr{Q}$ is a *terminal object* in $\mathscr{Q}$ if $\mathrm{Hom}^R_{\mathscr{Q}}(X, Y)$ is contractible for all objects $X$.
- *Left Hom objects* $\mathrm{Hom}^L_{\mathscr{Q}}(X, Y)$ and *initial objects* have the dual definition.

## Definition

- Let $K$ be a simplicial set and $k : K \to \mathcal{Q}$ a morphism. We can define a quasicategory $\mathcal{Q}_{/k}$ with objects $(X, \eta)$ an object $X$ in $\mathcal{Q}$ and a natural transformation $\eta : \mathbb{1}_X \Rightarrow k$, where $\mathbb{1}_X : K \to \mathcal{Q}$ is the constant functor with value $X$. A *limit* of $k : K \to \mathcal{Q}$ is a terminal object in $\mathcal{Q}_{/k}$. So, for example, a *fibre product* $X \times_{g,Z,h} Y$ in $\mathcal{Q}$ is a limit of the morphism

$$\begin{array}{ccccc} \underset{\Delta^0}{\bullet} & & & & \underset{\Delta^0}{\bullet} \\ & \searrow_{\Delta^1} & & \swarrow_{\Delta^1} & \\ & & \underset{\Delta^0}{\bullet} & & \end{array} \xrightarrow{k} \begin{array}{ccccc} X & & & & Y \\ & \searrow_{g} & & \swarrow_{h} & \\ & & Z. & & \end{array}$$

The theory of quasicategories is very well developed, and works really well.