Derived Differential Geometry

Lecture 1 of 14: Background material in algebraic geometry and category theory

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These slides, and references, etc., available at
http://people.maths.ox.ac.uk/∼joyce/DDG2015

Plan of talk:

1. Different kinds of spaces in algebraic geometry

Algebraic geometry studies spaces built using algebras of functions. Here are the main classes of spaces studied by algebraic geometers, in order of complexity, and difficulty of definition:

- Smooth varieties (e.g. Riemann surfaces, or algebraic complex manifolds such as $\mathbb{CP}^n$. Smooth means nonsingular.)
- Varieties (at their most basic, algebraic subsets of $\mathbb{C}^n$ or $\mathbb{CP}^n$. Can have singularities, e.g. $xy = 0$ in $\mathbb{C}^2$, singular at $(0,0)$.)
- Schemes (can be non-reduced, e.g. the scheme $x^2 = 0$ in $\mathbb{C}$ is not the same as the scheme $x = 0$ in $\mathbb{C}$.)
- Algebraic spaces (étale locally modelled on schemes.)
- Stacks. Each point $x \in X$ has a stabilizer group $\mathrm{Iso}(x)$, finite for Deligne–Mumford stacks, algebraic group for Artin stacks.
- Higher stacks.
- Derived stacks, including derived schemes.

1.1. The definition of schemes

Fix a field $\mathbb{K}$, e.g. $\mathbb{K} = \mathbb{C}$. An *affine* $\mathbb{K}$*-scheme* $X = \operatorname{Spec} A$ is basically a commutative $\mathbb{K}$-algebra A, but regarded as a geometric space in the following way. As a set, define X to be the set of all prime ideals $I \subset A$. If J is an ideal of A, define $V(J) \subseteq X$ to be the set of prime ideals $I \subseteq A$ with $J \subseteq I$. Then $\mathcal{T} = \{V(J) : J$ is an ideal in $A\}$ is a topology on X, the *Zariski topology*.
We can regard each $f \in A$ as a 'function' on X, where $f(I) = f + I$ in the quotient algebra A/I. For the subset $X(\mathbb{K}) \subseteq X$ of $\mathbb{K}$*-points* I with $A/I \cong \mathbb{K}$, f gives a genuine function $X(\mathbb{K}) \to \mathbb{K}$.
Thus, we have a topological space X called the *spectrum* $\operatorname{Spec} A$ of A, equipped with a sheaf of $\mathbb{K}$-algebras $\mathcal{O}_X$, and A is the algebra of functions on X.
A general $\mathbb{K}$-scheme X is a topological space X with a sheaf of $\mathbb{K}$-algebras $\mathcal{O}_X$, such that X may be covered by open sets $U \subseteq X$ with $(U, \mathcal{O}_X|_U)$ isomorphic to $\operatorname{Spec} A$ for some $\mathbb{K}$-algebra A.

Example 1.1

$\mathbb{C}^n$ is an affine $\mathbb{C}$-scheme, the spectrum of the polynomial algebra $A=\mathbb{C}[x_1,\ldots,x_n]$. Given polynomials $p_1,\ldots,p_k\in\mathbb{C}[x_1,\ldots,x_n]$, we can define an affine $\mathbb{C}$-subscheme $X\subseteq\mathbb{C}^n$ as the zero locus of $p_1,\ldots,p_k$, the spectrum of $B=\mathbb{C}[x_1,\ldots,x_n]/(p_1,\ldots,p_k)$. The $\mathbb{C}$-points $X(\mathbb{C})$ are $(x_1,\ldots,x_n)\in\mathbb{C}^n$ with $p_1(x_1,\ldots,x_n)=\cdots=p_k(x_1,\ldots,x_n)=0$. Note that the (nonreduced) scheme $x^2=0$ in $\mathbb{C}$ is not the same as the scheme $x=0$ in $\mathbb{C}$, as the algebras $\mathbb{C}[x]/(x^2)=\mathbb{C}\langle 1,x\rangle$ and $\mathbb{C}[x]/(x)=\mathbb{C}\langle 1\rangle$ are different.

To take a similar approach to manifolds M in differential geometry, we should consider the $\mathbb{R}$-algebra $C^\infty(M)$ of smooth functions $f:M\to\mathbb{R}$, and reconstruct M as the set of ideals $I\subset C^\infty(M)$ with $C^\infty(M)/I\cong\mathbb{R}$. In lecture 3 we will see that $C^\infty(M)$ is not just an $\mathbb{R}$-algebra, it has an algebraic structure called a C^∞-*ring*, and M is a scheme over C^∞-rings, a C^∞-*scheme*.

1.2. Some basics of category theory

To prepare for moduli functors and stacks, we first explain some ideas from category theory.

Definition

A *category* $\mathcal{C}$ consists of the following data:

- A family $\mathrm{Obj}(\mathcal{C})$ of *objects* $X,Y,Z,\ldots$ of $\mathcal{C}$. (Actually $\mathrm{Obj}(\mathcal{C})$ is a *class*, like a set but possibly larger.)
- For all objects X,Y in $\mathcal{C}$, a set $\mathrm{Hom}(X,Y)$ of *morphisms* f, written $f:X\to Y$.
- For all objects X,Y,Z in $\mathcal{C}$, a *composition map* $\circ:\mathrm{Hom}(Y,Z)\circ\mathrm{Hom}(X,Y)\to\mathrm{Hom}(X,Z)$, written $g\circ f:X\to Z$ for morphisms $f:X\to Y$ and $g:Y\to Z$. It is associative, $(h\circ g)\circ f=h\circ(g\circ f)$.
- For all objects X in $\mathcal{C}$ an *identity morphism* $\mathrm{id}_X\in\mathrm{Hom}(X,X)$, with $f\circ\mathrm{id}_X=\mathrm{id}_Y\circ f=f$ for all $f:X\to Y$.

Categories are everywhere in mathematics – whenever you have a class of mathematical objects, and a class of maps between them, you generally get a category. For example:

- The category **Sets** with objects sets, and morphisms maps.
- The category **Top** with objects topological spaces $X, Y, \ldots$ and morphisms continuous maps $f : X \to Y$.
- The category **Man** of smooth manifolds and smooth maps.
- The category $\mathbf{Sch}_{\mathbb{K}}$ of schemes over a field $\mathbb{K}$.

Definition

A category $\mathcal{C}$ is a *subcategory* of a category $\mathscr{D}$, written $\mathcal{C} \subset \mathscr{D}$, if $\mathrm{Obj}(\mathcal{C}) \subseteq \mathrm{Obj}(\mathscr{D})$, and for all $X, Y \in \mathrm{Obj}(\mathcal{C})$ we have $\mathrm{Hom}_{\mathcal{C}}(X, Y) \subseteq \mathrm{Hom}_{\mathscr{D}}(X, Y)$, and composition and identities in $\mathcal{C}, \mathscr{D}$ agree on $\mathrm{Hom}_{\mathcal{C}}(-,-)$. It is a *full* subcategory if $\mathrm{Hom}_{\mathcal{C}}(X, Y) = \mathrm{Hom}_{\mathscr{D}}(X, Y)$ for all X, Y in $\mathcal{C}$.

The category $\mathbf{Sch}_{\mathbb{K}}^{\mathbf{aff}}$ of affine $\mathbb{K}$-schemes is a full subcategory of $\mathbf{Sch}_{\mathbb{K}}$.

Functors between categories

Functors are the natural maps between categories.

Definition

Let $\mathcal{C}, \mathscr{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathscr{D}$ consists of the data:

- A map $F : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathscr{D})$.
- Maps $F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$ for all $X, Y \in \mathrm{Obj}(\mathcal{C})$, with $F(g \circ f) = F(g) \circ F(f)$ for all composable f, g in $\mathcal{C}$ and $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all X in $\mathcal{C}$.

- The *identity functor* $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ maps $X \mapsto X$ and $f \mapsto f$.
- There is a 'forgetful functor' $F : \mathbf{Man} \to \mathbf{Top}$ taking a manifold X to its underlying topological space $F(X)$, and a smooth map $f : X \to Y$ to its underlying continuous map.
- If $\mathcal{C} \subset \mathscr{D}$ is a subcategory, the *inclusion functor* $i : \mathcal{C} \hookrightarrow \mathscr{D}$.
- For $k \geqslant 0$, $H_k : \mathbf{Top} \to \mathbf{AbGp}$ (abelian groups) maps a topological space X to its k^{th} homology group $H_k(X; \mathbb{Z})$.

Our definition of functors are sometimes called *covariant functors*, in contrast to *contravariant functors* $F : \mathcal{C} \to \mathscr{D}$ which reverse the order of composition, $F(g \circ f) = F(f) \circ F(g)$, such as the cohomology functors $H^k : \mathbf{Top} \to \mathbf{AbGp}$. We prefer to write contravariant functors as (covariant) functors $F : \mathcal{C}^{\rm op} \to \mathscr{D}$, where $\mathcal{C}^{\rm op}$ is the *opposite category* to $\mathcal{C}$, the same as $\mathcal{C}$ but with order of composition reversed. For example, in scheme theory the *spectrum functor* maps $\mathrm{Spec} : (\mathbf{Alg}_{\mathbb{K}})^{\rm op} \to \mathbf{Sch}_{\mathbb{K}}^{\mathbf{aff}} \subset \mathbf{Sch}_{\mathbb{K}}$, where $\mathbf{Alg}_{\mathbb{K}}$ is the category of $\mathbb{K}$-algebras.

Natural transformations and natural isomorphisms

There is also a notion of morphism between functors:

Definition

Let $\mathcal{C}, \mathscr{D}$ be categories, and $F, G : \mathcal{C} \to \mathscr{D}$ be functors. A *natural transformation* η from F to G, written $\eta : F \Rightarrow G$, assigns the data of a morphism $\eta(X) : F(X) \to G(X)$ in $\mathscr{D}$ for all objects X in $\mathcal{C}$, such that $\eta(Y) \circ F(f) = G(f) \circ \eta(X) : F(X) \to G(Y)$ for all morphisms $f : X \to Y$ in $\mathcal{C}$.
We call η a *natural isomorphism* if $\eta(X)$ is an isomorphism (invertible morphism) in $\mathscr{D}$ for all X in $\mathcal{C}$.
Given natural transformations $\eta : F \Rightarrow G$, $\zeta : G \Rightarrow H$, the *composition* $\zeta \odot \eta : F \Rightarrow H$ is $(\zeta \odot \eta)(X) = \zeta(X) \circ \eta(X)$ for X in $\mathcal{C}$.
The *identity transformation* $\mathrm{id}_F : F \Rightarrow F$ is $\mathrm{id}_F(X) = \mathrm{id}_{F(X)} : F(X) \to F(X)$ for all X in $\mathcal{C}$.

Note that in the 'category of categories' $\mathfrak{Cat}$, we have objects categories $\mathcal{C}, \mathscr{D}, \ldots$, and morphisms (or 1-morphisms), functors $F, G : \mathcal{C} \to \mathscr{D}$, but also 'morphisms between morphisms' (or 2-morphisms), natural transformations $\eta : F \Rightarrow G$. This is our first example of a 2-*category*, defined in lecture 4.

In category theory, it is often important to think about when things are 'the same'. For objects X, Y in a category $\mathcal{C}$, there are two notions of when X, Y are 'the same': equality $X = Y$, and isomorphism $X \cong Y$, i.e. there are morphisms $f : X \to Y$, $g : Y \to X$ with $g \circ f = \mathrm{id}_X$, $f \circ g = \mathrm{id}_Y$. Usually isomorphism is better.

For functors $F, G : \mathcal{C} \to \mathscr{D}$, there are two notions of when F, G are 'the same': equality $F = G$, and natural isomorphism $F \cong G$, that is, there exists a natural isomorphism $\eta : F \Rightarrow G$. Usually natural isomorphism, the weaker, is better.

Equivalence of categories

For categories $\mathcal{C}, \mathscr{D}$, there are three notions of when $\mathcal{C}, \mathscr{D}$ are 'the same': strict equality $\mathcal{C} = \mathscr{D}$; strict isomorphism $\mathcal{C} \cong \mathscr{D}$, that is, there exist functors $F : \mathcal{C} \to \mathscr{D}$, $G : \mathscr{D} \to \mathcal{C}$ with $G \circ F = \mathrm{id}_{\mathcal{C}}$, $F \circ G = \mathrm{id}_{\mathscr{D}}$; and equivalence:

Definition

An *equivalence* between categories $\mathcal{C}, \mathscr{D}$ consists of functors $F : \mathcal{C} \to \mathscr{D}$, $G : \mathscr{D} \to \mathcal{C}$ and natural isomorphisms $\eta : G \circ F \Rightarrow \mathrm{id}_{\mathcal{C}}$, $\zeta : F \circ G \Rightarrow \mathrm{id}_{\mathscr{D}}$. We say that G is a *quasi-inverse* for F, and write $\mathcal{C} \simeq \mathscr{D}$ to mean that $\mathcal{C}, \mathscr{D}$ are equivalent.

Usually equivalence of categories, the weakest, is the best notion of when categories $\mathcal{C}, \mathscr{D}$ are 'the same'.

The Yoneda embedding

Let $\mathcal{C}, \mathscr{D}$ be categories. Then $\mathrm{Fun}(\mathcal{C}, \mathscr{D})$ is a category with objects functors $F : \mathcal{C} \to \mathscr{D}$, morphisms natural transformations $\eta : F \Rightarrow G$, composition $\zeta \odot \eta$, and identities id_F. A natural transformation $\eta : F \Rightarrow G$ is a natural isomorphism if and only if it is an isomorphism in $\mathrm{Fun}(\mathcal{C}, \mathscr{D})$.

Definition

Let $\mathcal{C}$ be any category. Then $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \textbf{Sets})$ is also a category. Define a functor $Y_{\mathcal{C}} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \textbf{Sets})$ called the *Yoneda embedding* by, for each X in $\mathcal{C}$, taking $Y_{\mathcal{C}}(X)$ to be the functor $\mathrm{Hom}(-, X) : \mathcal{C}^{\mathrm{op}} \to \textbf{Sets}$ mapping $Y \mapsto \mathrm{Hom}(Y, X)$ on objects $Y \in \mathcal{C}$, and mapping $\circ f : \mathrm{Hom}(Z, X) \to \mathrm{Hom}(Y, X)$ for all morphisms $f : Y \to Z$ in $\mathcal{C}$; and for each morphism $e : W \to X$ in $\mathcal{C}$, taking $Y_{\mathcal{C}}(e) : Y_{\mathcal{C}}(W) \to Y_{\mathcal{C}}(X)$ to be the natural transformation $e \circ : \mathrm{Hom}(-, W) \to \mathrm{Hom}(-, X)$.

The Yoneda Lemma

The *Yoneda Lemma* says $Y_{\mathcal{C}} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \textbf{Sets})$ is a *full* and *faithful* functor, i.e. the maps $Y_{\mathcal{C}} : \mathrm{Hom}_{\mathcal{C}}(W, X) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \textbf{Sets})}(Y_{\mathcal{C}}(W), Y_{\mathcal{C}}(X))$ are injective and surjective.

Call a functor $F : \mathcal{C}^{\mathrm{op}} \to \textbf{Sets}$ *representable* if F is naturally isomorphic to $Y_{\mathcal{C}}(X) = \mathrm{Hom}(-, X)$ for some $X \in \mathcal{C}$, which is then unique up to isomorphism. Write $\mathrm{Rep}(\mathcal{C})$ for the full subcategory of representable functors in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \textbf{Sets})$. Then $Y_{\mathcal{C}} : \mathcal{C} \to \mathrm{Rep}(\mathcal{C})$ is an equivalence of categories.

Basically, the idea here is that we should understand objects X in $\mathcal{C}$, up to isomorphism, by knowing the sets $\mathrm{Hom}(Y, X)$ for all $Y \in \mathcal{C}$, and the maps $\circ f : \mathrm{Hom}(Z, X) \to \mathrm{Hom}(Y, X)$ for all morphisms $f : Y \to Z$ in $\mathcal{C}$.

If $\mathcal{C} \subset \mathscr{D}$ is a subcategory, there is a functor $\mathscr{D} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \textbf{Sets})$ mapping $X \mapsto \mathrm{Hom}(\mathcal{C}, X)$ for $X \in \mathscr{D}$.

(Affine) schemes as functors $\mathbf{Alg}_{\mathbb{K}} \to \mathbf{Sets}$

Since $\mathrm{Spec} : (\mathbf{Alg}_{\mathbb{K}})^{\mathrm{op}} \to \mathbf{Sch}^{\mathbf{aff}}_{\mathbb{K}}$ is an equivalence of categories, and $(\mathbf{Alg}_{\mathbb{K}})^{\mathrm{op}}$ is equivalent to the full subcategory of representable functors in $\mathrm{Fun}(\mathbf{Alg}_{\mathbb{K}}, \mathbf{Sets})$, we see that $\mathbf{Sch}^{\mathbf{aff}}_{\mathbb{K}}$ is equivalent to the full subcategory of representable functors in $\mathrm{Fun}(\mathbf{Alg}_{\mathbb{K}}, \mathbf{Sets})$. There is also a natural functor $\mathbf{Sch}_{\mathbb{K}} \to \mathrm{Fun}(\mathbf{Alg}_{\mathbb{K}}, \mathbf{Sets})$, mapping a scheme X to the functor $A \mapsto \mathrm{Hom}_{\mathbf{Sch}_{\mathbb{K}}}(\mathrm{Spec}\, A, X)$. This functor is full and faithful because, as X can be covered by open subschemes $\mathrm{Spec}\, A \subseteq X$, we can recover X up to isomorphism from the collection of morphisms $\mathrm{Spec}\, A \to X$ for $A \in \mathbf{Alg}_{\mathbb{K}}$. Thus, $\mathbf{Sch}_{\mathbb{K}}$ is equivalent to a full subcategory of $\mathrm{Fun}(\mathbf{Alg}_{\mathbb{K}}, \mathbf{Sets})$. Since we consider equivalent categories to be 'the same', *we can identify* $\mathbf{Sch}_{\mathbb{K}}$ *with this subcategory of* $\mathrm{Fun}(\mathbf{Alg}_{\mathbb{K}}, \mathbf{Sets})$, and *we can consider schemes to be special functors* $\mathbf{Alg}_{\mathbb{K}} \to \mathbf{Sets}$.

1.3. Moduli spaces and moduli functors

Moduli spaces are a hugely important subject in both algebraic and differential geometry. They were also the motivation for inventing most of the classes of spaces we are discussing – algebraic spaces, stacks, derived stacks, ... – as interesting moduli spaces had these structures, and simpler spaces were not adequate to describe them. Suppose we want to study some class of geometric objects X up to isomorphism, e.g. Riemann surfaces of genus g. Write $\mathcal{M}$ for the set of isomorphism classes $[X]$ of such X. A set on its own is boring, so we would like to endow $\mathcal{M}$ with some geometric structure which captures properties of families $\{X_t : t \in T\}$ of the objects X we are interested in. For example, if we have a notion of continuous deformation $X_t : t \in [0,1]$ of objects X, then we should give $\mathcal{M}$ a topology such that the map $[0,1] \to \mathcal{M}$ mapping $t \mapsto [X_t]$ is continuous for all such families $X_t : t \in [0,1]$.

We would like the geometric structure we put on $\mathcal{M}$ to be as strong as possible (e.g. for Riemann surfaces not just a topological space, but a complex manifold, or a $\mathbb{C}$-scheme) to capture as much information as we can about families of objects $\{X_t : t \in T\}$.
To play the moduli space game, we must ask three questions:

(A) What kind of geometric structure should we try to put on $\mathcal{M}$ (e.g. topological space, complex manifold, $\mathbb{K}$-scheme, ...)?
(B) Does $\mathcal{M}$ actually have this structure?
(C) If it does, can we describe $\mathcal{M}$ in this class of geometric spaces completely, or approximately (e.g. if $\mathcal{M}$ is a complex manifold, can we compute its dimension, and Betti numbers $b^k(M)$)?

There are two main reasons people study moduli spaces. The first is *classification*: when you study some class of geometric objects X (e.g. vector bundles on curves), people usually consider that if you can fully describe the moduli space $\mathcal{M}$ (with whatever geometric structure is appropriate), then you have classified such objects.

The second reason is *invariants*. There are many important areas of mathematics (e.g. Gromov–Witten invariants) in which to study some space S (e.g. a symplectic manifold) we form moduli spaces $\mathcal{M}$ of secondary geometric objects X associated to S (e.g. J-holomorphic curves in S), and then we define invariants $I(S)$ by 'counting' $\mathcal{M}$, to get a number, a homology class, etc.
We want the invariants $I(S)$ to have nice properties (e.g. to be independent of the choice of almost complex structure J on S).
For this to hold it is *essential* that the geometric structure on $\mathcal{M}$ be of a very special kind (e.g. a compact oriented manifold), and the 'counting' be done in a very special way.
Theories of this type include Donaldson, Donaldson–Thomas, Gromov–Witten, and Seiberg–Witten invariants, Floer homology theories, and Fukaya categories in symplectic geometry.
This is actually a major motivation for Derived Differential Geometry: compact, oriented derived manifolds or derived orbifolds can be 'counted' in this way to define invariants.

Moduli schemes and representable functors

In algebraic geometry there is a standard method for defining moduli spaces as schemes, due to Grothendieck. Suppose we want to form a moduli scheme $\mathcal{M}$ of some class of geometric objects X over a field $\mathbb{K}$. Suppose too that we have a good notion of family $\{X_t : t \in T\}$ of such objects X over a base $\mathbb{K}$-scheme T. We then define a *moduli functor* $F : \mathbf{Alg}_{\mathbb{K}} \to \mathbf{Sets}$, by for each $A \in \mathbf{Alg}_{\mathbb{K}}$

$$F(A) = \bigl\{\text{iso. classes } [X_t : t \in \operatorname{Spec} A] \text{ of families } \{X_t : t \in \operatorname{Spec} A\}\bigr\},$$

and for each morphism $f : A \to A'$ in $\mathbf{Alg}_{\mathbb{K}}$, we define $F(f) : F(A) \to F(A')$ by

$$F(f) : [X_t : t \in \operatorname{Spec} A] \longmapsto [X_{\operatorname{Spec}(f)t'} : t' \in \operatorname{Spec}(A')].$$

If there exists a $\mathbb{K}$-scheme $\mathcal{M}$ (always unique up to isomorphism) such that F is naturally isomorphic to $\operatorname{Hom}(\operatorname{Spec} -, \mathcal{M})$, we say F is a *representable functor*, and $\mathcal{M}$ is a (*fine*) *moduli scheme*.

1.4. Algebraic spaces and (higher) stacks
When schemes are not enough

Unfortunately there are lots of interesting moduli problems in which one can define a moduli functor $F : \mathbf{Alg}_{\mathbb{K}} \to \mathbf{Sets}$, but F is not representable, and no moduli scheme exists.
Sometimes one can find a $\mathbb{K}$-scheme $\mathcal{M}$ which is a 'best approximation' to F (a *coarse moduli scheme*). But often, to describe the moduli space $\mathcal{M}$, we have to move out of schemes, into a larger class of spaces.
The simplest such enlargement is *algebraic spaces*, which are defined to be functors $F : \mathbf{Alg}_{\mathbb{K}} \to \mathbf{Sets}$ which can be presented as the quotient $\mathcal{M}/\sim$ of a scheme $\mathcal{M}$ by an étale equivalence relation $\sim$. For example, moduli spaces of simple complexes $\mathcal{E}^\bullet$ of coherent sheaves on a smooth projective $\mathbb{K}$-scheme S are algebraic spaces.

Introduction to stacks

A moduli $\mathbb{K}$-scheme $\mathcal{M}$ or moduli functor $F : \mathbf{Alg}_{\mathbb{K}} \to \mathbf{Sets}$ classifies objects X up to isomorphism, so that $\mathbb{K}$-points of $\mathcal{M}$ are isomorphism classes $[X]$ of objects X. For each X we have a group $\mathrm{Iso}(X)$ of isomorphisms $i : X \to X$.
Usually, if $\mathrm{Iso}(X)$ is nontrivial, then F is not representable, and $\mathcal{M}$ does not exist as either a scheme or an algebraic space. Roughly, the reason is that we should expect $\mathcal{M}$ to be modelled near $[X]$ on a quotient $[\mathcal{N}/\mathrm{Iso}(X)]$ for a scheme $\mathcal{N}$, but schemes and algebraic spaces are not closed under quotients by groups (though see GIT).
Stacks are a class of geometric spaces $\mathcal{M}$ in which the geometric structure at each point $[X] \in \mathcal{M}$ remembers the group $\mathrm{Iso}(X)$. They include *Deligne–Mumford stacks*, in which the groups $\mathrm{Iso}(X)$ are finite, and *Artin stacks*, in which the $\mathrm{Iso}(X)$ are algebraic $\mathbb{K}$-groups. For almost all classical moduli problems a moduli stack exists, even when a moduli scheme does not.

Groupoids and stacks

A *groupoid* is a category $\mathcal{C}$ in which all morphisms are isomorphisms. They form a category **Groupoids** in which objects are groupoids, and morphisms are functors between them.
Any set S can be regarded as a groupoid with objects $s \in S$, and only identity morphisms. This gives a full and faithful functor $\mathbf{Sets} \to \mathbf{Groupoids}$, so $\mathbf{Sets} \subset \mathbf{Groupoids}$ is a full subcategory.
You can also map (small) groupoids to sets by sending $\mathcal{C}$ to the set S of isomorphism classes in $\mathcal{C}$.
A *stack* is defined to be a functor $F : \mathbf{Alg}_{\mathbb{K}} \to \mathbf{Groupoids}$ satisfying some complicated conditions. Since (affine) schemes and algebraic spaces can all be regarded as functors $F : \mathbf{Alg}_{\mathbb{K}} \to \mathbf{Sets}$, and $\mathbf{Sets} \subset \mathbf{Groupoids}$, we can consider (affine) schemes and algebraic spaces as functors $F : \mathbf{Alg}_{\mathbb{K}} \to \mathbf{Groupoids}$, and then they are special examples of stacks.

Stacks as moduli functors

As for moduli schemes, there is a standard method for defining moduli stacks. Suppose we want to form a moduli stack $\mathcal{M}$ of some class of geometric objects X over a field $\mathbb{K}$. We define a *moduli functor* $F : \mathbf{Alg}_{\mathbb{K}} \to \mathbf{Groupoids}$, by for each $A \in \mathbf{Alg}_{\mathbb{K}}$

$$F(A) = \bigl\{\text{groupoid of families } \{X_t : t \in \operatorname{Spec} A\}, \text{ with morphisms isomorphisms of such families}\bigr\},$$

and for each morphism $f : A \to A'$ in $\mathbf{Alg}_{\mathbb{K}}$, we define $F(f) : F(A) \to F(A')$ to be the functor of groupoids mapping

$$F(f) : \{X_t : t \in \operatorname{Spec} A\} \longmapsto \{X_{\operatorname{Spec}(f)t'} : t' \in \operatorname{Spec}(A')\}.$$

If F satisfies the necessary conditions, then F *is* the moduli stack.

With some practice you can treat stacks as geometric spaces – they have points, a topology, 'atlases' which are schemes, and so on. Stacks X are often locally modelled on quotients Y/G, for Y a scheme, and G a group which is finite for Deligne–Mumford stacks, and an algebraic group for Artin stacks.

Above we saw that categories form a 2-category $\mathfrak{Cat}$, with objects categories, 1-morphisms functors, and 2-morphisms natural transformations. As groupoids are special categories, **Groupoids** is also a 2-category. Since all natural transformations of groupoids are natural isomorphisms, all 2-morphisms in **Groupoids** are invertible, i.e. it is a (2,1)-*category*.

Stacks $\subset \operatorname{Fun}(\mathbf{Alg}_{\mathbb{K}}, \mathbf{Groupoids})$ also form a (2,1)-category, with 2-morphisms defined using natural isomorphisms of groupoids.

Higher stacks

There are some moduli problems for which even stacks are not general enough. A typical example would be moduli spaces $\mathcal{M}$ of complexes $\mathcal{E}^\bullet$ in the derived category $D^b \operatorname{coh}(S)$ of coherent sheaves on a smooth projective scheme S. The point is that $\mathcal{M}$ classifies complexes $\mathcal{E}^\bullet$ not up to isomorphism, but up to a weaker notion of quasi-isomorphism. Really $D^b \operatorname{coh}(S)$ is an ∞-category. For such moduli problems we need *higher stacks*, which are functors $F : \mathbf{Alg}_{\mathbb{K}} \to \mathbf{SSets}$. Here $\mathbf{SSets}$ is the (∞-)category of *simplicial sets*, which are generalizations of groupoids, so that $\mathbf{Sets} \subset \mathbf{Groupoids} \subset \mathbf{SSets}$. Higher stacks form an ∞-*category*, meaning that there are not just objects, 1-morphisms, and 2-morphisms, but n-morphisms for all $n = 1, 2, \ldots$.

Derived Differential Geometry

Lecture 2 of 14: What is derived geometry?

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These slides, and references, etc., available at
http://people.maths.ox.ac.uk/∼joyce/DDG2015

Plan of talk:

2. What is derived geometry?

Derived geometry is the study of 'derived' spaces. It has two versions: Derived Algebraic Geometry (DAG), the study of derived schemes and derived stacks, and Derived Differential Geometry (DDG), the study of derived smooth real (i.e. C^∞) spaces, including derived manifolds and derived orbifolds.
DAG is older and more developed. It has a reputation for difficulty and abstraction, with foundational documents running to 1000's of pages (Lurie, Toën–Vezzosi). DDG is a new subject, just beginning, with few people working in it so far.
Today we begin with an introduction to DAG, to give some idea of what 'derived' spaces are, why they were introduced, and what they are useful for. An essential point is that derived geometry happens in *higher categories* (e.g. 2-categories or ∞-categories).

2.1. Derived schemes and derived stacks

In §1 we saw that in classical algebraic geometry, we have spaces

affine schemes ⊂ schemes ⊂ algebraic spaces ⊂ stacks ⊂ higher stacks,

which can be defined as classes of functors $F : \mathbf{Alg}_{\mathbb{K}} \to \mathbf{Sets}$ or **Groupoids** or **SSets**, where **Sets** ⊂ **Groupoids** ⊂ **SSets**.
Such a space X is completely described by knowing the family (set, or groupoid, or simplicial set) of all morphisms $f : \operatorname{Spec} A \to X$, for all $\mathbb{K}$-algebras A, plus the family of all commutative triangles

$$\begin{array}{lll} \operatorname{Spec} A' & & \\ \downarrow{\scriptstyle \operatorname{Spec}\alpha} & \searrow^{f'} & \\ \operatorname{Spec} A & \xrightarrow{\quad f \quad} & X, \end{array}$$

for all morphisms of $\mathbb{K}$-algebras $\alpha : A \to A'$.

To do derived geometry, instead of enlarging the target category **Sets**, we enlarge the domain category $\mathbf{Alg}_{\mathbb{K}}$. So, a *derived stack* over a field $\mathbb{K}$ is defined to be a functor $\mathbf{F} : \mathbf{CDGAlg}_{\mathbb{K}} \to \mathbf{SSets}$ satisfying complicated conditions, where $\mathbf{CDGAlg}_{\mathbb{K}}$ is the category of *commutative differential graded* $\mathbb{K}$*-algebras* (*cdgas*) in degrees $\leqslant 0$, which we explain shortly. An alternative definition, essentially equivalent when $\operatorname{char}\mathbb{K} = 0$, uses functors $\mathbf{F} : \mathbf{SAlg}_{\mathbb{K}} \to \mathbf{SSets}$, where $\mathbf{SAlg}_{\mathbb{K}}$ is the category of *simplicial* $\mathbb{K}$*-algebras*.
One might guess that derived schemes should be functors
$\mathbf{F} : \mathbf{CDGAlg}_{\mathbb{K}} \to \mathbf{Sets}$, and derived stacks functors
$\mathbf{F} : \mathbf{CDGAlg}_{\mathbb{K}} \to \mathbf{Groupoids}$, and derived higher stacks functors
$\mathbf{F} : \mathbf{CDGAlg}_{\mathbb{K}} \to \mathbf{SSets}$. In fact only functors
$\mathbf{F} : \mathbf{CDGAlg}_{\mathbb{K}} \to \mathbf{SSets}$ are considered. This is because Derived Algebraic Geometers always make things maximally complicated.

Any $\mathbb{K}$-algebra A can be regarded as a cdga $A^\bullet$ concentrated in degree 0, giving a full subcategory $\mathbf{Alg}_{\mathbb{K}} \subset \mathbf{CDGAlg}_{\mathbb{K}}$. Thus, any functor $\mathbf{F} : \mathbf{CDGAlg}_{\mathbb{K}} \to \mathbf{SSets}$ restricts to a functor $t_0(\mathbf{F}) = \mathbf{F}|_{\mathbf{Alg}_{\mathbb{K}}} : \mathbf{Alg}_{\mathbb{K}} \to \mathbf{SSets}$, called the *classical truncation* of $\mathbf{F}$. If $\mathbf{F}$ is a derived scheme, or derived Deligne–Mumford / Artin stack, or derived stack, then $t_0(\mathbf{F})$ is a scheme, or Deligne–Mumford / Artin stack, or higher stack, respectively.

So, derived stacks do not allow us to study a larger class of moduli problems, as algebraic spaces/stacks/higher stacks do. Instead, they give us a *richer geometric structure* on the moduli spaces we already knew about in classical algebraic geometry.

This is because a derived stack $\mathbf{X}$ knows about all morphisms $\operatorname{Spec} A^\bullet \to \mathbf{X}$ for all cdgas $A^\bullet$, but the corresponding classical stack $X = t_0(\mathbf{X})$ only knows about all morphisms $\operatorname{Spec} A \to X$ for all $\mathbb{K}$-algebras A, which is less information.

2.2. Commutative differential graded $\mathbb{K}$-algebras

Cdgas in derived geometry replace algebras in classical geometry.

Definition

Let $\mathbb{K}$ be a field. A *commutative differential graded $\mathbb{K}$-algebra* (*cdga*) $A^\bullet = (A^*, \mathrm{d})$ in degrees $\leqslant 0$ consists of a $\mathbb{K}$-vector space $A^* = \bigoplus_{k=0}^{-\infty} A^k$ graded in degrees $0, -1, -2, \ldots,$ together with $\mathbb{K}$-bilinear multiplication maps $\cdot : A^k \times A^l \to A^{k+l}$ for all $k, l \leqslant 0$ which are associative and supercommutative (i.e. $\alpha \cdot \beta = (-1)^{kl} \beta \cdot \alpha$ for all $\alpha \in A^k$, $\beta \in A^l$), an identity $1 \in A^0$ with $1 \cdot \alpha = \alpha \cdot 1 = \alpha$ for all $\alpha \in A^*$, and $\mathbb{K}$-linear differentials $\mathrm{d} : A^k \to A^{k+1}$ for all $k < 0$, which satisfy $\mathrm{d}^2 = 0$ and the Leibnitz rule $\mathrm{d}(\alpha \cdot \beta) = (\mathrm{d}\alpha) \cdot \beta + (-1)^k \alpha \cdot (\mathrm{d}\beta)$ for all $\alpha \in A^k$ and $\beta \in A^l$.

Write $H^k(A^\bullet) = \operatorname{Ker}(\mathrm{d} : A^k \to A^{k+1}) / \operatorname{Im}(\mathrm{d} : A^{k-1} \to A^k)$ for the cohomology of $A^\bullet$. Then $H^*(A^\bullet)$ is a graded $\mathbb{K}$-algebra, and $H^0(A^\bullet)$ an ordinary $\mathbb{K}$-algebra.

Example 2.1 (Our main example of cdgas and derived schemes)

Let $m, n \geqslant 0$, and consider the free graded $\mathbb{C}$-algebra $A^* = \mathbb{C}[x_1, \ldots, x_m; y_1, \ldots, y_n]$ generated by commutative variables $x_1, \ldots, x_m$ in degree 0, and anti-commutative variables $y_1, \ldots, y_n$ in degree -1. Then $A^k = \mathbb{C}[x_1, \ldots, x_m] \otimes_{\mathbb{C}} (\Lambda^{-k}\mathbb{C}^n)$ for $k = 0, -1, \ldots, -n$, and $A^k = 0$ otherwise.
Let $p_1, \ldots, p_n \in \mathbb{C}[x_1, \ldots, x_m]$ be complex polynomials in $x_1, \ldots, x_m$. Then as A^* is free, there are unique maps $\mathrm{d} : A^k \to A^{k+1}$ satisfying the Leibniz rule, such that $\mathrm{d}y_i = p_i(x_1, \ldots, x_m)$ for $i = 1, \ldots, n$. Also $\mathrm{d}^2 = 0$, so $A^\bullet = (A^*, \mathrm{d})$ is a cdga. We have $H^0(A^\bullet) = \mathbb{C}[x_1, \ldots, x_m]/(p_1, \ldots, p_n)$, where $(p_1, \ldots, p_n)$ is the ideal generated by $p_1, \ldots, p_n$. Hence $\mathrm{Spec}\, H^0(A^\bullet)$ is the subscheme of $\mathbb{C}^m$ defined by $p_1 = \cdots = p_n = 0$.
We interpret the derived scheme $\mathbf{Spec}\, A^\bullet$ as the derived subscheme of $\mathbb{C}^m$ defined by the equations $p_1 = \cdots = p_n = 0$.

What data does a derived scheme remember?

Consider the solutions X of $p_1 = \cdots = p_n = 0$ in $\mathbb{C}^m$ as (a) a variety, (b) a scheme, and (c) a derived scheme.
The variety X remembers only the set of solutions $(x_1, \ldots, x_m)$ in $\mathbb{C}^m$. So, for example, $x = 0$ and $x^2 = 0$ are the same variety in $\mathbb{C}$.
The scheme X remembers the ideal $(p_1, \ldots, p_n)$, so $x = 0$, $x^2 = 0$ are different schemes in $\mathbb{C}$ as $(x), (x^2)$ are distinct ideals in $\mathbb{C}[x]$.
But schemes forget dependencies between $p_1, \ldots, p_n$. So, for example, $x^2 = y^2 = 0$ with $n = 2$ and $x^2 = y^2 = x^2 + y^2 = 0$ with $n = 3$ are the same scheme in $\mathbb{C}^2$.
The derived scheme $\mathbf{X}$ *remembers information about the dependencies between* $p_1, \ldots, p_n$. For example $x^2 = y^2 = 0$ and $x^2 = y^2 = x^2 + y^2 = 0$ are different derived schemes in $\mathbb{C}^2$, as the two cdgas $A^\bullet, \tilde{A}^\bullet$ have $H^{-1}(A^\bullet) \not\cong H^{-1}(\tilde{A}^\bullet)$. In this case, $\mathbf{X}$ has a well-defined *virtual dimension* $\mathrm{vdim}\, \mathbf{X} = m - n$.

Bézout's Theorem and derived Bézout's Theorem

Let C, D be projective curves in $\mathbb{CP}^2$ of degrees m, n. If the classical scheme $X = C \cap D$ has dimension 0, then Bézout's Theorem says that the number of points in X counted with multiplicity (i.e. length(X)) is mn. But if $\dim X \neq 0$, counterexamples show you cannot recover mn from X.
Now consider the derived intersection $\mathbf{X} = C \cap D$. It is a proper, quasi-smooth derived scheme with $\mathrm{vdim}\, \mathbf{X} = 0$, even if $\dim X = 1$, and so has a 'virtual count' $[\mathbf{X}]_{\mathrm{virt}} \in \mathbb{Z}$, which is mn.
This is a derived version of Bézout's Theorem, without the transversality hypothesis $\dim C \cap D = 0$. It is possible as $\mathbf{X}$ remembers more about how C, D intersect. This illustrates:

General Principles of Derived Geometry

- Transversality is often not needed in derived geometry.
- Derived geometry is useful for Bézout-type 'counting' problems.

Patching together local models

Let X be a (say separated) classical $\mathbb{K}$-scheme. Then we can cover X by Zariski open subschemes $\mathrm{Spec}\, A \cong U \subseteq X$. Given two such $\mathrm{Spec}\, A \cong U$, $\mathrm{Spec}\, \tilde{A} \cong \tilde{U}$, we can compare them easily on the overlap $U \cap \tilde{U}$: there exist $f \in A$, $\tilde{f} \in \tilde{A}$ such that $U \cap \tilde{U}$ is identified with $\{f \neq 0\} \subseteq \mathrm{Spec}\, A$ and $\{\tilde{f} \neq 0\} \subseteq \mathrm{Spec}\, \tilde{A}$, and there is a canonical isomorphism of $\mathbb{K}$-algebras $A[f^{-1}] \cong \tilde{A}[\tilde{f}^{-1}]$, where $A[f^{-1}] = A[x]/(xf - 1)$ is the $\mathbb{K}$-algebra obtained by inverting f in A.
For a derived scheme $\mathbf{X}$, really $\mathbf{X}$ is a functor $\mathbf{CDGAlg}_{\mathbb{K}} \to \mathbf{SSets}$, but we can at least pretend that $\mathbf{X}$ is a space covered by Zariski open $\mathbf{Spec}\, A^\bullet \cong \mathbf{U} \subseteq \mathbf{X}$. Given two $\mathbf{Spec}\, A^\bullet \cong \mathbf{U}$, $\mathbf{Spec}\, \tilde{A}^\bullet \cong \tilde{\mathbf{U}}$, we can find $f \in A^0$, $\tilde{f} \in \tilde{A}^0$ such that $\mathbf{U} \cap \tilde{\mathbf{U}}$ is identified with $\{f \neq 0\} \subseteq \mathbf{Spec}\, A^\bullet$ and $\{\tilde{f} \neq 0\} \subseteq \mathbf{Spec}\, \tilde{A}^\bullet$. However, in general we do *not* have $A^\bullet[f^{-1}] \cong \tilde{A}^\bullet[\tilde{f}^{-1}]$ in $\mathbf{CDGAlg}_{\mathbb{K}}$. Instead, $A^\bullet[f^{-1}], \tilde{A}^\bullet[\tilde{f}^{-1}]$ are only *equivalent* cdgas, in a weak sense.

The problem is that $\mathbf{CDGAlg}_{\mathbb{K}}$ is really the *wrong category*. A *quasi-isomorphism* is a morphism $f : A^\bullet \to \tilde{A}^\bullet$ in $\mathbf{CDGAlg}_{\mathbb{K}}$ such that $H^*(f) : H^*(A^\bullet) \to H^*(\tilde{A}^\bullet)$ is an isomorphism on cohomology. The correct statement is that $A^\bullet[f^{-1}], \tilde{A}^\bullet[\tilde{f}^{-1}]$ should be isomorphic in a 'localized' category $\mathbf{CDGAlg}_{\mathbb{K}}[\mathcal{Q}^{-1}]$ in which all quasi-isomorphisms in $\mathbf{CDGAlg}_{\mathbb{K}}$ have inverses. This is difficult to work with, and should really be an ∞-category.

General Principles of Derived Geometry

- You can usually give nice local models for 'derived' spaces $\mathbf{X}$. However, the local models are glued together on overlaps not by isomorphisms, but by some mysterious equivalence relation.
- We often study categories $\mathcal{C}$ of differential graded objects $A^\bullet$, in which quasi-isomorphisms $\mathcal{Q}$ are to be inverted. The resulting $\mathcal{C}[\mathcal{Q}^{-1}]$ must be treated as an ∞-category, as too much information is lost by the ordinary category.

2.3. Fibre products

To explain why we need higher categories in derived geometry, we discuss fibre products in (ordinary) categories.

Definition

Let $\mathcal{C}$ be a category, and $g : X \to Z$, $h : Y \to Z$ be morphisms in $\mathcal{C}$. A *fibre product* (W, e, f) for g, h in $\mathcal{C}$ consists of an object W and morphisms $e : W \to X$, $f : W \to Y$ in $\mathcal{C}$ with $g \circ e = h \circ f$, with the *universal property* that if $e' : W' \to X$, $f' : W' \to Y$ are morphisms in $\mathcal{C}$ with $g \circ e' = h \circ f'$, then there is a unique morphism $b : W' \to W$ with $e' = e \circ b$ and $f' = f \circ b$.
We write $W = X \times_{g,Z,h} Y$ or $W = X \times_Z Y$.
In general, fibre products may or may not exist. If a fibre product exists, it is unique up to canonical isomorphism.

Given a fibre product $W = X \times_{g,Z,h} Y$, the commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\quad f \quad} & Y \\ \big\downarrow{\scriptstyle e} & & {\scriptstyle h}\big\downarrow \\ X & \xrightarrow{\quad g \quad} & Z \end{array}$$

is called a *Cartesian square*. Some examples:

- All fibre products exist in $\mathbf{Sch}_{\mathbb{K}}$.
- All fibre products $W = X \times_{g,Z,h} Y$ exist in **Top**. We can take $W = \{(x,y) \in X \times Y : g(x) = h(y)\}$, with the subspace topology.
- Not all fibre products exist in **Man**. If $g : X \to Z$, $h : Y \to Z$ are *transverse* then a fibre product $W = X \times_{g,Z,h} Y$ exists in **Man** with $\dim W = \dim X + \dim Y - \dim Z$.

Different kinds of spaces in algebraic geometry
What is derived geometry?
Derived schemes and derived stacks
Commutative differential graded $\mathbb{K}$-algebras
Fibre products
Outlook on Derived Differential Geometry

Intersections of subschemes, or submanifolds, are examples of fibre products. If $C, D \subseteq S$ are $\mathbb{K}$-subschemes of a $\mathbb{K}$-scheme S, then by the $\mathbb{K}$-subscheme $C \cap D$, we actually mean the fibre product $C \times_{i,S,j} D$ in $\mathbf{Sch}_{\mathbb{K}}$, with $i : C \hookrightarrow S$, $j : D \hookrightarrow S$ the inclusions.
Recall our 'derived Bézout's Theorem'. We claimed that given curves $C, D \subset \mathbb{CP}^2$ of degree m, n, there is a 'derived intersection' $\mathbf{X} = C \cap D$, which is quasi-smooth with dimension $\operatorname{vdim} \mathbf{X} = 0$, and has a 'virtual count' $[\mathbf{X}]_{\mathrm{virt}} \in \mathbb{Z}$, which is mn.
This statement *cannot be true* if $\mathbf{X}$ is the fibre product $C \times_{i,\mathbb{CP}^2,j} D$ in an ordinary category $\mathbf{dSch}_{\mathbb{C}}$ of derived $\mathbb{C}$-schemes. For example, if $C = D$ (or if C is a component of D), then in an ordinary category we must have $C \times_{\mathbb{CP}^2} D = C$, so that $\operatorname{vdim} \mathbf{X} = 1$.
However, it can be true if $\mathbf{dSch}_{\mathbb{C}}$ is a *higher category* (e.g. an ∞-category, or a 2-category), and fibre products in $\mathbf{dSch}_{\mathbb{C}}$ satisfy a more complicated universal property involving higher morphisms.

General Principles of Derived Geometry

- Derived geometric spaces should form higher categories (e.g. ∞-categories, or 2-categories), not ordinary categories.

In fact any higher category $\mathcal{C}$ has a *homotopy category* $\mathrm{Ho}(\mathcal{C})$, which is an ordinary category, where objects $\mathbf{X}$ of $\mathrm{Ho}(\mathcal{C})$ are objects of $\mathcal{C}$, and morphisms $[\mathbf{f}]:\mathbf{X}\to\mathbf{Y}$ in $\mathrm{Ho}(\mathcal{C})$ are 2-isomorphism classes of 1-morphisms $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$ in $\mathcal{C}$. So we can reduce to ordinary categories, but this loses too much information.

- The 'correct' fibre products (etc.) in $\mathcal{C}$ satisfy universal properties in $\mathcal{C}$ involving higher morphisms. This does not work in $\mathrm{Ho}(\mathcal{C})$, where no universal property holds.
- In $\mathrm{Ho}(\mathcal{C})$, morphisms $[\mathbf{f}]:\mathbf{X}\to\mathbf{Y}$ are *not local in* $\mathbf{X}$. That is, if $\mathbf{U},\mathbf{V}\subseteq\mathbf{X}$ are open with $\mathbf{X}=\mathbf{U}\cup\mathbf{V}$, then $[\mathbf{f}]$ is not determined by $[\mathbf{f}]|_{\mathbf{U}}$ and $[\mathbf{f}]|_{\mathbf{V}}$. To determine $\mathbf{f}$ up to 2-isomorphism you need to know the *choice* of 2-isomorphism $(\mathbf{f}|_{\mathbf{U}})|_{\mathbf{U}\cap\mathbf{V}}\to(\mathbf{f}|_{\mathbf{V}})|_{\mathbf{U}\cap\mathbf{V}}$, not just the existence.

2.4. Outlook on Derived Differential Geometry

There are several versions of 'derived manifolds' and 'derived orbifolds' in the literature, in order of increasing simplicity:

- Spivak's ∞-category $\mathbf{DerMan_{Spi}}$ of derived manifolds (2008).
- Borisov–Noël's ∞-category $\mathbf{DerMan_{BN}}$ of derived manifolds (2011,2012), which is equivalent to $\mathbf{DerMan_{Spi}}$.
- My d-manifolds and d-orbifolds (2010–2012), which form strict 2-categories $\mathbf{dMan},\mathbf{dOrb}$.
- My M-Kuranishi spaces and Kuranishi spaces (2014), which form a category $\mathbf{MKur}$ and a weak 2-category $\mathbf{Kur}$.

In fact the (M-)Kuranishi space approach is motivated by earlier work by Fukaya, Oh, Ohta and Ono in symplectic geometry (1999,2009-) whose 'Kuranishi spaces' are really a prototype kind of derived orbifold, from before the invention of DAG.

The first version, Spivak's $\mathbf{DerMan_{Spi}}$, was an application of Jacob Lurie's DAG machinery in differential geometry. It is complicated and difficult to work with. Borisov–Noël gave an equivalent (as an ∞-category) but simpler definition $\mathbf{DerMan_{BN}}$. D-manifolds $\mathbf{dMan}$ are nearly a 2-category truncation of $\mathbf{DerMan_{Spi}}$, $\mathbf{DerMan_{BN}}$; Borisov defines a 2-functor $\pi_2(\mathbf{DerMan_{BN}}) \to \mathbf{dMan}$ identifying equivalence classes of objects, and surjective on 2-isomorphism classes of 1-morphisms. There are equivalences of weak 2-categories $\mathbf{dMan} \simeq \mathbf{Kur_{trG}}$ and of (homotopy) categories $\mathrm{Ho}(\mathbf{dMan}) \simeq \mathbf{MKur} \simeq \mathrm{Ho}(\mathbf{Kur_{trG}})$, where $\mathbf{Kur_{trG}} \subset \mathbf{Kur}$ is the 2-category of M-Kuranishi spaces with trivial orbifold groups.
For practical purposes, the five models $\mathbf{DerMan_{Spi}}, \mathbf{DerMan_{BN}}, \mathbf{dMan}, \mathbf{MKur}, \mathbf{Kur_{trG}}$ of derived manifolds are all equivalent, e.g. equivalence classes of objects in all five are in natural bijection.
This course will mainly discuss the simplest models $\mathbf{dMan}, \mathbf{dOrb}, \mathbf{MKur}, \mathbf{Kur}$ of derived manifolds and derived orbifolds.

For $\mathbf{DerMan_{Spi}}, \mathbf{DerMan_{BN}}$, little has been done beyond the original definitions. For $\mathbf{dMan}, \mathbf{dOrb}$ (also for $\mathbf{MKur}, \mathbf{Kur}$) there is a well developed differential geometry, studying immersions, submersions, transverse fibre products, orientations, bordism, virtual cycles, definition from differential-geometric data, etc.
The 'derived geometry' in $\mathbf{dMan}, \mathbf{dOrb}, \mathbf{MKur}, \mathbf{Kur}$ is, by the standards of Derived Algebraic Geometry, very simple. The theory uses 2-categories, which are much simpler than any form of ∞-category, and uses ordinary sheaves rather than homotopy sheaves. This is possible because of nice features of the differential-geometric context: the existence of partitions of unity, and the Zariski topology being Hausdorff.
The theory is still long and complicated for other reasons: firstly, the need to do algebraic geometry over C^∞-rings, and secondly, to define categories of derived manifolds and derived orbifolds *with boundary*, and *with corners*, which are needed for applications.

Properties of d-manifolds

A d-manifold $\mathbf{X}$ is a topological space X with a geometric structure. A d-manifold $\mathbf{X}$ has a *virtual dimension* $\mathrm{vdim}\,\mathbf{X} \in \mathbb{Z}$, which can be negative. If $x \in X$ then there is a *tangent space* $T_x\mathbf{X}$ and an *obstruction space* $O_x\mathbf{X}$, both finite-dimensional over $\mathbb{R}$ with $\dim T_x\mathbf{X} - \dim O_x\mathbf{X} = \mathrm{vdim}\,\mathbf{X}$.
Manifolds **Man** embed in **dMan** as a full (2-)subcategory. A d-manifold $\mathbf{X}$ is (equivalent to) an ordinary manifold if and only if $O_x\mathbf{X} = 0$ for all $x \in X$.
A 1-morphism of d-manifolds $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is a continuous map $f : X \to Y$ with extra structure. If $x \in X$ with $f(x) = y$, then $\mathbf{f}$ induces functorial linear maps $T_x\mathbf{f} : T_x\mathbf{X} \to T_x\mathbf{Y}$ and $O_x\mathbf{f} : O_x\mathbf{X} \to O_y\mathbf{Y}$.

Fibre products of d-manifolds

Recall that smooth maps of manifolds $g : X \to Z$, $h : Y \to Z$ are *transverse* if for all $x \in X$, $y \in Y$ with $g(x) = h(y) = z \in Z$, then $T_xg \oplus T_yh : T_xX \oplus T_yY \to T_zZ$ is surjective. If g, h are transverse then a fibre product $W = X \times_{g,Z,h} Y$ exists in **Man**, with $\dim W = \dim X + \dim Y - \dim Z$.
Similarly, 1-morphisms $\mathbf{g} : \mathbf{X} \to \mathbf{Z}$, $\mathbf{h} : \mathbf{Y} \to \mathbf{Z}$ in **dMan** are *d-transverse* if for all $x \in X$, $y \in Y$ with $g(x) = h(y) = z \in Z$, then $O_x\mathbf{g} \oplus O_y\mathbf{h} : O_x\mathbf{X} \oplus O_y\mathbf{Y} \to O_z\mathbf{Z}$ is surjective.

Theorem 2.2

Suppose $\mathbf{g} : \mathbf{X} \to \mathbf{Z}$, $\mathbf{h} : \mathbf{Y} \to \mathbf{Z}$ *are d-transverse 1-morphisms in* **dMan**. *Then a fibre product* $\mathbf{W} = \mathbf{X} \times_{\mathbf{g},\mathbf{Z},\mathbf{h}} \mathbf{Y}$ *exists in the 2-category* **dMan**, *with* $\mathrm{vdim}\,\mathbf{W} = \mathrm{vdim}\,\mathbf{X} + \mathrm{vdim}\,\mathbf{Y} - \mathrm{vdim}\,\mathbf{Z}$.

Note that the fibre product $\mathbf{W} = \mathbf{X} \times_{\mathbf{Z}} \mathbf{Y}$ is characterized by a universal property involving 2-morphisms in **dMan**, which has no analogue in the ordinary category $\mathrm{Ho}(\mathbf{dMan})$. So we need a 2-category (or other higher category) for Theorem 2.2 to work.
D-transversality is a weak assumption. For example, if $\mathbf{Z}$ is a manifold then $O_z\mathbf{Z} = 0$ for all z, and any $\mathbf{g}, \mathbf{h}$ are d-transverse, so:

Corollary 2.3

Suppose $\mathbf{g} : \mathbf{X} \to Z$, $\mathbf{h} : \mathbf{Y} \to Z$ *are* 1*-morphisms in* **dMan**, *with* Z *a manifold. Then a fibre product* $\mathbf{W} = \mathbf{X} \times_{\mathbf{g},Z,\mathbf{h}} \mathbf{Y}$ *exists in* **dMan**, *with* $\mathrm{vdim}\,\mathbf{W} = \mathrm{vdim}\,\mathbf{X} + \mathrm{vdim}\,\mathbf{Y} - \dim Z$.

This is really useful. For instance, if $g : X \to Z$, $h : Y \to Z$ are smooth maps of manifolds then a fibre product $\mathbf{W} = X \times_{g,Z,h} Y$ exists in **dMan** *without any transversality assumptions at all*.

General Principles of Derived Geometry

- Transversality is often not needed in derived geometry.

Application to moduli spaces

Almost any moduli space used in any enumerative invariant problem over $\mathbb{R}$ or $\mathbb{C}$ has a d-manifold or d-orbifold structure, natural up to equivalence. There are truncation functors to d-manifolds and d-orbifolds from structures currently used — FOOO Kuranishi spaces, polyfolds, $\mathbb{C}$-schemes or Deligne–Mumford $\mathbb{C}$-stacks with obstruction theories.
Combining these truncation functors with known results gives d-manifold/d-orbifold structures on many moduli spaces.
If $P(u) = 0$ is a nonlinear elliptic equation on a compact manifold, then the moduli space $\mathcal{M}$ of solutions u has the structure of a d-manifold $\boldsymbol{\mathcal{M}}$, where if $u \in \mathcal{M}$ is a solution and $\mathcal{L}_uP : C^{k+d,\alpha}(E) \to C^{k,\alpha}(F)$ is the (Fredholm) linearization of P at u, then $T_u\boldsymbol{\mathcal{M}} = \mathrm{Ker}(\mathcal{L}_uP)$ and $O_u\boldsymbol{\mathcal{M}} = \mathrm{Coker}(\mathcal{L}_uP)$.