Derived Differential Geometry

Lecture 13 of 14: Existence of derived manifold or orbifold structures on moduli spaces

Dominic Joyce, Oxford University
Summer 2015

These slides, and references, etc., available at
http://people.maths.ox.ac.uk/∼joyce/DDG2015

Plan of talk:

13. Putting derived orbifold structures on moduli spaces

Suppose we have a moduli space $\mathcal{M}$ of some objects in differential geometry, or complex algebraic geometry, and we would like to make $\mathcal{M}$ into a derived manifold or derived orbifold (Kuranishi space) $\boldsymbol{\mathcal{M}}$, possibly with corners; either in order to form a virtual class/virtual chain for $\boldsymbol{\mathcal{M}}$ as in §12, or for some other reason. How do we go about this? There are two obvious methods:

(A) To somehow directly construct the derived orbifold $\boldsymbol{\mathcal{M}}$.

(B) Suppose we already know, e.g. by a theorem in the literature, that $\mathcal{M}$ carries some other geometric structure $\mathcal{G}$, such as a $\mathbb{C}$-scheme with perfect obstruction theory. Then we may be able to apply a 'truncation functor', a theorem saying that topological spaces X with geometric structure $\mathcal{G}$ can be made into derived manifolds or orbifolds $\mathbf{X}$.

Which moduli problems give derived manifolds or orbifolds?

For a moduli space $\mathcal{M}$ of geometric objects E to form a derived manifold or orbifold $\boldsymbol{\mathcal{M}}$, roughly we need:

(a) Objects E should have at most finite symmetry groups (other than multiples of the identity in linear problems);

(b) Objects E can have deformations and obstructions, but no 'higher obstructions'; and

(c) Some global conditions on $\mathcal{M}$: Hausdorff, constant dimension.

In Differential Geometry, moduli spaces $\mathcal{M}$ of solutions of *nonlinear elliptic equations on compact manifolds* are almost automatically derived manifolds or orbifolds, as we explain in §13.1. This is a large class, which includes many important problems.

In Complex Algebraic Geometry, the deformation theory of objects E in $\mathcal{M}$ is usually understood either in terms of Ext groups $\mathrm{Ext}^i(E,E)$ for $i=0,1,\ldots,$ or sheaf cohomology groups $H^i(\Theta_E)$ of some sheaf Θ_E. Here $\mathrm{Ext}^0(E,E)$ or $H^0(\Theta_E)$ is the Lie algebra of the symmetry group of E; $\mathrm{Ext}^1(E,E)$ or $H^1(\Theta_E)$ the tangent space $T_E\boldsymbol{\mathcal{M}}$; $\mathrm{Ext}^2(E,E)$ or $H^2(\Theta_E)$ the obstruction space $O_E\boldsymbol{\mathcal{M}}$; and $\mathrm{Ext}^i(E,E)$ or $H^i(\Theta_E)$ for $i>2$ the 'higher obstruction spaces'. So to get a derived manifold or orbifold $\boldsymbol{\mathcal{M}}$, we need $\mathrm{Ext}^i(E,E)=0$ or $H^i(\Theta_E)=0$ for $i=0$ and $i>2$.
In linear problems we may restrict to the 'trace-free' part $\mathrm{Ext}^i(E,E)_0$.
We get $\mathrm{Ext}^0(E,E)=0$ or $H^0(\Theta_E)=0$ by restricting to moduli spaces of 'stable' objects E.
$\mathrm{Ext}^i(E,E)=0$ or $H^i(\Theta_E)=0$ for $i>2$ may occur for dimensional reasons. It is automatic for E living on a curve or algebraic surface. For E on some classes of 3-folds (Calabi–Yau, Fano), we may have $\mathrm{Ext}^3(E,E)=0$ by Serre duality or vanishing theorems.

Briefly, the following classes of complex algebraic moduli spaces can usually be made into derived manifolds or orbifolds:

- Moduli spaces of Deligne–Mumford stable curves Σ in a smooth complex algebraic variety Y of any dimension.
- Moduli spaces of stable coherent sheaves / vector bundles / principal bundles on a Riemann surface, complex algebraic surface, Calabi–Yau 3-fold, Fano 3-fold, or Calabi–Yau 4-fold.

In Derived Algebraic Geometry, the main condition for a derived $\mathbb{C}$-stack $\mathbf{X}$ to be a derived manifold or orbifold is that it should be a locally finitely presented derived $\mathbb{C}$-scheme or Deligne–Mumford $\mathbb{C}$-stack which is *quasi-smooth*, i.e. has cotangent complex $\mathbb{L}_{\mathbf{X}}$ perfect in the interval $[-1,0]$.

13.1. D-manifolds and nonlinear elliptic equations

Elliptic equations are a class of p.d.e.s. They are determined (have the same number of equations as unknowns) and satisfy a nondegeneracy condition. Moduli problems with gauge symmetries are often elliptic after 'gauge-fixing'.

Elliptic equations are studied using functional analysis. For example, let Y be a compact manifold, $E, F \to Y$ be vector bundles, and $P : C^\infty(E) \to C^\infty(F)$ a linear partial differential operator of order k. For P to be elliptic we need $\mathrm{rank}\, E = \mathrm{rank}\, F$, and an invertibility condition on the k^{th} order derivatives in P.

Extend P to Hölder spaces $P : C^{k+l,\alpha}(E) \to C^{l,\alpha}(F)$ or Sobolev spaces $P : L^p_{k+l}(E) \to L^p_l(F)$. Then Y compact and P elliptic implies these maps are Fredholm maps between Banach spaces, with $\mathrm{Ker}\, P$, $\mathrm{Coker}\, P$ finite-dimensional, and the *index* $\mathrm{ind} P = \dim \mathrm{Ker}\, P - \dim \mathrm{Coker}\, P$ is given in terms of algebraic topology by the Atiyah–Singer Index Theorem.

Theorem 13.1

Let $\mathcal{V}$ be a Banach manifold, $\mathcal{E} \to \mathcal{V}$ a Banach vector bundle, and $s : \mathcal{V} \to \mathcal{E}$ a smooth Fredholm section, with constant Fredholm index $n \in \mathbb{Z}$. Then there is a d-manifold $\mathbf{X}$, unique up to equivalence in **dMan***, with topological space $X = s^{-1}(0)$ and $\mathrm{vdim}\, \mathbf{X} = n$. If instead $\mathcal{V}$ is a Banach orbifold, or has boundary or corners, then the same thing holds with $\mathbf{X}$ a d-orbifold or Kuranishi space, or with boundary or corners.*

Note that this basically says we can do 'standard model' d-manifolds $\mathbf{S}_{\mathcal{V},\mathcal{E},s}$ for (infinite-dimensional) Banach manifolds $\mathcal{V}$ and Banach vector bundles $\mathcal{E}$, with Fredholm sections s.

To prove Theorem 13.1, near each $x \in s^{-1}(0)$ we use the Implicit Function Theorem for Banach spaces and Fredholmness to show $s^{-1}(0)$ is locally modelled on $\tilde{s}^{-1}(0)$ for $\tilde{V}$ a manifold, $\tilde{E} \to \tilde{V}$ a vector bundle, and $\tilde{s} \in C^\infty(\tilde{E})$. Then we combine these Kuranishi neighbourhoods $(\tilde{V}, \tilde{E}, \tilde{s})$ into a d-manifold/Kuranishi structure on X.

Nonlinear elliptic equations, when written as maps between suitable Hölder or Sobolev spaces, become the zeroes $s=0$ of Fredholm sections s of a (possibly trivial) Banach vector bundle $\mathcal{E}\rightarrow\mathcal{V}$ over a Banach manifold (or Banach space) $\mathcal{V}$. Thus we have:

Corollary 13.2

Let $\mathcal{M}$ be a moduli space of solutions of a nonlinear elliptic equation on a compact manifold, with fixed topological invariants. Then $\mathcal{M}$ extends to a d-manifold $\boldsymbol{\mathcal{M}}$.

The virtual dimension $\boldsymbol{\mathcal{M}}$ at $x\in\boldsymbol{\mathcal{M}}$ is the index of the (Fredholm) linearization of the nonlinear elliptic equation at x, which is given by the A–S Index Theorem. We require *fixed topological invariants* so this dimension is constant over $\boldsymbol{\mathcal{M}}$.

Note that Corollary 13.2 does *not* include problems involving dividing by a gauge group, since such gauge groups typically act only continuously on the Banach manifold. Nonetheless, a similar result should hold for nonlinear elliptic equations modulo gauge.

Example 13.3

Let (X,g), (Y,h) be Riemannian manifolds, with X compact. The moduli space $\mathcal{M}$ of *harmonic maps* $f:X\rightarrow Y$ is defined by a nonlinear elliptic equation, and so becomes a d-manifold $\boldsymbol{\mathcal{M}}$, with $\mathrm{vdim}\,\boldsymbol{\mathcal{M}}=0$. For instance, when $X=\mathcal{S}^1$, $\boldsymbol{\mathcal{M}}$ is the moduli space of parametrized closed geodesics in (Y,h).

Example 13.4

Let (Σ,j) be a Riemann surface, and (Y,J) a manifold with almost complex structure. Then the moduli space $\mathcal{M}(\beta)$ of (j,J)-holomorphic maps $u:\Sigma\rightarrow Y$ with $u_*([\Sigma])=\beta\in H_2(Y;\mathbb{Z})$ is defined by an elliptic equation, and is a d-manifold $\boldsymbol{\mathcal{M}}(\beta)$. Note that (Σ,j) is a *fixed, nonsingular* Riemann surface. Moduli spaces in which (Σ,j) is allowed to vary (and especially, allowed to become singular) are more complicated.

13.2. Truncation functors from other structures
Fukaya–Oh–Ohta–Ono Kuranishi spaces

Fukaya–Ono 1999 and Fukaya–Oh–Ohta–Ono 2009 defined their version of Kuranishi spaces, which we call *FOOO Kuranishi spaces*.

Theorem 13.5

Let $\mathbf{X}$ *be a FOOO Kuranishi space. Then we can define a Kuranishi space* $\mathbf{X}'$ *in the sense of* §8*, canonical up to equivalence in the* 2*-category* **Kur***, with the same topological space as* $\mathbf{X}$*.*
The same holds for other Kuranishi-space-like structures in the literature, such as McDuff–Wehrheim's 'Kuranishi atlases', 2012.

Therefore any moduli space which has been proved to carry a FOOO Kuranishi space structure (many J-holomorphic curve moduli spaces) is also a Kuranishi space/d-orbifold in our sense.
FOOO Kuranishi spaces do not form a category, so Theorem 13.5 does not give a 'truncation functor'.

Hofer–Wysocki–Zehnder's polyfolds

Polyfolds, due to Hofer, Wysocki and Zehnder (2005–2015+), are a rival theory to FOOO Kuranishi spaces. They do form a category. Polyfolds remember much more information than Kuranishi spaces.

Theorem 13.6

There is a functor $\Pi_{\mathbf{PolFS}}^{\mathbf{dOrb^c}}:\mathbf{PolFS}\to \mathrm{Ho}(\mathbf{Kur})$*, where* **PolFS** *is a category whose objects are triples* $(\mathcal{V},\mathcal{E},s)$ *of a polyfold* $\mathcal{V}$*, a fillable strong polyfold bundle* $\mathcal{E}$ *over* $\mathcal{V}$*, and an sc-smooth Fredholm section* s *of* $\mathcal{E}$ *with constant Fredholm index.*

Here $\mathrm{Ho}(\mathbf{Kur})$ is the homotopy category of the 2-category **Kur**. Combining the theorem with constructions of polyfold structures on moduli spaces (e.g. HWZ arXiv:1107.2097, J-holomorphic curves for G–W invariants), gives d-orbifold structures on moduli spaces.

$\mathbb{C}$-schemes and $\mathbb{C}$-stacks with obstruction theories

In algebraic geometry, the standard method of forming virtual cycles is to use a proper scheme or Deligne–Mumford stack equipped with a *perfect obstruction theory* (Behrend–Fantechi). They are used to define algebraic Gromov–Witten invariants, Donaldson–Thomas invariants of Calabi–Yau 3-folds, ...

Theorem 13.7

There is a functor $\Pi_{\mathbf{SchObs}}^{\mathbf{dMan}} : \mathbf{Sch}_{\mathbb{C}}\mathbf{Obs} \to \mathrm{Ho}(\mathbf{dMan})$, *where* $\mathbf{Sch}_{\mathbb{C}}\mathbf{Obs}$ *is a category whose objects are triples* $(X, E^\bullet, \phi)$, *for* X *a separated, second countable* $\mathbb{C}$*-scheme and* $\phi : E^\bullet \to \mathbb{L}_X$ *a perfect obstruction theory on* X *with constant virtual dimension.*
The analogue holds for $\Pi_{\mathbf{StaObs}}^{\mathbf{dOrb}} : \mathbf{Sta}_{\mathbb{C}}\mathbf{Obs} \to \mathrm{Ho}(\mathbf{dOrb})$, *replacing* $\mathbb{C}$*-schemes by Deligne–Mumford* $\mathbb{C}$*-stacks, and d-manifolds by d-orbifolds (or equivalently Kuranishi spaces, using* $\mathrm{Ho}(\mathbf{Kur})$*).*

So, many $\mathbb{C}$-algebraic moduli spaces are d-manifolds or d-orbifolds.

Derived $\mathbb{C}$-schemes and Deligne–Mumford $\mathbb{C}$-stacks

Theorem 13.8

There is a functor $\Pi_{\mathbf{dSch}}^{\mathbf{dMan}} : \mathrm{Ho}(\mathbf{dSch}_{\mathbb{C}}^{\mathbf{qs}}) \to \mathrm{Ho}(\mathbf{dMan})$, *where* $\mathrm{Ho}(\mathbf{dSch}_{\mathbb{C}}^{\mathbf{qs}})$ *is the homotopy category of the* ∞*-category of derived* $\mathbb{C}$*-schemes* $\mathbf{X}$, *where* $\mathbf{X}$ *is assumed locally finitely presented, separated, second countable, of constant virtual dimension, and* **quasi-smooth**, *that is,* $\mathbb{L}_{\mathbf{X}}$ *is perfect in the interval* $[-1, 0]$.
The analogue holds for $\Pi_{\mathbf{dSta}}^{\mathbf{dOrb}} : \mathrm{Ho}(\mathbf{dSta}_{\mathbb{C}}^{\mathbf{qs}}) \to \mathrm{Ho}(\mathbf{dOrb})$, *replacing derived* $\mathbb{C}$*-schemes by derived Deligne–Mumford* $\mathbb{C}$*-stacks, and d-manifolds by d-orbifolds (or Kuranishi spaces).*

Actually this follows from Theorem 13.7, since if $\mathbf{X}$ is a quasi-smooth derived $\mathbb{C}$-scheme then the classical truncation $X = t_0(\mathbf{X})$ is a $\mathbb{C}$-scheme with perfect obstruction theory $\mathbb{L}_i : \mathbb{L}_{\mathbf{X}}|_X \to \mathbb{L}_X$, for $i : X \hookrightarrow \mathbf{X}$ the inclusion.

−2-shifted symplectic derived $\mathbb{C}$-schemes

Theorem 13.9 (Borisov–Joyce arXiv:1504.00690)

Suppose $\mathbf{X}$ *is a derived* $\mathbb{C}$*-scheme with a* −2*-shifted symplectic structure* $\omega_{\mathbf{X}}$ *in the sense of Pantev–Toën–Vaquié–Vezzosi* arXiv:1111.3209. *Then we can define a d-manifold* $\mathbf{X}_{\rm dm}$ *with the same underlying topological space, and virtual dimension* $\mathrm{vdim}_{\mathbb{R}}\,\mathbf{X}_{\rm dm} = \frac{1}{2}\,\mathrm{vdim}_{\mathbb{R}}\,\mathbf{X}$*, i.e. half the expected dimension.*

Note that $\mathbf{X}$ is not quasi-smooth, $\mathbb{L}_{\mathbf{X}}$ lies in the interval $[-2, 0]$, so this does not follow from Theorem 13.8. Also $\mathbf{X}_{\rm dm}$ is only canonical up to bordisms fixing the underlying topological space. Derived moduli schemes or stacks of coherent sheaves on a Calabi–Yau m-fold are $(2-m)$-shifted symplectic, so this gives:

Corollary 13.10

Stable moduli schemes of coherent sheaves $\mathcal{M}$ *with fixed Chern character on a Calabi–Yau* 4*-fold can be made into d-manifolds* $\boldsymbol{\mathcal{M}}$*.*

13.3. D-orbifolds as representable 2-functors

Disclaimer: the rest of this lecture is work in progress (or more honestly, not yet begun). I'm fairly confident it will work eventually.

Recall the Grothendieck approach to moduli spaces in algebraic geometry from §1.3, using *moduli functors*. Write $\mathbf{Sch}_{\mathbb{C}}$ for the category of $\mathbb{C}$-schemes, and $\mathbf{Sch}^{\mathbf{aff}}_{\mathbb{C}}$ for the subcategory of affine $\mathbb{C}$-schemes. Any $\mathbb{C}$-scheme X defines a functor $\mathrm{Hom}(-, X) : \mathbf{Sch}^{\mathbf{op}}_{\mathbb{C}} \to \mathbf{Sets}$ mapping each $\mathbb{C}$-scheme S to the set $\mathrm{Hom}(S, X)$, where $\mathbf{Sch}^{\mathbf{op}}_{\mathbb{C}}$ is the *opposite category* to $\mathbf{Sch}_{\mathbb{C}}$ (reverse directions of morphisms). By the Yoneda Lemma, the $\mathbb{C}$-scheme X is determined up to isomorphism by the functor $\mathrm{Hom}(-, X)$ up to natural isomorphism. This is still true if we restrict to $\mathbf{Sch}^{\mathbf{aff}}_{\mathbb{C}}$. Thus, given a functor $F : (\mathbf{Sch}^{\mathbf{aff}}_{\mathbb{C}})^{\mathbf{op}} \to \mathbf{Sets}$, we can ask if there exists a $\mathbb{C}$-scheme X (necessarily unique up to canonical isomorphism) with $F \cong \mathrm{Hom}(-, X)$. If so, we call F a *representable functor*.

Classical stacks

As in §1.4, to extend this from $\mathbb{C}$-schemes to Deligne–Mumford or Artin $\mathbb{C}$-stacks, we consider functors $F : (\mathbf{Sch}^{\mathbf{aff}}_{\mathbb{C}})^{\mathbf{op}} \to \mathbf{Groupoids}$, where a groupoid is a category all of whose morphisms are isomorphisms. (We can regard a set as a category all of whose morphisms are identities, so replacing **Sets** by **Groupoids** is a generalization.)

A *stack* is a functor $F : (\mathbf{Sch}^{\mathbf{aff}}_{\mathbb{C}})^{\mathbf{op}} \to \mathbf{Groupoids}$ satisfying a sheaf-type condition: if S is an affine $\mathbb{C}$-scheme and $\{S_i : i \in I\}$ an open cover of S (in some algebraic topology) then we should be able to reconstruct $F(S)$ from $F(S_i)$, $F(S_i \cap S_j)$, $F(S_i \cap S_j \cap S_k)$, $i, j, k \in I$, and the functors between them.
A Deligne–Mumford or Artin $\mathbb{C}$-stack is a stack $F : (\mathbf{Sch}^{\mathbf{aff}}_{\mathbb{C}})^{\mathbf{op}} \to \mathbf{Groupoids}$ satisfying extra geometric conditions.

Grothendieck's moduli schemes

Suppose we have an algebro-geometric moduli problem (e.g. vector bundles on a smooth projective $\mathbb{C}$-scheme Y) for which we want to form a moduli scheme. Grothendieck tells us that we should define a *moduli functor* $F : (\mathbf{Sch}^{\mathbf{aff}}_{\mathbb{C}})^{\mathbf{op}} \to \mathbf{Sets}$, such that for each affine $\mathbb{C}$-scheme S, $F(S)$ is the set of isomorphism classes of families of the relevant objects over S (e.g. vector bundles over $Y \times S$). Then we should try to prove F is a representable functor, using some criteria for representability. If it is, $F \cong \mathrm{Hom}(-, \mathcal{M})$, where $\mathcal{M}$ is the (*fine*) *moduli scheme*.

To form a *moduli stack*, we define $F : (\mathbf{Sch}^{\mathbf{aff}}_{\mathbb{C}})^{\mathbf{op}} \to \mathbf{Groupoids}$, so that for each affine $\mathbb{C}$-scheme S, $F(S)$ is the groupoid of families of objects over S, with morphisms isomorphisms of families, and try to show F satisfies the criteria to be an Artin stack.

D-orbifolds as representable 2-functors

D-orbifolds **dOrb** (or Kuranishi spaces **Kur**) are a 2-category with all 2-morphisms invertible. Thus, if $\mathbf{S},\mathbf{X}\in\mathbf{dOrb}$ then $\mathbf{Hom}(\mathbf{S},\mathbf{X})$ is a groupoid, and $\mathbf{Hom}(-,\mathbf{X}):\mathbf{dOrb^{op}}\rightarrow\mathbf{Groupoids}$ is a 2-functor, which determines $\mathbf{X}$ up to equivalence in **dOrb**. This is still true if we restrict to the 2-category $\mathbf{SMod}\subset\mathbf{dOrb}$ of standard model d-manifolds, a good analogue of affine schemes. Thus, we can consider 2-functors $F:\mathbf{SMod^{op}}\rightarrow\mathbf{Groupoids}$, and ask whether there exists a d-orbifold $\mathbf{X}$ (unique up to equivalence) with $F\simeq\mathbf{Hom}(-,\mathbf{X})$. If so, we call F a *representable 2-functor*.

Why use $\mathbf{SMod^{op}}$ as the domain of the functor? A d-orbifold $\mathbf{X}$ also induces a functors $\mathrm{Hom}(-,\mathbf{X}):\mathcal{C}^{\mathbf{op}}\rightarrow\mathbf{Groupoids}$ for $\mathcal{C}=\mathbf{Man},\mathbf{Orb},\mathbf{C^\infty Sch},\mathbf{C^\infty Sta},\mathbf{dMan},\mathbf{dOrb},\mathbf{dSpa},\mathbf{dSta},\ldots$. We want $\mathcal{C}$ large enough that $\mathbf{dOrb}\hookrightarrow\mathbf{Funct}(\mathcal{C}^{\mathbf{op}},\mathbf{Groupoids})$ is an embedding, but otherwise as small as possible, as we must prove things for all objects in $\mathcal{C}$, so a smaller $\mathcal{C}$ saves work.

Criteria for representable 2-functors

Let $F:\mathbf{SMod^{op}}\rightarrow\mathbf{Groupoids}$ be a functor. When is F representable (that is, $F\simeq\mathrm{Hom}(-,\mathbf{X})$ for some d-orbifold $\mathbf{X}$)? It is good to have usable *criteria for representability*, such that if one can show the criteria hold in an example, then we know F is representable (even without constructing the d-orbifold $\mathbf{X}$).
I expect there are nice criteria of the form:

(A) F satisfies a sheaf-type condition, i.e. F is a *stack*;

(B) the 'coarse topological space' $\mathcal{M}=F(\text{point})/\text{isos}$ of F is Hausdorff and second countable, and each point x of $\mathcal{M}$ has finite stabilizer group $\mathrm{Aut}(x)$; and

(C) F admits a 'Kuranishi neighbourhood' of dimension $n\in\mathbb{Z}$ near each $x\in\mathcal{M}$, a local model with a universal property.

Functors satisfying (A) (*stacks*) are a kind of geometric space, even if they are not d-orbifolds. They have points, and a topology, and one can work locally on them.

13.4. Moduli 2-functors in differential geometry

Suppose we are given a moduli problem in differential geometry (e.g. J-holomorphic curves in a symplectic manifold) and we want to form a moduli space $\mathcal{M}$ as a d-orbifold. I propose that we should define a *moduli 2-functor* $F : \mathbf{SMod}^{\mathbf{op}} \to \mathbf{Groupoids}$, such that for each standard model d-manifold $\mathbf{S}$, $F(\mathbf{S})$ is the category of families of the relevant objects over $\mathbf{S}$. Then we should try to prove F satisfies (A)–(C), and so is represented by a d-orbifold $\mathcal{M}$; here (A),(B) will usually be easy, and (C) the difficult part.

If F is represented by $\mathcal{M}$, then there will automatically exist a *universal family* of objects over $\mathcal{M}$.

Example: moduli functors of J-holomorphic curves

Let (M,ω) be a symplectic manifold, and J an almost complex structure on M. Suppose we want to construct $F : \mathbf{SMod}^{\mathbf{op}} \to \mathbf{Groupoids}$ representing the moduli space of J-holomorphic maps $u : \Sigma \to M$, where (Σ, j) is a nonsingular genus g Riemann surface, and $[u(\Sigma)] = \beta \in H_2(M;\mathbb{Z})$.

Then, for each standard model d-manifold $\mathbf{S}$, we must construct a groupoid $F(\mathbf{S})$ of families of J-holomorphic maps $u : \Sigma \to M$ over the base $\mathbf{S}$. There is a natural way to do this:

- Objects of $F(\mathbf{S})$ are quadruples $(\mathbf{X}, \boldsymbol{\pi}, \mathbf{u}, j)$, where $\mathbf{X}$ is a d-manifold with $\mathrm{vdim}\,\mathbf{X} = \mathrm{vdim}\,\mathbf{S} + 2$, $\boldsymbol{\pi} : \mathbf{X} \to \mathbf{S}$ a proper submersion of d-manifolds with $\boldsymbol{\pi}^{-1}(s)$ a genus g surface for all $s \in \mathbf{S}$, $\mathbf{u} : \mathbf{X} \to M$ is a 1-morphism with $[\mathbf{u}(\boldsymbol{\pi}^{-1}(s))] = \beta$ for all $s \in \mathbf{S}$, and $j : \mathbb{T}_{\boldsymbol{\pi}} \to \mathbb{T}_{\boldsymbol{\pi}}$ is bundle linear with $j^2 = -\mathrm{id}$ and $\mathbf{u}^*(J) \circ \mathrm{d}\mathbf{u} = \mathrm{d}\mathbf{u} \circ j$, for $\mathbb{T}_{\boldsymbol{\pi}}$ the relative tangent bundle of $\boldsymbol{\pi}$.

- Morphisms $[\mathbf{i},\eta,\zeta]:(\mathbf{X},\boldsymbol{\pi},\mathbf{u},j)\to(\mathbf{X}',\boldsymbol{\pi}',\mathbf{u}',j')$ in $F(\mathbf{S})$ are $\sim$-equivalence classes $[\mathbf{i},\eta,\zeta]$ of triples $(\mathbf{i},\eta,\zeta)$, where $\mathbf{i}:\mathbf{X}\to\mathbf{X}'$ is an equivalence in **dMan**, and $\eta:\boldsymbol{\pi}\Rightarrow\boldsymbol{\pi}'\circ\mathbf{i}$, $\zeta:\mathbf{u}\Rightarrow\mathbf{u}'\circ\mathbf{i}$ are 2-morphisms, and $H^0(\mathrm{d}\mathbf{i})$ identifies j,j', and $(\mathbf{i},\eta,\zeta)\sim(\tilde{\imath},\tilde\eta,\tilde\zeta)$ if there exists a 2-morphism $\alpha:\mathbf{i}\Rightarrow\tilde{\imath}$ with $\tilde\eta=(\mathrm{id}_{\boldsymbol{\pi}'}*\alpha)\odot\eta$ and $\tilde\zeta=(\mathrm{id}_{\mathbf{u}'}*\alpha)\odot\zeta$.
- If $\mathbf{f}:\mathbf{T}\to\mathbf{S}$ is a 1-morphism in **SMod**, the functor $F(\mathbf{f}):F(\mathbf{S})\to F(\mathbf{T})$ acts by $F(\mathbf{f}):(\mathbf{X},\boldsymbol{\pi},\mathbf{u},j)\mapsto(\mathbf{X}\times_{\boldsymbol{\pi},\mathbf{S},\mathbf{f}}\mathbf{T},\boldsymbol{\pi}_{\mathbf{T}},\mathbf{u}\circ\boldsymbol{\pi}_{\mathbf{X}},\boldsymbol{\pi}_{\mathbf{X}}^*(j))$ on objects and in a natural way on morphisms, with $\mathbf{X}\times_{\boldsymbol{\pi},\mathbf{S},\mathbf{f}}\mathbf{T}$ the fibre product in **dMan**.
- If $\mathbf{f},\mathbf{g}:\mathbf{T}\to\mathbf{S}$ are 1-morphisms and $\theta:\mathbf{f}\Rightarrow\mathbf{g}$ a 2-morphism in **SMod**, then $F(\theta):F(\mathbf{f})\Rightarrow F(\mathbf{g})$ is a natural isomorphism of functors, $F(\theta):(\mathbf{X},\boldsymbol{\pi},\mathbf{u},j)\mapsto[\mathbf{i},\eta,\zeta]$ for $(\mathbf{X},\boldsymbol{\pi},\mathbf{u},j)$ in $F(\mathbf{S})$, where $[\mathbf{i},\eta,\zeta]:(\mathbf{X}\times_{\boldsymbol{\pi},\mathbf{S},\mathbf{f}}\mathbf{T},\boldsymbol{\pi}_{\mathbf{T}},\mathbf{u}\circ\boldsymbol{\pi}_{\mathbf{X}},\boldsymbol{\pi}_{\mathbf{X}}^*(j))\to(\mathbf{X}\times_{\boldsymbol{\pi},\mathbf{S},\mathbf{g}}\mathbf{T},\boldsymbol{\pi}_{\mathbf{T}},\mathbf{u}\circ\boldsymbol{\pi}_{\mathbf{X}},\boldsymbol{\pi}_{\mathbf{X}}^*(j))$ in $F(\mathbf{T})$, with $\mathbf{i}:\mathbf{X}\times_{\boldsymbol{\pi},\mathbf{S},\mathbf{f}}\mathbf{T}\to\mathbf{X}\times_{\boldsymbol{\pi},\mathbf{S},\mathbf{g}}\mathbf{T}$ induced by $\theta:\mathbf{f}\Rightarrow\mathbf{g}$.

Conjecture 13.11

The moduli functor F : $\mathbf{SMod}^{\mathbf{op}}\to\mathbf{Groupoids}$ *above is represented by a d-orbifold.*

Some remarks:

- I may have got the treatment of almost complex structures in the definition of F wrong — this is a first guess.
- I expect to be able to prove Conjecture 13.11 (perhaps after correcting the definition). The proof won't be specific to J-holomorphic curves — there should be a standard method for proving representability of moduli functors of solutions of nonlinear elliptic equations with gauge symmetries, which would also work for many other classes of moduli problems.
- Proving Conjecture 13.11 will involve verifying the representability criteria (A)–(C) above for F.

- The definition of F involves fibre products $\mathbf{X} \times_{\boldsymbol{\pi},\mathbf{S},\mathbf{f}} \mathbf{T}$ in **dMan**, which exist as $\boldsymbol{\pi} : \mathbf{X} \to \mathbf{S}$ is a submersion. Existence of suitable fibre products is *crucial* for the representable 2-functor approach. This becomes complicated when boundaries and corners are involved – see §11.
- Current definitions of differential-geometric moduli spaces (e.g. Kuranishi spaces, polyfolds) are generally very long, complicated ad hoc constructions, with no obvious naturality. In contrast, if we allow differential geometry over d-manifolds, my approach gives you a short, natural definition of the moduli functor F (only 2 slides above give a nearly complete definition!), followed by a long proof that F is representable. The effort moves from a construction to a theorem.
- Can write $\mathbf{X}, \mathbf{S}$ as 'standard model' d-manifolds, as in §5, and $\boldsymbol{\pi}, \mathbf{f}, \eta, \zeta, \ldots$ as 'standard model' 1- and 2-morphisms. Thus, can express F in terms of Kuranishi neighbourhoods and classical differential geometry.

Putting derived orbifold structures on moduli spaces
J-holomorphic curves and Gromov–Witten invariants
D-manifolds and nonlinear elliptic equations
Truncation functors from other structures
D-orbifolds as representable 2-functors
Moduli 2-functors in differential geometry

- The definition of F involves *only finite-dimensional families of smooth objects*, with no analysis, Banach spaces, etc. (But the proof of (C) will involve analysis and Banach spaces.) This enables us to sidestep some analytic problems.
- In some problems, there will be several moduli spaces, with morphisms between them. E.g. if we include *marked points* in our J-holomorphic curves (do this by modifying objects $(\mathbf{X}, \boldsymbol{\pi}, \mathbf{u}, j)$ in $F(\mathbf{S})$ to include morphisms $\mathbf{z}_1, \ldots, \mathbf{z}_k : \mathbf{S} \to \mathbf{X}$ with $\boldsymbol{\pi} \circ \mathbf{z}_i \simeq \mathbf{id}_{\mathbf{S}}$), then we can have 'forgetful functors' between moduli spaces forgetting some of the marked points. Such forgetful functors appear as 2-*natural transformations* $\Theta : F \Rightarrow G$ between moduli functors $F, G : \mathbf{SMod}^{\mathbf{op}} \to \mathbf{Groupoids}$. If F, G are representable, forgetful functors induce 1-morphisms between the d-orbifolds.

Derived Differential Geometry

Lecture 14 of 14: J-holomorphic curves and Gromov–Witten invariants

Dominic Joyce, Oxford University
Summer 2015

These slides, and references, etc., available at
http://people.maths.ox.ac.uk/∼joyce/DDG2015

Plan of talk:

14. J-holomorphic curves and Gromov–Witten invariants

14.1. J-holomorphic curves

An *almost complex structure* J on a $2n$-manifold S is a tensor J_a^b on S with $J_a^b J_b^c = -\delta_a^c$. For $v \in C^\infty(TS)$ define $(Jv)^b = J_a^b v^a$. Then $J^2 = -1$, so J makes the tangent spaces T_pS into complex vector spaces. If J is integrable then (S, J) is a complex manifold.
Now let (S, ω) be a symplectic manifold. An almost complex structure J on S is *compatible with* ω if $g = g_{ab} = \omega_{ac} J_b^c$ is symmetric and positive definite (i.e. a Riemannian metric) on S. If J is integrable then (S, J, g, ω) is Kähler.
Every symplectic manifold (S, ω) admits compatible almost complex structures J, and the (infinite-dimensional) family of such almost complex structures is contractible. So, in particular, given J_0, J_1, there exists a smooth family $J_t : t \in [0, 1]$ of compatible almost complex structures on (S, ω) interpolating between J_0 and J_1.

Let (S, ω) be symplectic, with almost complex structure J. A *pseudoholomorphic curve* or *J-holomorphic curve* in S is a Riemann surface (Σ, j) (almost always compact, sometimes singular) with a smooth map $u : \Sigma \to S$ such that $J \circ \mathrm{d}u = \mathrm{d}u \circ j : T\Sigma \to u^*(TS)$.
Moduli spaces of J-holomorphic curves $\overline{\mathcal{M}}$ in S behave a lot like moduli spaces of curves in complex manifolds, or smooth complex varieties; they do not really care that J is not integrable.
The importance of the symplectic structure is that

$$\mathrm{Area}_g(u(\Sigma)) = \int_{u(\Sigma)} \omega = [\omega] \cdot u_*([\Sigma]),$$

where $u_*([\Sigma]) \in H_2(S; \mathbb{Z})$ and $[\omega] \in H^2_{\mathrm{dR}}(S; \mathbb{R})$, and the area is computed using $g_{ab} = \omega_{ac} J_b^c$. Thus, J-holomorphic curves $u : \Sigma \to S$ in a fixed homology class in $H_2(S; \mathbb{Z})$ have a fixed, and hence bounded, area in S. This helps to ensure moduli spaces $\overline{\mathcal{M}}$ of J-holomorphic curves are compact (as areas of curves cannot go to infinity at noncompact ends of $\overline{\mathcal{M}}$), which is crucial.

Several important areas of symplectic geometry — Gromov–Witten invariants, Lagrangian Floer cohomology, Fukaya categories, contact homology, Symplectic Field Theory, ... — work as follows:

(a) Given a symplectic manifold (S, ω) (etc.), choose compatible J and define moduli spaces $\bar{\mathcal{M}}$ of J-holomorphic curves in S.
(b) Show $\bar{\mathcal{M}}$ is a compact, oriented Kuranishi space (or similar), possibly with corners.
(c) Form a virtual class / virtual chain $[\bar{\mathcal{M}}]_{\rm virt}$ for $\bar{\mathcal{M}}$.
(d) Do homological algebra with these $[\bar{\mathcal{M}}]_{\rm virt}$ to define Gromov–Witten invariants, Lagrangian Floer cohomology, etc.
(e) Prove the results are independent of the choice of J (up to isomorphism), so depend only on (S, ω) (etc.).
(f) Use the machine you have created to prove interesting stuff about symplectic manifolds, Lagrangian submanifolds, ...

We will explain Gromov–Witten invariants.

J-holomorphic curves with marked points

Let (S, ω) be a symplectic manifold, and J an almost complex structure on S compatible with ω. The obvious way to define moduli spaces of J-holomorphic curves is as sets of isomorphism classes $[\Sigma, u]$ of pairs (Σ, u), where Σ is a Riemann surface, and $u : \Sigma \to S$ is J-holomorphic.
But we will do something more complicated. We consider moduli spaces of J-holomorphic curves *with marked points*.
Our moduli spaces $\bar{\mathcal{M}}_{g,m}(S, J, \beta)$ will be sets of isomorphism classes $[\Sigma, \vec{z}, u]$ of triples $(\Sigma, \vec{z}, u)$, where Σ is a Riemann surface, $\vec{z} = (z_1, \ldots, z_m)$ with $z_1, \ldots, z_m$ points of Σ called *marked points*, and $u : \Sigma \to S$ is J-holomorphic. The point of this is that we then have *evaluation maps* ${\rm ev}_i : \bar{\mathcal{M}}_{g,m}(S, J, \beta) \to S$ for $i = 1, \ldots, m$ mapping ${\rm ev}_i : [\Sigma, \vec{z}, u] \mapsto u(z_i)$.

Moduli spaces of nonsingular curves

We first discuss moduli spaces of Riemann surfaces without maps to a symplectic manifold. Fix $g, m \geqslant 0$. Consider pairs $(\Sigma, \vec{z})$, where Σ is a compact, nonsingular Riemann surface with genus g, and $\vec{z} = (z_1, \ldots, z_m)$ are distinct points of Σ. An *isomorphism* between $(\Sigma, \vec{z})$ and $(\Sigma', \vec{z}')$ is a biholomorphism $f : \Sigma \to \Sigma'$ with $f(z_i) = z_i'$ for $i = 1, \ldots, m$. Write $[\Sigma, \vec{z}]$ for the *isomorphism class* of $(\Sigma, \vec{z})$, that is, the equivalence class of $(\Sigma, \vec{z})$ under the equivalence relation of isomorphism.

The *automorphism group* $\mathrm{Aut}(\Sigma, \vec{z})$ is the group of automorphisms f from $(\Sigma, \vec{z})$ to $(\Sigma, \vec{z})$. We call $(\Sigma, \vec{z})$ *stable* if $\mathrm{Aut}(\Sigma, \vec{z})$ is finite. Otherwise $(\Sigma, \vec{z})$ is *unstable*. In fact $(\Sigma, \vec{z})$ is stable iff $g = 0$ and $m \geqslant 3$, or $g = 1$ and $m \geqslant 1$, or $g \geqslant 2$, that is, if $2g + m \geqslant 3$. But if we allow *singular* Σ then there can be unstable $(\Sigma, \vec{z})$ for any g, m. We will exclude unstable $(\Sigma, \vec{z})$ as they would make our moduli spaces non-Hausdorff.

Define $\mathcal{M}_{g,m}$ to be the set of isomorphism classes $[\Sigma, \vec{z}]$ of stable pairs $(\Sigma, \vec{z})$ with Σ nonsingular of genus g and m marked points $\vec{z} = (z_1, \ldots, z_m)$. By studying the deformation theory of pairs $(\Sigma, \vec{z})$ one can prove:

Theorem

$\mathcal{M}_{g,m}$ has the structure of a complex orbifold of complex dimension $3g + m - 3$. It is Hausdorff, but noncompact in general.

Here a *complex orbifold* M is a complex manifold with only quotient singularities. That is, M is locally modelled on $\mathbb{C}^n/\Gamma$ for Γ a finite group acting linearly on $\mathbb{C}^n$. The orbifold singularities of $\mathcal{M}_{g,m}$ come from $[\Sigma, \vec{z}]$ with $\mathrm{Aut}(\Sigma, \vec{z})$ nontrivial; $\mathcal{M}_{g,m}$ is locally modelled near $[\Sigma, \vec{z}]$ on $\mathbb{C}^{3g+m-3}/\mathrm{Aut}(\Sigma, \vec{z})$.

In Gromov–Witten theory, we must work with orbifolds rather than manifolds. This means that G–W invariants are *rational numbers* rather than integers, since the 'number of points' in the 0-orbifold $\{0\}/\Gamma$ should be $1/|\Gamma|$.

To compute $\dim \mathcal{M}_{g,m}$, suppose for simplicity that $g \geqslant 2$. Then we find that

$$T_{[\Sigma,\vec{z}]}\mathcal{M}_{g,m} \cong H^1(T\Sigma) \oplus \bigoplus_{i=1}^{m} T_{z_i}\Sigma,$$

where the sheaf cohomology group $H^1(T\Sigma)$ parametrizes deformations of the complex structure of Σ, and $T_{z_i}\Sigma$ parametrizes variations of the marked point z_i. Thus $\dim_{\mathbb{C}} \mathcal{M}_{g,m}$ $= h^1(T\Sigma) + m$. But $H^0(T\Sigma) = 0$ as $g \geqslant 2$ and $H^k(T\Sigma) = 0$ for $k \geqslant 2$ as $\dim \Sigma = 1$, so $\dim H^1(T\Sigma) = -\chi(T\Sigma)$, and $\chi(T\Sigma) = 3 - 3g$ by the Riemann–Roch Theorem.

10.3. Examples

- $\mathcal{M}_{0,m} = \emptyset$ for $m = 0, 1, 2$ since $\mathrm{Aut}(\mathbb{CP}_1, \vec{z})$ is infinite, e.g. it is $\mathrm{PSL}(2, \mathbb{C})$ for $m = 0$.
- $\mathcal{M}_{0,3}$ is a single point, since any genus 0 curve with 3 marked points is isomorphic to $\bigl(\mathbb{CP}^1, ([1,0],[1,1],[0,1])\bigr)$.
- Suppose $[\Sigma, \vec{z}] \in \mathcal{M}_{0,4}$. Then there is a unique isomorphism $f : \Sigma \to \mathbb{CP}^1$ taking z_1, z_2, z_3 to $[1,0],[1,1],[0,1]$ respectively. Set $f(z_4) = [1, \lambda]$, for $\lambda \in \mathbb{C} \setminus \{0,1\}$. This gives an isomorphism $\mathcal{M}_{0,4} \cong \mathbb{C} \setminus \{0,1\}$. So $\mathcal{M}_{0,4}$ is *noncompact*, the complement of 3 points in $\mathbb{CP}^1$.

- Suppose $[\Sigma,\vec{z}]\in\mathcal{M}_{1,1}$. Choose a basis α,β for $H_1(\Sigma;\mathbb{Z})$. Then there is a unique $\lambda\in\mathbb{C}\setminus\mathbb{R}$ and an isomorphism $f:\Sigma\rightarrow\mathbb{C}/\langle 1,\lambda\rangle_{\mathbb{Z}}$ with $f(z_1)=0$, such that f identifies α with the loop $[0,1]$ and β with the loop $\lambda[0,1]$ in $\mathbb{C}/\langle 1,\lambda\rangle_{\mathbb{Z}}$. Choices of bases α,β for $H_1(\Sigma;\mathbb{Z})$ are parametrized by $GL(2;\mathbb{Z})$. So $\mathcal{M}_{1,1}\cong(\mathbb{C}\setminus\mathbb{R})/GL(2;\mathbb{Z})$. This is a noncompact complex 1-orbifold with two special orbifold points, one with group $\mathbb{Z}_4$ from $\lambda=i$, and one with group $\mathbb{Z}_6$ from $\lambda=e^{2\pi i/6}$. Every other point actually has stabilizer group $\mathbb{Z}_2$.

14.2. Compactification and Deligne–Mumford stable curves

To do Gromov–Witten theory, we need *compact* moduli spaces. So we need a *compactification* $\bar{\mathcal{M}}_{g,m}$ of $\mathcal{M}_{g,m}$. This must satisfy:

- $\bar{\mathcal{M}}_{g,m}$ is a compact, Hausdorff topological space containing $\mathcal{M}_{g,m}$ as an open subset.
- points of $\bar{\mathcal{M}}_{g,m}\setminus\mathcal{M}_{g,m}$ should be interpreted as *singular* Riemann surfaces with marked points.
- $\bar{\mathcal{M}}_{g,m}$ is a complex orbifold.

In general, when compactifying moduli spaces, the compactification should be as close to being a smooth, oriented manifold as we can manage. In this case, we can make it a complex orbifold.

In algebraic geometry there are often several different ways of compactifying moduli spaces. In this case there is a clear best way to do it, using *Deligne–Mumford stable curves*.

A *prestable Riemann surface* Σ is a compact connected complex variety of dimension 1 whose only singular points are finitely many *nodes*, modelled on (0,0) in $\{(x,y)\in\mathbb{C}^2 : xy=0\}$. Each such singular Σ is the limit as $t\to 0$ of a family of nonsingular Riemann surfaces Σ_t for $0<|t|<\epsilon$ modelled on $\{(x,y)\in\mathbb{C}^2 : xy=t\}$ near each node of Σ.
We call Σ_t a *smoothing* of Σ. The *genus* of Σ is the genus of its smoothings Σ_t.
A *prestable Riemann surface* $(\Sigma,\vec z)$ *with marked points* is a prestable Σ with $\vec z=(z_1,\ldots,z_m)$, where $z_1,\ldots,z_m$ are distinct smooth points (not nodes) of Σ. Define isomorphisms and $\mathrm{Aut}(\Sigma,\vec z)$ as in the nonsingular case. We call $(\Sigma,\vec z)$ *stable* if $\mathrm{Aut}(\Sigma,\vec z)$ is finite.
The D–M moduli space $\overline{\mathcal{M}}_{g,m}$ is the set of isomorphism classes $[\Sigma,\vec z]$ of stable pairs $(\Sigma,\vec z)$, where Σ is a prestable Riemann surface of genus g, and $\vec z=(z_1,\ldots,z_m)$ are distinct smooth points of Σ.

Theorem

$\overline{\mathcal{M}}_{g,m}$ is a compact, Hausdorff complex orbifold of complex dimension $3g+m-3$.

The moduli spaces $\overline{\mathcal{M}}_{g,m}$ are very well-behaved, because of exactly the right choice of definition of singular curve. With (nearly) any other notion of singular curve, we would have lost compactness, or Hausdorffness, or smoothness.
The $\overline{\mathcal{M}}_{g,m}$ have been intensively studied, lots is known about their cohomology, etc.
Note that as $\overline{\mathcal{M}}_{g,m}$ is complex, it is *oriented* as a real orbifold.

Example

$\bar{\mathcal{M}}_{0,4}$ is $\mathbb{CP}^1$, with $\bar{\mathcal{M}}_{0,4} \setminus \mathcal{M}_{0,4}$ three points. These correspond to two $\mathbb{CP}^1$'s joined by a node, with two marked points in each $\mathbb{CP}^1$.

14.3. Moduli spaces of stable maps

Now let (S,ω) be a compact symplectic manifold, and J an almost complex structure compatible with ω. Fix $g,m\geqslant 0$ and $\beta\in H_2(S;\mathbb{Z})$. Consider triples $(\Sigma,\vec z,u)$ where $(\Sigma,\vec z)$ is a prestable Riemann surface of genus g (possibly singular) with marked points, and $u:\Sigma\to S$ a J-holomorphic map, with $u_*([\Sigma])=\beta$ in $H_2(S;\mathbb{Z})$.
An *isomorphism* between $(\Sigma,\vec z,u)$ and $(\Sigma',\vec z',u')$ is a biholomorphism $f:\Sigma\to\Sigma'$ with $f(z_i)=z_i'$ for $i=1,\ldots,m$ and $u'\circ f\equiv u$.
The *automorphism group* $\mathrm{Aut}(\Sigma,\vec z,u)$ is the set of isomorphisms from $(\Sigma,\vec z,u)$ to itself. We call $(\Sigma,\vec z,u)$ *stable* if $\mathrm{Aut}(\Sigma,\vec z,u)$ is finite. The *moduli space* $\bar{\mathcal{M}}_{g,m}(S,J,\beta)$ is the set of isomorphism classes $[\Sigma,\vec z,u]$ of stable triples $(\Sigma,\vec z,u)$, for Σ of genus g with m marked points $\vec z$, and $u_*([\Sigma])=\beta$ in $H_2(S;\mathbb{Z})$.
We also write $\mathcal{M}_{g,m}(S,J,\beta)$ for the subset of $[\Sigma,\vec z,u]$ with Σ nonsingular.

For $i = 1, \ldots, m$ define *evaluation maps* $\mathrm{ev}_i : \bar{\mathcal{M}}_{g,m}(S, J, \beta) \to S$ by $\mathrm{ev}_i : [\Sigma, \vec{z}, u] \mapsto u(z_i)$.
Define $\pi : \bar{\mathcal{M}}_{g,m}(S, J, \beta) \to \bar{\mathcal{M}}_{g,m}$ for $2g + m \geqslant 3$ by $\pi : [\Sigma, \vec{z}, u] \mapsto [\Sigma, \vec{z}]$, provided $(\Sigma, \vec{z})$ is stable. (If $(\Sigma, \vec{z})$ is unstable, map to the *stabilization* of $(\Sigma, \vec{z})$.)
There is a natural topology on $\bar{\mathcal{M}}_{g,m}(S, J, \beta)$ due to Gromov, called the C^∞ *topology*. It is derived from the notion of smooth family of prestable $(\Sigma, \vec{z})$ used to define the topology on $\bar{\mathcal{M}}_{g,m}$, and the C^∞ topology on smooth maps $u : \Sigma \to S$.

Theorem 14.1

$\bar{\mathcal{M}}_{g,m}(S, J, \beta)$ is a compact, Hausdorff topological space. Also $\mathrm{ev}_1, \ldots, \mathrm{ev}_m, \pi$ are continuous.

Both compactness and Hausdorffness in Theorem 14.1 are nontrivial. Hausdorffness really follows from the Hausdorffness of $\bar{\mathcal{M}}_{g,m}$. Compactness follows from the compactness of S, the compactness of $\bar{\mathcal{M}}_{g,m}$, the fixed homology class β, and the fact that J is compatible with a symplectic form ω, which bounds areas of curves.

Theorem 14.2 (Fukaya–Ono 1999; Hofer–Wysocki–Zehnder 2011)

We can make $\bar{\mathcal{M}}_{g,m}(S, J, \beta)$ into a compact, oriented Kuranishi space $\bar{\boldsymbol{\mathcal{M}}}_{g,m}(S, J, \beta)$, without boundary, of virtual dimension

$$2d = 2\big(c_1(S) \cdot \beta + (n-3)(1-g) + m\big), \tag{14.1}$$

where $\dim S = 2n$. Also $\mathrm{ev}_1, \ldots, \mathrm{ev}_m, \pi$ become 1-morphisms $\mathbf{ev}_i : \bar{\boldsymbol{\mathcal{M}}}_{g,m}(S, J, \beta) \to S$ and $\boldsymbol{\pi} : \bar{\boldsymbol{\mathcal{M}}}_{g,m}(S, J, \beta) \to \bar{\mathcal{M}}_{g,m}$.

For J-holomorphic maps $u : \Sigma \to S$ from a fixed Riemann surface Σ, or even from a varying, nonsingular Riemann surface Σ, this is fairly straightforward, given the technology we already discussed. Including singular curves is more difficult.

14.4. Virtual classes and Gromov–Witten invariants

We have now defined a compact, oriented Kuranishi space $\bar{\boldsymbol{\mathcal{M}}}_{g,m}(S,J,\beta)$ of dimension $2d$ in (14.1), and a 1-morphism

$$\mathbf{ev}_1\times\cdots\times\mathbf{ev}_m\times\boldsymbol{\pi}:\bar{\boldsymbol{\mathcal{M}}}_{g,m}(S,J,\beta)\longrightarrow S^m\times\bar{\mathcal{M}}_{g,m}$$

if $2g+m\geqslant 3$, where $S^m\times\bar{\mathcal{M}}_{g,m}$ is an orbifold, or

$$\mathbf{ev}_1\times\cdots\times\mathbf{ev}_m:\bar{\boldsymbol{\mathcal{M}}}_{g,m}(S,J,\beta)\to S^m$$

if $2g+m<3$, where S^m is a manifold.
As in §12.2, we can define a virtual class $[\bar{\boldsymbol{\mathcal{M}}}_{g,m}(S,J,\beta)]_{\rm virt}$ in $H_{2d}\big(S^m\times\bar{\mathcal{M}}_{g,m};\mathbb{Q}\big)$ or $H_{2d}\big(S^m;\mathbb{Q}\big)$.

Theorem 14.3 (Fukaya–Ono 1999)

These virtual classes $[\bar{\boldsymbol{\mathcal{M}}}_{g,m}(S,J,\beta)]_{\rm virt}$ are independent of the choice of almost complex structure J compatible with ω. They are also unchanged by continuous deformations of ω.

Sketch proof.

Let J_0,J_1 be possible almost complex structures. Choose a smooth family $J_t:t\in[0,1]$ of compatible almost complex structures joining them. Write $\bar{\mathcal{M}}_{g,m}(S,J_t:t\in[0,1],\beta)$ for the union of $\bar{\mathcal{M}}_{g,m}(S,J_t,\beta)$ over $t\in[0,1]$. This becomes a compact oriented Kuranishi space with boundary of virtual dimension $2d+1$, whose boundary is $\bar{\boldsymbol{\mathcal{M}}}_{g,m}(S,J_1,\beta)\amalg-\bar{\boldsymbol{\mathcal{M}}}_{g,m}(S,J_0,\beta)$.
Construct a virtual chain for $\bar{\boldsymbol{\mathcal{M}}}_{g,m}(S,J_t:t\in[0,1],\beta)$. This is a $(2d+1)$-chain on $S^m\times\bar{\mathcal{M}}_{g,m}$ whose boundary is the difference of virtual cycles for $\bar{\boldsymbol{\mathcal{M}}}_{g,m}(S,J_1,\beta)$ and $\bar{\boldsymbol{\mathcal{M}}}_{g,m}(S,J_0,\beta)$. Thus these cycles are homologous, and their homology classes, the virtual classes $[\bar{\boldsymbol{\mathcal{M}}}_{g,m}(S,J_i,\beta)]_{\rm virt}$, are the same. The same argument works for continuous deformations of ω. □

Gromov–Witten invariants

Gromov–Witten invariants are basically the virtual classes $[\overline{\mathcal{M}}_{g,m}(S,J,\beta)]_{\rm virt}$. But it is conventional to define them as *maps on cohomology*, rather than as homology classes. We follow Cox and Katz §7. Since $\overline{\mathcal{M}}_{g,m}$ is a compact oriented orbifold of real dimension $6g+2m-6$, Poincaré duality gives an isomorphism

$$H_l(\overline{\mathcal{M}}_{g,m};\mathbb{Q}) \cong H^{6g+2m-6-l}(\overline{\mathcal{M}}_{g,m};\mathbb{Q}). \qquad (14.2)$$

For $g,m \geqslant 0$ and $\beta \in H_2(S;\mathbb{Z})$, the *Gromov–Witten invariant*

$$\langle I_{g,m,\beta}\rangle : H^*(S;\mathbb{Q})^{\otimes^m} \to \mathbb{Q}$$

is the linear map corresponding to the virtual cycle $[\overline{\mathcal{M}}_{g,m}(S,J,\beta)]_{\rm virt}$ in $H_{2d}(S^m;\mathbb{Q})$ under the Künneth isomorphism

$$H_*(S^m;\mathbb{Q}) \cong \bigl(H^*(S;\mathbb{Q})^{\otimes^m}\bigr)^*$$

This is zero on $H^{k_1}(S;\mathbb{Q})\otimes\cdots\otimes H^{k_m}(S;\mathbb{Q})$ unless $k_1+\cdots+k_m=2d$.

Putting derived orbifold structures on moduli spaces
J-holomorphic curves and Gromov–Witten invariants
J-holomorphic curves
Compactification and Deligne–Mumford stable curves
Moduli spaces of stable maps
Virtual classes and Gromov–Witten invariants

Gromov–Witten classes

For $2g+m \geqslant 3$ and $\beta \in H_2(S;\mathbb{Z})$, the *Gromov–Witten class*

$$I_{g,m,\beta} : H^*(S;\mathbb{Q})^{\otimes^m} \to H^*(\overline{\mathcal{M}}_{g,m};\mathbb{Q})$$

is the linear map corresponding to $[\overline{\mathcal{M}}_{g,m}(S,J,\beta)]_{\rm virt}$ under

$$H_*(S^m\times\overline{\mathcal{M}}_{g,m};\mathbb{Q}) \cong \bigl(H^*(S;\mathbb{Q})^{\otimes^m}\bigr)^* \otimes H^{6g+2m-6-*}(\overline{\mathcal{M}}_{g,m};\mathbb{Q}),$$

using Künneth again and (14.2). The relation between G–W invariants $\langle I_{g,m,\beta}\rangle$ and G–W classes $I_{g,m,\beta}$ is

$$\langle I_{g,m,\beta}\rangle = \int_{\overline{\mathcal{M}}_{g,m}} I_{g,m,\beta},$$

that is, $\langle I_{g,m,\beta}\rangle$ is the contraction of $I_{g,m,\beta}$ with the fundamental class $[\overline{\mathcal{M}}_{g,m}]$. Gromov–Witten classes satisfy a system of axioms (Kontsevich and Manin 1994). Using them we can define *quantum cohomology* of symplectic manifolds.

The Splitting Axiom

Suppose $[\Sigma^1, \vec{z}^1] \in \overline{\mathcal{M}}_{g^1,m^1+1}$ and $[\Sigma^2, \vec{z}^2] \in \overline{\mathcal{M}}_{g^2,m^2+1}$. Then we can glue Σ_1, Σ_2 together at marked points $z^1_{m^1+1}, z^2_{m^2+1}$ to get Σ with a node at $z^1_{m^1+1} = z^2_{m^2+1}$.

The Splitting Axiom

This Σ has genus $g = g^1 + g^2$ and $m = m^1 + m^2$ remaining marked points $z^1_1, \ldots, z^1_{m^1}$ from $(\Sigma^1, \vec{z}^1)$ and $z^2_1, \ldots, z^2_{m^2}$ from $(\Sigma^2, \vec{z}^2)$. Define $\vec{z} = (z^1_1, \ldots, z^1_{m^1}, z^2_1, \ldots, z^2_{m^2})$. Then $[\Sigma, \vec{z}] \in \overline{\mathcal{M}}_{g,m}$. This defines a map

$$\varphi : \overline{\mathcal{M}}_{g^1,m^1+1} \times \overline{\mathcal{M}}_{g^2,m^2+1} \to \overline{\mathcal{M}}_{g,m}.$$

Choose a basis $(T_i)_{i=1}^N$ for $H^*(S;\mathbb{Q})$, and let $(T^j)_{j=1}^N$ be the dual basis under the cup product, that is, $T_i \cup T^j = \delta_i^j$.
Then the Splitting Axiom says that

$$\varphi^*\bigl(I_{g,m,\beta}(\alpha^1_1, \ldots, \alpha^1_{m^1}, \alpha^2_1, \ldots, \alpha^2_{m^2})\bigr) = \sum_{\beta=\beta^1+\beta^2} \sum_{i=1}^N I_{g^1,m^1,\beta^1}(\alpha^1_1, \ldots, \alpha^1_{m^1}, T_i) \otimes I_{g^2,m^2,\beta^2}(\alpha^2_1, \ldots, \alpha^2_{m^2}, T^i).$$

The Splitting Axiom

Here is how to understand this. Let Δ_S be the diagonal $\{(p,p): p \in S\}$ in $S \times S$. Then $\sum_{i=1}^N T_i \otimes T^i$ in $H^*(S;\mathbb{Q}) \otimes H^*(S;\mathbb{Q})$ is Poincaré dual to $[\Delta_S]$ in $H_*(S \times S;\mathbb{Q}) \cong H_*(S;\mathbb{Q}) \otimes H_*(S;\mathbb{Q})$.

Thus the term

$$\sum_{i=1}^N I_{g^1,m^1,\beta^1}(\alpha_1^1,\ldots,\alpha_{m^1}^1,T_i) \otimes I_{g^2,m^2,\beta^2}(\alpha_1^2,\ldots,\alpha_{m^2}^2,T^i)$$

'counts' pairs of curves $u^1 : \Sigma^1 \to S$ and $u^2 : \Sigma^2 \to S$, with genera g^1, g^2 and homology classes β^1, β^2, such that $u^a(\Sigma^a)$ intersects cycles $C_1^a,\ldots,C_{m^a}^a$ Poincaré dual to $\alpha_1^a,\ldots,\alpha_{m^a}^a$, and also $u^1(\Sigma^1) \times u^2(\Sigma^2)$ intersects the diagonal Δ_S in $S \times S$. This last condition means that $u^1(\Sigma^1)$ and $u^2(\Sigma^2)$ intersect in S. But then we can glue Σ^1, Σ^2 at their intersection point to get a nodal curve Σ, genus $g = g^1 + g^2$, class $\beta = \beta^1 + \beta^2$.