Derived Differential Geometry

Lecture 3 of 14: C^∞-Algebraic Geometry

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These slides, and references, etc., available at
http://people.maths.ox.ac.uk/∼joyce/DDG2015

Plan of talk:

3. C^∞-Algebraic Geometry

Our goal is to define the 2-category of d-manifolds **dMan**.
First consider an algebro-geometric version of what we want to do.
A good algebraic analogue of smooth manifolds are *complex algebraic manifolds*, that is, separated smooth $\mathbb{C}$-schemes S of pure dimension. These form a full subcategory $\mathbf{AlgMan}_{\mathbb{C}}$ in the category $\mathbf{Sch}_{\mathbb{C}}$ of $\mathbb{C}$-schemes, and can roughly be characterized as the (sufficiently nice) objects S in $\mathbf{Sch}_{\mathbb{C}}$ whose cotangent complex $\mathbb{L}_S$ is a vector bundle (i.e. perfect in the interval $[0, 0]$).

To make a derived version of this, we first define an ∞-category $\mathbf{DerSch}_{\mathbb{C}}$ of *derived* $\mathbb{C}$*-schemes*, and then define the ∞-category $\mathbf{DerAlgMan}_{\mathbb{C}}$ of *derived complex algebraic manifolds* to be the full ∞-subcategory of objects $\mathbf{S}$ in $\mathbf{DerSch}_{\mathbb{C}}$ which are *quasi-smooth* (have cotangent complex $\mathbb{L}_S$ perfect in the interval $[-1, 0]$), and satisfy some other niceness conditions (separated, etc.).

Thus, we have 'classical' categories $\mathbf{AlgMan}_{\mathbb{C}} \subset \mathbf{Sch}_{\mathbb{C}}$, and related 'derived' ∞-categories $\mathbf{DerAlgMan}_{\mathbb{C}} \subset \mathbf{DerSch}_{\mathbb{C}}$.
David Spivak, a student of Jacob Lurie, defined an ∞-category $\mathbf{DerMan}_{\mathbf{Spi}}$ of 'derived smooth manifolds' using a similar structure: he considered 'classical' categories $\mathbf{Man} \subset \mathbf{C^\infty Sch}$ and related 'derived' ∞-categories $\mathbf{DerMan}_{\mathbf{Spi}} \subset \mathbf{DerC^\infty Sch}$. Here $\mathbf{C^\infty Sch}$ is C^∞*-schemes*, and $\mathbf{DerC^\infty Sch}$ *derived* C^∞*-schemes*. That is, before we can 'derive', we must first embed **Man** into a larger category of C^∞-schemes, singular generalizations of manifolds.
Our set-up is a simplification of Spivak's. I consider 'classical' categories $\mathbf{Man} \subset \mathbf{C^\infty Sch}$ and related 'derived' 2-categories $\mathbf{dMan} \subset \mathbf{dSpa}$, where **dMan** is *d-manifolds*, and **dSpa** *d-spaces*.
Here $\mathbf{dMan}, \mathbf{dSpa}$ are roughly 2-category truncations of Spivak's $\mathbf{DerMan}, \mathbf{DerC^\infty Sch}$ — see Borisov arXiv:1212.1153.
This lecture will introduce classical C^∞-schemes.

3.1. C^∞-rings

Algebraic geometry (based on algebra and polynomials) has excellent tools for studying singular spaces – the theory of schemes. In contrast, conventional differential geometry (based on smooth real functions and calculus) deals well with nonsingular spaces – manifolds – but poorly with singular spaces.
There is a little-known theory of schemes in differential geometry, C^∞-*schemes*, going back to Lawvere, Dubuc, Moerdijk and Reyes, ... in synthetic differential geometry in the 1960s-1980s.
C^∞-schemes are essentially *algebraic* objects, on which smooth real functions and calculus make sense.
The theory works by replacing commutative rings or $\mathbb{K}$-algebras in algebraic geometry by algebraic objects called C^∞-*rings*.

Definition 3.1 (First definition of C^∞-ring)

A C^∞-*ring* is a set $\mathfrak{C}$ together with n-fold operations $\Phi_f : \mathfrak{C}^n \to \mathfrak{C}$ for all smooth maps $f : \mathbb{R}^n \to \mathbb{R}$, $n \geqslant 0$, satisfying:
Let $m, n \geqslant 0$, and $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$ and $g : \mathbb{R}^m \to \mathbb{R}$ be smooth functions. Define $h : \mathbb{R}^n \to \mathbb{R}$ by

$$h(x_1, \ldots, x_n) = g(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1 \ldots, x_n)),$$

for $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Then for all $c_1, \ldots, c_n$ in $\mathfrak{C}$ we have

$$\Phi_h(c_1, \ldots, c_n) = \Phi_g(\Phi_{f_1}(c_1, \ldots, c_n), \ldots, \Phi_{f_m}(c_1, \ldots, c_n)).$$

Also defining $\pi_j : (x_1, \ldots, x_n) \mapsto x_j$ for $j = 1, \ldots, n$ we have $\Phi_{\pi_j} : (c_1, \ldots, c_n) \mapsto c_j$.
A *morphism* of C^∞-rings is a map of sets $\phi : \mathfrak{C} \to \mathfrak{D}$ with $\Phi_f \circ \phi^n = \phi \circ \Phi_f : \mathfrak{C}^n \to \mathfrak{D}$ for all smooth $f : \mathbb{R}^n \to \mathbb{R}$. Write $\mathbf{C^\infty Rings}$ for the category of C^∞-rings.

Definition 3.2 (Second definition of C^∞-ring)

Write **Euc** for the full subcategory of manifolds **Man** with objects $\mathbb{R}^n$ for $n = 0, 1, \ldots$. That is, **Euc** is the category with objects Euclidean spaces $\mathbb{R}^n$, and morphisms smooth maps $f : \mathbb{R}^m \to \mathbb{R}^n$.
A *C^∞-ring* is a product-preserving functor F : **Euc** $\to$ **Sets**. That is, F is a functor with functorial identifications
$F(\mathbb{R}^{m+n}) = F(\mathbb{R}^m \times \mathbb{R}^n) \cong F(\mathbb{R}^m) \times F(\mathbb{R}^n)$ for all $m, n \geqslant 0$.
A morphism $\phi : F \to G$ of C^∞-rings F, G is a natural transformation of functors $\phi : F \Rightarrow G$.

Definitions 3.1 and 3.2 are equivalent as follows. Given F : **Euc** $\to$ **Sets** as above, define a set $\mathfrak{C} = F(\mathbb{R})$. As F is product-preserving, $F(\mathbb{R}^n) \cong F(\mathbb{R})^n = \mathfrak{C}^n$ for all $n \geqslant 0$. If $f : \mathbb{R}^n \to \mathbb{R}$ is smooth then $F(f) : F(\mathbb{R}^n) \to F(\mathbb{R})$ is identified with a map $\Phi_f : \mathfrak{C}^n \to \mathfrak{C}$. Then $\bigl(\mathfrak{C}, \Phi_{f,\, f:\mathbb{R}^n \to \mathbb{R}\ C^\infty}\bigr)$ is a C^∞-ring as in Definition 3.1. Conversely, given $\mathfrak{C}$ we define F with $F(\mathbb{R}^n) = \mathfrak{C}^n$.

Manifolds as C^∞-rings

Let X be a manifold, and write $\mathfrak{C} = C^\infty(X)$ for the set of smooth functions $c : X \to \mathbb{R}$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth. Define $\Phi_f : C^\infty(X)^n \to C^\infty(X)$ by $\Phi_f(c_1, \ldots, c_n)(x) = f\bigl(c_1(x), \ldots, c_n(x)\bigr)$ for $x \in X$. These make $C^\infty(X)$ into a C^∞-ring as in Definition 3.1.
Define F : **Euc** $\to$ **Sets** by $F(\mathbb{R}^n) = \mathrm{Hom}_{\mathbf{Man}}(X, \mathbb{R}^n)$ and $F(f) = f\circ : \mathrm{Hom}_{\mathbf{Man}}(X, \mathbb{R}^m) \to \mathrm{Hom}_{\mathbf{Man}}(X, \mathbb{R}^n)$ for $f : \mathbb{R}^m \to \mathbb{R}^n$ smooth. Then F is a C^∞-ring as in Definition 3.2.
If $f : X \to Y$ is smooth map of manifolds then $f^* : C^\infty(Y) \to C^\infty(X)$ is a morphism of C^∞-rings; conversely, if $\phi : C^\infty(Y) \to C^\infty(X)$ is a morphism of C^∞-rings then $\phi = f^*$ for some unique smooth $f : X \to Y$. This gives a *full and faithful functor* $F : \mathbf{Man} \to \mathbf{C^\infty Rings}^{\mathrm{op}}$ by $F : X \mapsto C^\infty(X)$, $F : f \mapsto f^*$.
Thus, we can think of manifolds as examples of C^∞-rings. But there are many more C^∞-rings than manifolds. For example, $C^0(X)$ is a C^∞-ring for any topological space X.

C^∞-rings as $\mathbb{R}$-algebras, ideals, and quotient C^∞-rings

Every C^∞-ring $\mathfrak{C}$ is a commutative $\mathbb{R}$-algebra, where addition is $c+d=\Phi_f(c,d)$ for $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=x+y$, and multiplication is $c\cdot d=\Phi_g(c,d)$ for $g:\mathbb{R}^2\to\mathbb{R}$, $g(x,y)=xy$, multiplication by $\alpha\in\mathbb{R}$ is $\alpha\, c=\Phi_h(c)$ for $h:\mathbb{R}\to\mathbb{R}$, $h(x)=\alpha x$. An ideal $I\subseteq\mathfrak{C}$ in a C^∞-ring $\mathfrak{C}$ is an ideal in $\mathfrak{C}$ as an $\mathbb{R}$-algebra. Then the quotient vector space $\mathfrak{C}/I$ is a commutative $\mathbb{R}$-algebra.

Proposition 3.3

If $\mathfrak{C}$ is a C^∞-ring and $I\subseteq\mathfrak{C}$ an ideal, then there is a unique C^∞-ring structure on $\mathfrak{C}/I$ such that the projection $\pi:\mathfrak{C}\to\mathfrak{C}/I$ is a morphism of C^∞-rings.

Definition

A C^∞-ring $\mathfrak{C}$ is called *finitely generated* if $\mathfrak{C}\cong C^\infty(\mathbb{R}^n)/I$ for some ideal $I\subseteq C^\infty(\mathbb{R}^n)$.

Proof of Proposition 3.3

Let $f:\mathbb{R}^n\to\mathbb{R}$ be smooth, and $c_1+I,\ldots,c_n+I\in\mathfrak{C}/I$. For $\pi:\mathfrak{C}\to\mathfrak{C}/I$ to be a morphism of C^∞-rings, we are forced to set

$$\Phi_f(c_1+I,\ldots,c_n+I)=\Phi_f(c_1,\ldots,c_n)+I,$$

which determines the C^∞-ring structure on $\mathfrak{C}/I$ completely. The only thing to prove is that this is well-defined. That is, if $c_1',\ldots,c_n'\in\mathfrak{C}$ with $c_i-c_i'\in I$, so that $c_1+I=c_1'+I,\ldots,c_n+I=c_n'+I$, we must show that

$$\Phi_f(c_1,\ldots,c_n)-\Phi_f(c_1',\ldots,c_n')\in I.$$

Proof of Proposition 3.3

Lemma 3.4 (Hadamard's Lemma)

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is smooth. Then there exist smooth $g_i : \mathbb{R}^{2n} \to \mathbb{R}$ for $i = 1, \ldots, n$ such that for all x_j, y_j we have

$$f(x_1,\ldots,x_n)-f(y_1,\ldots,y_n) = \sum_{i=1}^{n} g_i(x_1,\ldots,x_n,y_1,\ldots,y_n)\cdot(x_i-y_i).$$

Note that $g_i(x_1,\ldots,x_n,x_1,\ldots,x_n) = \frac{\partial f}{\partial x_i}(x_1,\ldots,x_n)$, so Hadamard's Lemma gives an algebraic interpretation of partial derivatives. The definition of C^∞-ring implies that

$$\Phi_f(c_1,\ldots,c_n)-\Phi_f(c_1',\ldots,c_n') = \sum_{i=1}^{n} \Phi_{g_i}(c_1,\ldots,c_n,c_1',\ldots,c_n')\cdot(c_i-c_i'),$$

which lies in I as $c_i - c_i' \in I$, as we have to prove.

Example 3.5 (Finitely presented C^∞-rings. Compare Example 1.1.)

Suppose $p_1, \ldots, p_k : \mathbb{R}^n \to \mathbb{R}$ are smooth functions. Then $C^\infty(\mathbb{R}^n)$ is a C^∞-ring, and so an $\mathbb{R}$-algebra. Write $I = (p_1, \ldots, p_k)$ for the ideal in $C^\infty(\mathbb{R}^n)$ generated by $p_1, \ldots, p_k$. Then $C^\infty(\mathbb{R}^n)/(p_1, \ldots, p_k)$ is a C^∞-ring, by Proposition 3.3. We think of it as the C^∞-ring of functions on the smooth space $X = \{(x_1,\ldots,x_n) \in \mathbb{R}^n : p_i(x_1,\ldots,x_n) = 0,\ i = 1,\ldots,k\}$. Note that X may be singular.

Example 3.6

Let $I \subset C^\infty(\mathbb{R})$ be the ideal of all smooth $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = 0$ for all $x \geqslant 0$. Then I is *not finitely generated*, so $C^\infty(\mathbb{R})$ is *not noetherian* as an $\mathbb{R}$-algebra. This is one way in which C^∞-algebraic geometry behaves worse than ordinary algebraic geometry. We think of $C^\infty(\mathbb{R})/I$ as the C^∞-ring of smooth functions $f : [0, \infty) \to \mathbb{R}$.

Definition

A C^∞-ring $\mathfrak{C}$ is a C^∞-*local ring* if as an $\mathbb{R}$-algebra, $\mathfrak{C}$ has a unique maximal ideal $\mathfrak{m}$, with $\mathfrak{C}/\mathfrak{m} \cong \mathbb{R}$.

Example 3.7

Let X be a manifold, and $x \in X$. Write $C_x^\infty(X)$ for the C^∞-ring of germs of smooth functions $f : X \to \mathbb{R}$ at x. That is, elements of $C_x^\infty(X)$ are $\sim$-equivalence classes $[U, f]$ of pairs (U, f), where $x \in U \subseteq X$ is open and $f : U \to \mathbb{R}$ is smooth, and $(U, f) \sim (U', f')$ if there exists open $x \in U'' \subseteq U \cap U'$ with $f|_{U''} = f'|_{U''}$. Then $C_x^\infty(X)$ is a C^∞-local ring.

Definition

An ideal $I \subseteq C^\infty(\mathbb{R}^n)$ is called *fair* if for $f \in C^\infty(\mathbb{R}^n)$, $\pi_x(f) \in \pi_x(I)$ for all $x \in \mathbb{R}^n$ implies that $f \in I$, where $\pi_x : C^\infty(\mathbb{R}^n) \to C_x^\infty(\mathbb{R}^n)$ is the projection. A C^∞-ring $\mathfrak{C}$ is called *fair* if $\mathfrak{C} \cong C^\infty(\mathbb{R}^n)/I$ for $I \subseteq C^\infty(\mathbb{R}^n)$ a fair ideal.

Modules over C^∞-rings

Definition

Let $\mathfrak{C}$ be a C^∞-ring. A *module* over $\mathfrak{C}$ is a module over $\mathfrak{C}$ as an $\mathbb{R}$-algebra.

You might expect that the definition of module should involve the operations Φ_f as well as the $\mathbb{R}$-algebra structure, but it does not.

Example 3.8

Let X be a manifold, and $E \to X$ a vector bundle. Then $C^\infty(X)$ is a C^∞-ring, and the vector space $C^\infty(E)$ of smooth sections of E is a module over $C^\infty(X)$.

Cotangent modules

Definition

Let $\mathfrak{C}$ be a C^∞-ring, and M a $\mathfrak{C}$-module. A *C^∞-derivation* is an $\mathbb{R}$-linear map $\mathrm{d} : \mathfrak{C} \to M$ such that whenever $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth map and $c_1, \ldots, c_n \in \mathfrak{C}$, we have

$$\mathrm{d}\Phi_f(c_1, \ldots, c_n) = \textstyle\sum_{i=1}^n \Phi_{\frac{\partial f}{\partial x_i}}(c_1, \ldots, c_n) \cdot \mathrm{d}c_i.$$

Note that d is *not* a morphism of $\mathfrak{C}$-modules. We call such a pair $\Omega_\mathfrak{C}, \mathrm{d}_\mathfrak{C}$ a *cotangent module* for $\mathfrak{C}$ if it has the universal property that for any $\mathfrak{C}$-module M and C^∞-derivation $\mathrm{d} : \mathfrak{C} \to M$, there is a unique morphism of $\mathfrak{C}$-modules $\phi : \Omega_\mathfrak{C} \to M$ with $\mathrm{d} = \phi \circ \mathrm{d}_\mathfrak{C}$.

Every C^∞-ring has a cotangent module, unique up to isomorphism.

Example 3.9

Let X be a manifold, with cotangent bundle T^*X. Then $C^\infty(T^*X)$ is a cotangent module for the C^∞-ring $C^\infty(X)$.

3.2. Sheaves

Sheaves are a central idea in algebraic geometry.

Definition

Let X be a topological space. A *presheaf of sets* $\mathcal{E}$ on X consists of a set $\mathcal{E}(U)$ for each open $U \subseteq X$, and a restriction map $\rho_{UV} : \mathcal{E}(U) \to \mathcal{E}(V)$ for all open $V \subseteq U \subseteq X$, such that:

(i) $\mathcal{E}(\emptyset) = *$ is one point;
(ii) $\rho_{UU} = \mathrm{id}_{\mathcal{E}(U)}$ for all open $U \subseteq X$; and
(iii) $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$ for all open $W \subseteq V \subseteq U \subseteq X$.

We call $\mathcal{E}$ a *sheaf* if also whenever $U \subseteq X$ is open and $\{V_i : i \in I\}$ is an open cover of U, then:

(iv) If $s, t \in \mathcal{E}(U)$ with $\rho_{UV_i}(s) = \rho_{UV_i}(t)$ for all $i \in I$, then $s = t$;
(v) If $s_i \in \mathcal{E}(V_i)$ for all $i \in I$ with $\rho_{V_i(V_i \cap V_j)}(s_i) = \rho_{V_j(V_i \cap V_j)}(s_j)$ in $\mathcal{E}(V_i \cap V_j)$ for all $i, j \in I$, then there exists $s \in \mathcal{E}(U)$ with $\rho_{UV_i}(s) = s_i$ for all $i \in I$. This s is unique by (iv).

Definition

Let $\mathcal{E},\mathcal{F}$ be (pre)sheaves on X. A *morphism* $\phi:\mathcal{E}\to\mathcal{F}$ consists of a map $\phi(U):\mathcal{E}(U)\to\mathcal{F}(U)$ for all open $U\subseteq X$, such that $\rho_{UV}\circ\phi(U)=\phi(V)\circ\rho_{UV}:\mathcal{E}(U)\to\mathcal{F}(V)$ for all open $V\subseteq U\subseteq X$. Then sheaves form a category.

If $\mathcal{C}$ is any category in which direct limits exist, such as the categories of sets, rings, vector spaces, C^∞-rings, ..., then we can define (pre)sheaves $\mathcal{E}$ of objects in $\mathcal{C}$ on X in the obvious way, and morphisms $\phi:\mathcal{E}\to\mathcal{F}$ by taking $\mathcal{E}(U)$ to be an object in $\mathcal{C}$, and $\rho_{UV}:\mathcal{E}(U)\to\mathcal{E}(V)$, $\phi(U):\mathcal{E}(U)\to\mathcal{F}(U)$ to be morphisms in $\mathcal{C}$, and $\mathcal{E}(\emptyset)$ to be a terminal object in $\mathcal{C}$ (e.g. the zero ring). So in particular, we can define *sheaves of* C^∞*-rings* on X.
Almost any class of functions on X, or sections of a bundle on X, will form a sheaf on X. To be a sheaf means to be 'local on X', determined by its behaviour on any cover of small open sets.

Stalks of sheaves

Definition

Let X be a topological space, and $\mathcal{E}$ a (pre)sheaf of sets (or C^∞-rings, etc.) on X, and $x\in X$. The *stalk* $\mathcal{E}_x$ of $\mathcal{E}$ at x is

$$\mathcal{E}_x=\varinjlim_{x\in U\subseteq X}\mathcal{E}(U),$$

where the direct limit (as a set, or C^∞-ring, etc.) is over all open $U\subseteq X$ with $x\in U$ using $\rho_{UV}:\mathcal{E}(U)\to\mathcal{E}(V)$ for open $x\in V\subseteq U\subseteq X$. That is, for all open $x\in U\subseteq X$ we have a morphism $\pi_x:\mathcal{E}(U)\to\mathcal{E}_x$, such that for all $x\in V\subseteq U\subseteq X$ we have $\pi_x=\pi_x\circ\rho_{UV}$, and $\mathcal{E}_x$ is universal with this property.

Example 3.10

Let X be a manifold. Define a sheaf of C^∞-rings $\mathcal{O}_X$ on X by $\mathcal{O}_X(U)=C^\infty(U)$ for all open $U\subseteq X$, as a C^∞-ring, and $\rho_{UV}:C^\infty(U)\to C^\infty(V)$, $\rho_{UV}:f\mapsto f|_V$ for all open $V\subseteq U\subseteq X$. The stalk $\mathcal{O}_{X,x}$ at $x\in X$ is $C^\infty_x(X)$ from Example 3.7.

Sheafification and pullbacks

Definition

Let X be a topological space and $\mathcal{E}$ a presheaf (of sets, C^∞-rings, etc.) on X. A *sheafification* of $\mathcal{E}$ is a sheaf $\mathcal{E}'$ and a morphism of presheaves $\pi : \mathcal{E} \to \mathcal{E}'$, with the universal property that any morphism $\phi : \mathcal{E} \to \mathcal{F}$ with $\mathcal{F}$ a sheaf factorizes uniquely as $\phi = \phi' \circ \pi$ for $\phi' : \mathcal{E}' \to \mathcal{F}$.

Any presheaf $\mathcal{E}$ has a sheafification $\mathcal{E}'$, unique up to canonical isomorphism, and the stalks satisfy $\mathcal{E}_x \cong \mathcal{E}'_x$.

Definition

Let $f : X \to Y$ be a continuous map of topological spaces, and $\mathcal{E}$ a sheaf on Y. Define a presheaf $\mathcal{P}f^{-1}(\mathcal{E})$ on X by

$$\mathcal{P}f^{-1}(\mathcal{E}) = \varinjlim_{V \supseteq f(U)} \mathcal{E}(V),$$

where the direct limit is over open $V \subseteq Y$ with $f(U) \subseteq V$. Define the *pullback sheaf* $f^{-1}(\mathcal{E})$ to be the sheafification of $\mathcal{P}f^{-1}(\mathcal{E})$.

3.3. C^∞-schemes

We can now define C^∞-schemes almost exactly as for schemes in algebraic geometry, but replacing rings or $\mathbb{K}$-algebras by C^∞-rings.

Definition

A *C^∞-ringed space* $\underline{X} = (X, \mathcal{O}_X)$ is a topological space X with a sheaf of C^∞-rings $\mathcal{O}_X$. It is called a *local C^∞-ringed space* if the stalks $\mathcal{O}_{X,x}$ are C^∞-local rings for all $x \in X$.

A *morphism* $\underline{f} : \underline{X} \to \underline{Y}$ of C^∞-ringed spaces is $\underline{f} = (f, f^\sharp)$, where $f : X \to Y$ is a continuous map of topological spaces, and $f^\sharp : f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$ is a morphism of sheaves of C^∞-rings on X.

Write $\mathbf{C^\infty RS}$ for the category of C^∞-ringed spaces, and $\mathbf{LC^\infty RS}$ for the full subcategory of local C^∞-ringed spaces.

Definition

The *global sections functor* $\Gamma : \mathbf{LC^\infty RS} \to \mathbf{C^\infty Rings}^{\mathrm{op}}$ maps $\Gamma : (X, \mathcal{O}_X) \mapsto \mathcal{O}_X(X)$. It has a right adjoint, the *spectrum functor* $\mathrm{Spec} : \mathbf{C^\infty Rings}^{\mathrm{op}} \to \mathbf{LC^\infty RS}$. That is, for each C^∞-ring $\mathfrak{C}$ we construct a local C^∞-ringed space $\underline{X} = \mathrm{Spec}\,\mathfrak{C}$. Points $x \in X$ are $\mathbb{R}$-algebra morphisms $x : \mathfrak{C} \to \mathbb{R}$ (this implies x is a C^∞-ring morphism). Then each $c \in \mathfrak{C}$ defines a map $c : X \to \mathbb{R}$. We give X the weakest topology such that these $c : X \to \mathbb{R}$ are continuous for all $c \in \mathfrak{C}$. We don't use prime ideals.

In algebraic geometry, $\mathrm{Spec} : \mathbf{Rings^{op}} \to \mathbf{LRS}$ is full and faithful. In C^∞-algebraic geometry, it is full but not faithful, that is, Spec forgets some information, as we don't use prime ideals. But on the subcategory $\mathbf{C^\infty Rings^{fa}}$ of *fair* C^∞-rings, Spec is full and faithful.

Definition

A local C^∞-ringed space $\underline{X}$ is called an *affine C^∞-scheme* if $\underline{X} \cong \mathrm{Spec}\,\mathfrak{C}$ for some C^∞-ring $\mathfrak{C}$. We call $\underline{X}$ a *C^∞-scheme* if X can be covered by open subsets U with $(U, \mathcal{O}_X|_U)$ an affine C^∞-scheme. Write $\mathbf{C^\infty Sch}$ for the full subcategory of C^∞-schemes in $\mathbf{LC^\infty RS}$.

If X is a manifold, define a C^∞-scheme $\underline{X} = (X, \mathcal{O}_X)$ by $\mathcal{O}_X(U) = C^\infty(U)$ for all open $U \subseteq X$. Then $\underline{X} \cong \mathrm{Spec}\,C^\infty(X)$. This defines a full and faithful embedding $\mathbf{Man} \hookrightarrow \mathbf{C^\infty Sch}$. So we can regard manifolds as examples of C^∞-schemes.

Think of a C^∞-ringed space $\underline{X}$ as a topological space X with a notion of 'smooth function' $f : U \to \mathbb{R}$ for open $U \subseteq X$, i.e. $f \in \mathcal{O}_X(U)$. If $\underline{X}$ is a local C^∞-ringed space then the notion of 'value of f in $\mathbb{R}$ at a point $x \in U$' makes sense, since we can compose the maps $f \in \mathcal{O}_X(U) \xrightarrow{\pi_x} \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/\mathfrak{m} \cong \mathbb{R}$. If $\underline{X}$ is a C^∞-scheme, then for small open $U \subseteq X$ we can locally reconstruct the sheaf $\mathcal{O}_X|_U$ from the C^∞-ring $\mathcal{O}_X(U)$.

All *fibre products* exist in $\mathbf{C^\infty Sch}$. In manifolds **Man**, fibre products $X \times_{g,Z,h} Y$ need exist only if $g : X \to Z$ and $h : Y \to Z$ are transverse. When g, h are not transverse, the fibre product $X \times_{g,Z,h} Y$ exists in $\mathbf{C^\infty Sch}$, but may not be a manifold.

We also define *vector bundles* and *quasicoherent sheaves* on a C^∞-scheme $\underline{X}$, as sheaves of $\mathcal{O}_X$-modules, and write $\mathrm{qcoh}(\underline{X})$ for the abelian category of quasicoherent sheaves. A C^∞-scheme $\underline{X}$ has a well-behaved *cotangent sheaf* $T^*\underline{X}$.

Differences with ordinary Algebraic Geometry

- In algebraic geometry, central examples of schemes such as $\mathbb{CP}^n$ are not affine. In C^∞-algebraic geometry, most interesting C^∞-schemes are affine (e.g. all manifolds), except for non-Hausdorff C^∞-schemes. But scheme theory is still useful, to glue things from local data.
- The topology on C^∞-schemes is finer than the Zariski topology on schemes – affine schemes are always Hausdorff. No need to introduce the étale topology.
- Can find smooth functions supported on (almost) any open set.
- (Almost) any open cover has a subordinate partition of unity.
- Our C^∞-rings $\mathfrak{C}$ are generally *not noetherian* as $\mathbb{R}$-algebras. So ideals I in $\mathfrak{C}$ may not be finitely generated, even in $C^\infty(\mathbb{R}^n)$. This means there is not a well-behaved notion of coherent sheaf.

Derived Differential Geometry

Lecture 4 of 14: 2-categories, d-spaces, and d-manifolds

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Summer 2015

These slides, and references, etc., available at
http://people.maths.ox.ac.uk/∼joyce/DDG2015

Plan of talk:

4. 2-categories, d-spaces, and d-manifolds

Our goal is to define the 2-category of d-manifolds **dMan**. To do this we will define a 2-category **dSpa** of 'd-spaces', a kind of derived C^∞-scheme, and then define d-manifolds $\mathbf{dMan} \subset \mathbf{dSpa}$ to be a special kind of d-space, just as manifolds $\mathbf{Man} \subset \mathbf{C^\infty Sch}$ are a special kind of C^∞-scheme.

First we introduce 2-*categories*. There are two kinds, strict 2-categories and weak 2-categories. We will meet both, as d-manifolds and d-orbifolds $\mathbf{dMan}, \mathbf{dOrb}$ are strict 2-categories, but Kuranishi spaces **Kur** are a weak 2-category. Every weak 2-category $\mathcal{C}$ is equivalent as a weak 2-category to a strict 2-category $\mathcal{C}'$ (weak 2-categories can be 'strictified'), so there is no fundamental difference, but weak 2-categories have more notation.

4.1. 2-categories

A 2-category $\mathcal{C}$ has *objects* $X, Y, \ldots$, 1-*morphisms* $f, g : X \to Y$ (morphisms), and 2-*morphisms* $\eta : f \Rightarrow g$ (morphisms between morphisms). Here are some examples to bear in mind:

Example 4.1

(a) The strict 2-category $\mathfrak{Cat}$ has objects categories $\mathcal{C}, \mathscr{D}, \ldots,$ 1-morphisms functors $F, G : \mathcal{C} \to \mathscr{D}$, and 2-morphisms natural transformations $\eta : F \Rightarrow G$.

(b) The strict 2-category $\mathbf{Top^{ho}}$ of *topological spaces up to homotopy* has objects topological spaces $X, Y, \ldots,$ 1-morphisms continuous maps $f, g : X \to Y$, and 2-morphisms isotopy classes $[H] : f \Rightarrow g$ of homotopies H from f to g. That is, $H : X \times [0,1] \to Y$ is continuous with $H(x,0) = f(x)$, $H(x,1) = g(x)$, and $H, H' : X \times [0,1] \to Y$ are isotopic if there exists continuous $I : X \times [0,1]^2 \to Y$ with $I(x,s,0) = H(x,s)$, $I(s,x,1) = H'(x,s)$, $I(x,0,t) = f(x)$, $I(x,1,t) = g(x)$.

Definition

A (*strict*) 2-*category* $\mathcal{C}$ consists of a proper class of *objects* $\mathrm{Obj}(\mathcal{C})$, for all $X, Y \in \mathrm{Obj}(\mathcal{C})$ a category $\mathrm{Hom}(X, Y)$, for all X in $\mathrm{Obj}(\mathcal{C})$ an object id_X in $\mathrm{Hom}(X, X)$ called the *identity 1-morphism*, and for all X, Y, Z in $\mathrm{Obj}(\mathcal{C})$ a functor $\mu_{X,Y,Z} : \mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$. These must satisfy the *identity property*, that

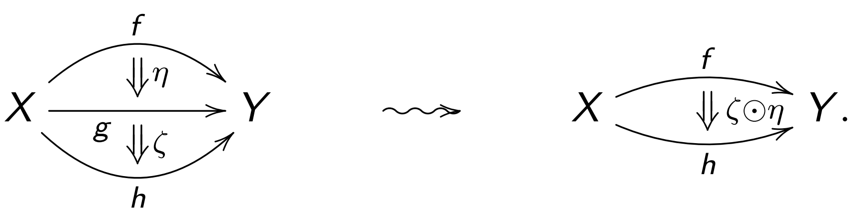
$$\mu_{X,X,Y}(\mathrm{id}_X, -) = \mu_{X,Y,Y}(-, \mathrm{id}_Y) = \mathrm{id}_{\mathrm{Hom}(X,Y)} \qquad (4.1)$$

as functors $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Y)$, and the *associativity property*, that

$$\mu_{W,Y,Z} \circ (\mu_{W,X,Y} \times \mathrm{id}) = \mu_{W,X,Z} \circ (\mathrm{id} \times \mu_{X,Y,Z}) \qquad (4.2)$$

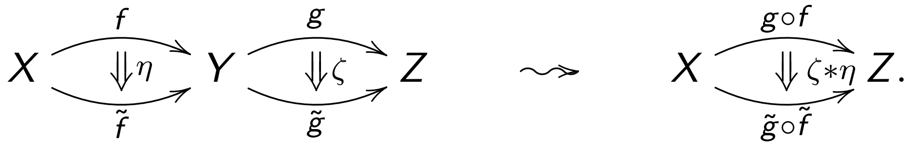
as functors $\mathrm{Hom}(W, X) \times \mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(W, X)$. Objects f of $\mathrm{Hom}(X, Y)$ are called 1-*morphisms*, written $f : X \to Y$. For 1-morphisms $f, g : X \to Y$, morphisms $\eta \in \mathrm{Hom}_{\mathrm{Hom}(X,Y)}(f, g)$ are called 2-*morphisms*, written $\eta : f \Rightarrow g$.

There are three kinds of composition in a 2-category, satisfying various associativity relations. If $f : X \to Y$ and $g : Y \to Z$ are 1-morphisms then $\mu_{X,Y,Z}(f, g)$ is the *horizontal composition of 1-morphisms*, written $g \circ f : X \to Z$. If $f, g, h : X \to Y$ are 1-morphisms and $\eta : f \Rightarrow g$, $\zeta : g \Rightarrow h$ are 2-morphisms then composition of η, ζ in $\mathrm{Hom}(X, Y)$ gives the *vertical composition of 2-morphisms* of η, ζ, written $\zeta \odot \eta : f \Rightarrow h$, as a diagram



(4.3)

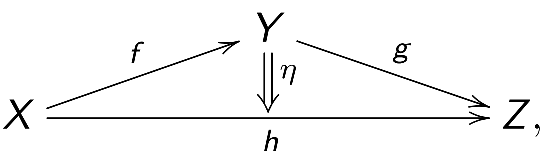
And if $f, \tilde f : X \to Y$ and $g, \tilde g : Y \to Z$ are 1-morphisms and $\eta : f \Rightarrow \tilde f$, $\zeta : g \Rightarrow \tilde g$ are 2-morphisms then $\mu_{X,Y,Z}(\eta, \zeta)$ is the *horizontal composition of 2-morphisms*, written $\zeta * \eta : g \circ f \Rightarrow \tilde g \circ \tilde f$, as a diagram



(4.4)

There are also two kinds of identity: *identity 1-morphisms* $\mathrm{id}_X : X \to X$ and *identity 2-morphisms* $\mathrm{id}_f : f \Rightarrow f$.
A 2-morphism is a 2-*isomorphism* if it is invertible under vertical composition. A 2-category is called a (2,1)-*category* if all 2-morphisms are 2-isomorphisms. For example, stacks in algebraic geometry form a (2,1)-category.

In a 2-category $\mathfrak{C}$, there are three notions of when objects X, Y in $\mathfrak{C}$ are 'the same': *equality* $X = Y$, and *isomorphism*, that is we have 1-morphisms $f : X \to Y$, $g : Y \to X$ with $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$, and *equivalence*, that is we have 1-morphisms $f : X \to Y$, $g : Y \to X$ and 2-isomorphisms $\eta : g \circ f \Rightarrow \mathrm{id}_X$ and $\zeta : f \circ g \Rightarrow \mathrm{id}_Y$. Usually equivalence is the correct notion.
Commutative diagrams in 2-categories should in general only commute *up to (specified) 2-isomorphisms*, rather than strictly. A simple example of a commutative diagram in a 2-category $\mathfrak{C}$ is



which means that X, Y, Z are objects of $\mathfrak{C}$, $f : X \to Y$, $g : Y \to Z$ and $h : X \to Z$ are 1-morphisms in $\mathfrak{C}$, and $\eta : g \circ f \Rightarrow h$ is a 2-isomorphism.

Definition (Fibre products in 2-categories. Compare §2.3.)

Let $\mathcal{C}$ be a strict 2-category and $g : X \to Z$, $h : Y \to Z$ be 1-morphisms in $\mathcal{C}$. A *fibre product* $X \times_Z Y$ in $\mathcal{C}$ is an object W, 1-morphisms $\pi_X : W \to X$ and $\pi_Y : W \to Y$ and a 2-isomorphism $\eta : g \circ \pi_X \Rightarrow h \circ \pi_Y$ in $\mathcal{C}$ with the following universal property: suppose $\pi'_X : W' \to X$ and $\pi'_Y : W' \to Y$ are 1-morphisms and $\eta' : g \circ \pi'_X \Rightarrow h \circ \pi'_Y$ is a 2-isomorphism in $\mathcal{C}$. Then there exists a 1-morphism $b : W' \to W$ and 2-isomorphisms $\zeta_X : \pi_X \circ b \Rightarrow \pi'_X$, $\zeta_Y : \pi_Y \circ b \Rightarrow \pi'_Y$ such that the following diagram commutes:

$$\begin{array}{ccc} g \circ \pi_X \circ b & \xRightarrow{\eta * \mathrm{id}_b} & h \circ \pi_Y \circ b \\ {\scriptstyle \mathrm{id}_g * \zeta_X} \Downarrow & & \Downarrow {\scriptstyle \mathrm{id}_h * \zeta_Y} \\ g \circ \pi'_X & \xRightarrow{\eta'} & h \circ \pi'_Y. \end{array}$$

Furthermore, if $\tilde{b}, \tilde{\zeta}_X, \tilde{\zeta}_Y$ are alternative choices of b, ζ_X, ζ_Y then there should exist a unique 2-isomorphism $\theta : \tilde{b} \Rightarrow b$ with

$$\tilde{\zeta}_X = \zeta_X \odot (\mathrm{id}_{\pi_X} * \theta) \quad \text{and} \quad \tilde{\zeta}_Y = \zeta_Y \odot (\mathrm{id}_{\pi_Y} * \theta).$$

If a fibre product $X \times_Z Y$ exists, it is unique up to equivalence.

Weak 2-categories

A *weak 2-category*, or *bicategory*, is like a strict 2-category, except that the equations of functors (4.1), (4.2) are required to hold not up to equality, but up to specified natural isomorphisms. That is, a weak 2-category $\mathcal{C}$ consists of data $\mathrm{Obj}(\mathcal{C}), \mathrm{Hom}(X, Y), \mu_{X,Y,Z}$, id_X as above, but in place of (4.1), a natural isomorphism

$$\alpha : \mu_{W,Y,Z} \circ (\mu_{W,X,Y} \times \mathrm{id}) \Longrightarrow \mu_{W,X,Z} \circ (\mathrm{id} \times \mu_{X,Y,Z}),$$

and in place of (4.2), natural isomorphisms

$$\beta : \mu_{X,X,Y}(\mathrm{id}_X, -) \Longrightarrow \mathrm{id}, \quad \gamma : \mu_{X,Y,Y}(-, \mathrm{id}_Y) \Longrightarrow \mathrm{id},$$

satisfying some identities. That is, composition of 1-morphisms is associative *only up to specified 2-isomorphisms*, so for 1-morphisms $e : W \to X$, $f : X \to Y$, $g : Y \to Z$ we have a 2-isomorphism

$$\alpha_{g,f,e} : (g \circ f) \circ e \Longrightarrow g \circ (f \circ e).$$

Similarly identities $\mathrm{id}_X, \mathrm{id}_Y$ work up to 2-isomorphism, so for each $f : X \to Y$ we have 2-isomorphisms

$$\beta_f : f \circ \mathrm{id}_X \Longrightarrow f, \qquad \gamma_f : \mathrm{id}_Y \circ f \Longrightarrow f.$$

4.2. Differential graded C^∞-rings

As in §2, to define derived $\mathbb{K}$-schemes, we replaced commutative $\mathbb{K}$-algebras by commutative differential graded $\mathbb{K}$-algebras (or simplicial $\mathbb{K}$-algebras). So, to define derived C^∞-schemes, we should replace C^∞-rings by *differential graded C^∞-rings* (or perhaps simplicial C^∞-rings, as in Spivak and Borisov–Noël).

Definition

A *differential graded C^∞-ring* (or *dg C^∞-ring*) $\mathfrak{C}^\bullet = (\mathfrak{C}^*, \mathrm{d})$ is a commutative differential graded $\mathbb{R}$-algebra $(\mathfrak{C}^*, \mathrm{d})$ in degrees $\leqslant 0$, as in §2.2, together with the structure $(\Phi_f)_{f:\mathbb{R}^n\to\mathbb{R}\ C^\infty}$ of a C^∞-ring on $\mathfrak{C}^0$, such that the $\mathbb{R}$-algebra structures on $\mathfrak{C}^0$ from the C^∞-ring and the cdga over $\mathbb{R}$ agree.
A *morphism* $\phi : \mathfrak{C}^\bullet \to \mathfrak{D}^\bullet$ of dg C^∞-rings is maps $\phi^k : \mathfrak{C}^k \to \mathfrak{D}^k$ for all $k \leqslant 0$, such that $(\phi^k)_{k\leqslant 0}$ is a morphism of cdgas over $\mathbb{R}$, and $\phi^0 : \mathfrak{C}^0 \to \mathfrak{D}^0$ is a morphism of C^∞-rings.

Then dg C^∞-rings form an (∞-)category **DGC$^\infty$Rings**.
One could use dg C^∞-rings to define 'derived C^∞-schemes' and 'derived C^∞-stacks' as functors F : **DGC$^\infty$Rings** $\to$ **SSets**. An alternative is to use *simplicial C^∞-rings* **SC$^\infty$Rings**, as in Spivak 2008, Borisov–Noel 2011, and Borisov 2012.

Example 4.2 (Kuranishi neighbourhoods. Compare Example 2.1.)

Let V be a smooth manifold, and $E \to V$ a smooth real vector bundle of rank n, and $s : V \to E$ a smooth section. Define a dg C^∞-ring $\mathfrak{C}^\bullet$ as follows: take $\mathfrak{C}^0 = C^\infty(V)$, with its natural $\mathbb{R}$-algebra and C^∞-ring structures. Set $\mathfrak{C}^k = C^\infty(\Lambda^{-k}E^*)$ for $k = -1, -2, \ldots, -n$, and $\mathfrak{C}^k = 0$ for $k < -n$. The multiplication $\mathfrak{C}^k \times \mathfrak{C}^l \to \mathfrak{C}^{k+l}$ are multiplication by functions in $C^\infty(V)$ if $k = 0$ or $l = 0$, and wedge product $\wedge : \Lambda^{-k}E^* \times \Lambda^{-l}E^* \to \Lambda^{-k-l}E^*$ if $k, l < 0$. The differential $\mathrm{d} : \mathfrak{C}^k \to \mathfrak{C}^{k+1}$ is contraction with s, $s\cdot : \Lambda^{-k}E^* \to \Lambda^{-k-1}E^*$.

Square zero dg C^∞-rings

We will use only a special class of dg C^∞-rings called *square zero dg C^∞-rings*, which form a 2-category $\mathbf{SZC^\infty Rings}$.

Definition

A dg C^∞-ring $\mathfrak{C}^\bullet$ is *square zero* if $\mathfrak{C}^i = 0$ for $i < -1$ and $\mathfrak{C}^{-1} \cdot \mathrm{d}[\mathfrak{C}^{-1}] = 0$. Then $\mathfrak{C}$ is $\mathfrak{C}^{-1} \xrightarrow{\mathrm{d}} \mathfrak{C}^0$, and $\mathrm{d}[\mathfrak{C}^{-1}]$ is a square zero ideal in the (ordinary) C^∞-ring $\mathfrak{C}^0$, and $\mathfrak{C}^{-1}$ is a module over the 'classical' C^∞-ring $H^0(\mathfrak{C}^\bullet) = \mathfrak{C}^0/\mathrm{d}[\mathfrak{C}^{-1}]$.
A 1-*morphism* $\alpha^\bullet : \mathfrak{C}^\bullet \to \mathfrak{D}^\bullet$ in $\mathbf{SZC^\infty Rings}$ is maps $\alpha^0 : \mathfrak{C}^0 \to \mathfrak{D}^0$, $\alpha^{-1} : \mathfrak{C}^{-1} \to \mathfrak{D}^{-1}$ preserving all the structure.
Then $H^0(\alpha^\bullet) : H^0(\mathfrak{C}) \to H^0(\mathfrak{D})$ is a morphism of C^∞-rings.
For 1-morphisms $\alpha^\bullet, \beta^\bullet : \mathfrak{C}^\bullet \to \mathfrak{D}^\bullet$ a 2-*morphism* $\eta : \alpha^\bullet \Rightarrow \beta^\bullet$ is a linear $\eta : \mathfrak{C}^0 \to \mathfrak{D}^{-1}$ with $\beta^0 = \alpha^0 + \mathrm{d} \circ \eta$ and $\beta^{-1} = \alpha^{-1} + \eta \circ \mathrm{d}$.
There is an embedding of (2-)categories $\mathbf{C^\infty Rings} \subset \mathbf{SZC^\infty Rings}$ as the (2-)subcategory of $\mathfrak{C}^\bullet$ with $\mathfrak{C}^{-1} = 0$.

There is a truncation functor $T : \mathbf{DGC^\infty Rings} \to \mathbf{SZC^\infty Rings}$, where if $\mathfrak{C}^\bullet$ is a dg C^∞-ring, then $\mathfrak{D}^\bullet = T(\mathfrak{C}^\bullet)$ is the square zero C^∞-ring with

$$\mathfrak{D}^0 = \mathfrak{C}^0/[\mathrm{d}\mathfrak{C}^{-1}]^2, \quad \mathfrak{D}^{-1} = \mathfrak{C}^{-1}/[\mathrm{d}\mathfrak{C}^{-2} + (\mathrm{d}\mathfrak{C}^{-1}) \cdot \mathfrak{C}^{-1})].$$

Applied to Example 4.2 this gives:

Example 4.3 (Kuranishi neighbourhoods. Compare Example 4.2.)

Let V be a manifold, $E \to V$ a vector bundle, and $s : V \to E$ a smooth section. Associate a square zero dg C^∞-ring $\mathfrak{C}^{-1} \xrightarrow{\mathrm{d}} \mathfrak{C}^0$ to the 'Kuranishi neighbourhood' (V, E, s) by

$$\mathfrak{C}^0 = C^\infty(V)/I_s^2, \qquad \mathfrak{C}^{-1} = C^\infty(E^*)/I_s \cdot C^\infty(E^*),$$
$$\mathrm{d}(\epsilon + I_s \cdot C^\infty(E^*)) = \epsilon(s) + I_s^2,$$

where $I_s = C^\infty(E^*) \cdot s \subset C^\infty(V)$ is the ideal generated by s.

These will be the local models for d-manifolds.

Cotangent complexes in the 2-category setting

Let $\mathfrak{C}^\bullet$ be a square zero dg C^∞-ring. Define the *cotangent complex* $\mathbb{L}_{\mathfrak{C}}^{-1} \xrightarrow{\mathrm{d}_{\mathfrak{C}}} \mathbb{L}_{\mathfrak{C}}^{0}$ to be the 2-term complex of $H^0(\mathfrak{C}^\bullet)$-modules

$$\mathfrak{C}^{-1} \xrightarrow{\;\mathrm{d}_{\mathrm{DR}}\circ\mathrm{d}\;} \Omega_{\mathfrak{C}^0} \otimes_{\mathfrak{C}^0} H^0(\mathfrak{C}^\bullet),$$

regarded as an element of the 2-category of 2-term complexes of $H^0(\mathfrak{C}^\bullet)$-modules, with $\Omega_{\mathfrak{C}^0}$ the cotangent module of the C^∞-ring $\mathfrak{C}^0$, as in §3.1. Let $\alpha^\bullet, \beta^\bullet : \mathfrak{C}^\bullet \to \mathfrak{D}^\bullet$ be 1-morphisms and $\eta : \alpha^\bullet \Rightarrow \beta^\bullet$ a 2-morphism in **SZC$^\infty$Rings**. Then $H^0(\alpha^\bullet) = H^0(\beta^\bullet)$, so we may regard $\mathfrak{D}^{-1}$ as an $H^0(\mathfrak{C}^\bullet)$-module. And $\eta : \mathfrak{C}^0 \to \mathfrak{D}^{-1}$ is a derivation, so it factors through an $H^0(\mathfrak{C}^\bullet)$-linear map $\hat\eta : \Omega_{\mathfrak{C}^0} \otimes_{\mathfrak{C}^0} H^0(\mathfrak{C}^\bullet) \to \mathfrak{D}^{-1}$. We have a diagram

$$\begin{array}{ccc} \mathbb{L}_{\mathfrak{C}}^{-1} & \xrightarrow{\quad\quad \mathrm{d}_{\mathfrak{C}} \quad\quad} & \mathbb{L}_{\mathfrak{C}}^{0} \\ {\scriptstyle \mathbb{L}_\alpha^{-1}}\downarrow\downarrow{\scriptstyle \mathbb{L}_\beta^{-1}} & \swarrow{\scriptstyle \hat\eta} & {\scriptstyle \mathbb{L}_\alpha^{0}}\downarrow\downarrow{\scriptstyle \mathbb{L}_\beta^{0}} \\ \mathbb{L}_{\mathfrak{D}}^{-1} & \xrightarrow{\quad\quad \mathrm{d}_{\mathfrak{D}} \quad\quad} & \mathbb{L}_{\mathfrak{D}}^{0}. \end{array}$$

So 1-morphisms induce morphisms, and 2-morphisms homotopies, of cotangent complexes.

4.3. D-spaces

D-spaces are our notion of derived C^∞-scheme:

Definition

A *d-space* $\mathbf{X}$ is a topological space X with a sheaf of square zero dg-C^∞-rings $\mathcal{O}_{\mathbf{X}}^\bullet = \mathcal{O}_X^{-1} \xrightarrow{\mathrm{d}} \mathcal{O}_{\mathbf{X}}^0$, such that $\underline{X} = (X, H^0(\mathcal{O}_{\mathbf{X}}^\bullet))$ and $(X, \mathcal{O}_{\mathbf{X}}^0)$ are C^∞-schemes, and $\mathcal{O}_X^{-1}$ is quasicoherent over $\underline{X}$. We call $\underline{X}$ the *underlying classical C^∞-scheme*.

We require that the topological space X should be Hausdorff and second countable, and the underlying classical C^∞-scheme $\underline{X}$ should be *locally fair*, i.e. covered by open $\operatorname{Spec}\mathfrak{C} \cong \underline{U} \subseteq \underline{X}$ for $\mathfrak{C}$ a fair C^∞-ring. Basically this means $\underline{X}$ is locally finite-dimensional.

Note that $\mathcal{O}_{\mathbf{X}}^\bullet$ is an ordinary (strict) sheaf of square zero dg C^∞-rings, using only the objects and 1-morphisms in **SZC$^\infty$Rings**, and not (as usual in DAG) a homotopy sheaf using 2-isomorphisms $\rho_{VW} \circ \rho_{UV} \Rightarrow \rho_{UW}$ for open $W \subseteq V \subseteq U \subseteq X$.

Definition

A 1-*morphism* $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$ of d-spaces $\mathbf{X},\mathbf{Y}$ is $\mathbf{f}=(f,f^\sharp)$, where $f:X\to Y$ is a continuous map of topological spaces, and $f^\sharp:f^{-1}(\mathcal{O}^\bullet_{\mathbf{Y}})\to\mathcal{O}^\bullet_{\mathbf{X}}$ is a morphism of sheaves of square zero dg C^∞-rings on X. Then $\underline{f}=(f,H^0(f^\sharp)):\underline{X}\to\underline{Y}$ is a morphism of the underlying classical C^∞-schemes.

Definition

Let $\mathbf{f},\mathbf{g}:\mathbf{X}\to\mathbf{Y}$ be 1-morphisms of d-spaces, and suppose the continuous maps $f,g:X\to Y$ are equal. We have morphisms $f^\sharp,g^\sharp:f^{-1}(\mathcal{O}^\bullet_{\mathbf{Y}})\to\mathcal{O}^\bullet_{\mathbf{X}}$ of sheaves of square zero dg C^∞-rings. That is, $f^\sharp,g^\sharp$ are sheaves on X of 1-morphisms in $\mathbf{SZC^\infty Rings}$. A 2-*morphism* $\eta:\mathbf{f}\Rightarrow\mathbf{g}$ is a sheaf on X of 2-morphisms $\eta:f^\sharp\Rightarrow g^\sharp$ in $\mathbf{SZC^\infty Rings}$. That is, for each open $U\subseteq X$, we have a 2-morphism $\eta(U):f^\sharp(U)\Rightarrow g^\sharp(U)$ in $\mathbf{SZC^\infty Rings}$, with $\mathrm{id}_{\rho_{UV}}*\eta(U)=\eta(V)*\mathrm{id}_{\rho_{UV}}$ for all open $V\subseteq U\subseteq X$.

With the obvious notions of composition of 1- and 2-morphisms, and identities, d-spaces form a strict 2-category $\mathbf{dSpa}$, in which all 2-morphisms are 2-isomorphisms.
C^∞-schemes include into d-spaces as those $\mathbf{X}$ with $\mathcal{O}_X^{-1}=0$.
Thus we have inclusions of (2-)categories $\mathbf{Man}\subset\mathbf{C^\infty Sch}\subset\mathbf{dSpa}$, so manifolds are examples of d-spaces.
The *cotangent complex* $\mathbb{L}^\bullet_{\mathbf{X}}$ of $\mathbf{X}$ is the sheaf of cotangent complexes of $\mathcal{O}^\bullet_{\mathbf{X}}$, a 2-term complex $\mathbb{L}^{-1}_{\mathbf{X}}\xrightarrow{\mathrm{d}_{\mathbf{X}}}\mathbb{L}^0_{\mathbf{X}}$ of quasicoherent sheaves on $\underline{X}$. Such complexes form a 2-category $\mathrm{qcoh}^{[-1,0]}(\underline{X})$.

Theorem 4.4

All fibre products exist in the 2-category $\mathbf{dSpa}$.

The proof is by construction: given 1-morphisms $\mathbf{g}:\mathbf{X}\to\mathbf{Z}$ and $\mathbf{h}:\mathbf{Y}\to\mathbf{Z}$, we write down an explicit d-space $\mathbf{W}$, 1-morphisms $\mathbf{e}:\mathbf{W}\to\mathbf{X}$, $\mathbf{f}:\mathbf{W}\to\mathbf{Y}$ and 2-isomorphism $\eta:\mathbf{g}\circ\mathbf{e}\Rightarrow\mathbf{h}\circ\mathbf{f}$, and verify by hand that it satisfies the universal property in §4.1.

Gluing d-spaces by equivalences

Theorem 4.5

Let $\mathbf{X}, \mathbf{Y}$ be d-spaces, $\emptyset \neq \mathbf{U} \subseteq \mathbf{X}$, $\emptyset \neq \mathbf{V} \subseteq \mathbf{Y}$ open d-subspaces, and $\mathbf{f} : \mathbf{U} \to \mathbf{V}$ an equivalence in the 2-category **dSpa**. *Suppose the topological space $Z = X \cup_{U=V} Y$ made by gluing X, Y using f is Hausdorff. Then there exist a d-space $\mathbf{Z}$, unique up to equivalence in* **dSpa**, *open $\hat{\mathbf{X}}, \hat{\mathbf{Y}} \subseteq \mathbf{Z}$ with $\mathbf{Z} = \hat{\mathbf{X}} \cup \hat{\mathbf{Y}}$, equivalences $\mathbf{g} : \mathbf{X} \to \hat{\mathbf{X}}$ and $\mathbf{h} : \mathbf{Y} \to \hat{\mathbf{Y}}$, and a 2-morphism $\eta : \mathbf{g}|_{\mathbf{U}} \Rightarrow \mathbf{h} \circ \mathbf{f}$.*

The proof is again by explicit construction. First we glue the classical C^∞-schemes $\underline{X}, \underline{Y}$ on $\underline{U} \subseteq \underline{X}, \underline{V} \subseteq \underline{Y}$ by the isomorphism $\underline{f} : \underline{U} \to \underline{V}$ to get a C^∞-scheme $\underline{Z}$. The definition of $\mathbf{Z}$ involves choosing a smooth partition of unity on $\underline{Z}$ subordinate to the open cover $\{\underline{U}, \underline{V}\}$. This is possible in the world of C^∞-schemes, but would not work in conventional (derived) algebraic geometry.

Theorem 4.6

Suppose I is an indexing set, and $<$ is a total order on I, and $\mathbf{X}_i$ for $i \in I$ are d-spaces, and for all $i < j$ in I we are given open d-subspaces $\mathbf{U}_{ij} \subseteq \mathbf{X}_i$, $\mathbf{U}_{ji} \subseteq \mathbf{X}_j$ and an equivalence $\mathbf{e}_{ij} : \mathbf{U}_{ij} \to \mathbf{U}_{ji}$, such that for all $i < j < k$ in I we have a 2-commutative diagram

$$\begin{array}{ccccc} & & \mathbf{U}_{ji} \cap \mathbf{U}_{jk} & & \\ & \nearrow^{\mathbf{e}_{ij}|_{\mathbf{U}_{ij} \cap \mathbf{U}_{ik}}} & \Downarrow \eta_{ijk} & \searrow^{\mathbf{e}_{jk}|_{\mathbf{U}_{ji} \cap \mathbf{U}_{jk}}} & \\ \mathbf{U}_{ij} \cap \mathbf{U}_{ik} & & \xrightarrow{\mathbf{e}_{ik}|_{\mathbf{U}_{ij} \cap \mathbf{U}_{ik}}} & & \mathbf{U}_{ki} \cap \mathbf{U}_{kj}. \end{array} \qquad (4.5)$$

Define the quotient topological space $Z = (\coprod_{i \in I} X_i)/\sim$, where $\sim$ is generated by $x_i \sim x_j$ if $i < j$, $x_i \in U_{ij} \subseteq X_i$ and $x_j \in U_{ji} \subseteq X_j$ with $e_{ij}(x_i) = x_j$. Suppose Z is Hausdorff and second countable. Then there exist a d-space $\mathbf{Z}$ and a 1-morphism $\mathbf{f}_i : \mathbf{X}_i \to \mathbf{Z}$ which is an equivalence with an open d-subspace $\hat{\mathbf{X}}_i \subseteq \mathbf{Z}$ for all $i \in I$, where $\mathbf{Z} = \bigcup_{i \in I} \hat{\mathbf{X}}_i$, such that $\mathbf{f}_i(\mathbf{U}_{ij}) = \hat{\mathbf{X}}_i \cap \hat{\mathbf{X}}_j$ for $i < j$ in I, and there exists a 2-morphism $\zeta_{ij} : \mathbf{f}_j \circ \mathbf{e}_{ij} \Rightarrow \mathbf{f}_i|_{\mathbf{U}_{ij}}$. The d-space $\mathbf{Z}$ is unique up to equivalence, and is independent of choice of η_{ijk}.

Theorem 4.6 generalizes Theorem 4.5 to gluing many d-spaces by equivalences. It is important that the 2-isomorphisms η_{ijk} in (4.5) are only required to exist, they need not satisfy any conditions on quadruple overlaps, etc., and $\mathbf{Z}$ is independent of the choice of η_{ijk}. Because of this, Theorem 4.6 actually makes sense as a statement in the homotopy category $\mathrm{Ho}(\mathbf{dSpa})$. The analogue is false for gluing by equivalences for orbifolds **Orb**, d-orbifolds **dOrb**, and d-stacks **dSta**.

4.4. D-manifolds

Definition

A *d-manifold* $\mathbf{X}$ of *virtual dimension* $n \in \mathbb{Z}$ is a d-space $\mathbf{X}$ such that $\mathbf{X}$ is covered by open d-subspaces $\mathbf{Y} \subset \mathbf{X}$ with equivalences $\mathbf{Y} \simeq U \times_{g,W,h} V$, where U, V, W are manifolds with $\dim U + \dim V - \dim W = n$, regarded as d-spaces by $\mathbf{Man} \subset \mathbf{C^\infty Sch} \subset \mathbf{dSpa}$, and $g : U \to W$, $h : V \to W$ are smooth maps, and $U \times_{g,W,h} V$ is the fibre product in the 2-category **dSpa**. Write **dMan** for the full 2-subcategory of d-manifolds in **dSpa**.

Note that the fibre product $U \times_W V$ exists by Theorem 4.4, and must be taken in **dSpa** as a 2-category, not as an ordinary category Alternatively, we can write the local models as $\mathbf{Y} \simeq V \times_{0,E,s} V$, where V is a manifold, $E \to V$ a vector bundle, $s : V \to E$ a smooth section, and $n = \dim V - \operatorname{rank} E$. Then (V, E, s) is a *Kuranishi neighbourhood* on $\mathbf{X}$, as in Fukaya–Oh–Ohta–Ono.

Thus, a d-manifold $\mathbf{X}$ is a 'derived' geometric space covered by simple, differential-geometric local models: they are fibre products $U \times_{g,W,h} V$ for smooth maps of manifolds $g : U \to W$, $h : V \to W$, or they are the zeroes $s^{-1}(0)$ of a smooth section $s : V \to E$ of a vector bundle $E \to V$ over a manifold V.

However, as usual in derived geometry, the way in which these local models are glued together (by equivalences in the 2-category **dSpa**) is more mysterious, is weaker than isomorphisms, and takes some work to understand. We discuss this later in the course.

If $\mathbf{g} : \mathbf{X} \to \mathbf{Z}$, $\mathbf{h} : \mathbf{Y} \to \mathbf{Z}$ are 1-morphisms in **dMan**, then Theorem 4.4 says that a fibre product $\mathbf{W} = \mathbf{X} \times_{\mathbf{g},\mathbf{Z},\mathbf{h}} \mathbf{Y}$ exists in **dSpa**. If $\mathbf{W}$ is a d-manifold (which is a local question on $\mathbf{W}$) then $\mathbf{W}$ is also a fibre product in **dMan**. So we will give be able to give useful criteria for existence of fibre products in **dMan**.

Theorems 4.5 and 4.6 immediately lift to results on gluing by equivalences in **dMan**, taking $\mathbf{U}, \mathbf{V}, \mathbf{X}_i$ to be d-manifolds of a fixed virtual dimension $n \in \mathbb{Z}$. Thus, we can define d-manifolds by gluing together local models by equivalences. This is very useful, as natural examples (e.g. moduli spaces) are often presented in terms of local models somehow glued on overlaps.

I chose to use square zero dg C^∞-rings to define **dSpa**, **dMan** (rather than, say, general dg C^∞-rings) as they are very 'small' — they are essentially the minimal extension of classical C^∞-rings which remembers the 'derived' information I care about (in particular, sufficient to form virtual cycles for derived manifolds).

This has the advantage of making the theory simpler than it could have been, e.g. by using 2-categories rather than ∞-categories, whilst still having good properties, e.g. 'correct' fibre products and gluing by equivalences. A possible disadvantage is that they forget 'higher obstructions', which occur in some moduli problems.