Derived Differential Geometry

Lecture 9 of 14: Differential Geometry of derived manifolds and orbifolds

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These slides, and references, etc., available at
http://people.maths.ox.ac.uk/∼joyce/DDG2015

Plan of talk:

9. Differential Geometry of derived manifolds and orbifolds

Here are some important topics in ordinary differential geometry:

- Immersions, embeddings, and submanifolds.
  The Whitney Embedding Theorem.
- Submersions.
- Orientations.
- Transverse fibre products.
- Manifolds with boundary and corners.
- (Oriented) bordism groups.
- Fundamental classes of compact oriented manifolds in homology.

The next four lectures will explain how all these extend to derived manifolds and orbifolds.

9.1. Orbifold groups, tangent and obstruction spaces for d-orbifolds and Kuranishi spaces

In §5.4 and §6.3 we explained that a d-manifold / M-Kuranishi space $\mathbf{X}$ has functorial tangent spaces $T_x\mathbf{X}$ and obstruction spaces $O_x\mathbf{X}$ for $x \in \mathbf{X}$. We now discuss the orbifold versions of these, which are not quite functorial. Let $\mathbf{X}$ be a d-orbifold or Kuranishi space, and $x \in \mathbf{X}$. Then we can define the *orbifold group* $G_x\mathbf{X}$, a finite group, and the *tangent space* $T_x\mathbf{X}$ and *obstruction space* $O_x\mathbf{X}$, both finite-dimensional real representations of $G_x\mathbf{X}$. If (V, E, Γ, s, ψ) is a Kuranishi neighbourhood on $\mathbf{X}$ with $x \in \operatorname{Im}\psi$, and $v \in s^{-1}(0) \subseteq V$ with $\psi(v\Gamma) = x$ then we may write

$$\begin{aligned} G_x\mathbf{X} &= \mathrm{Stab}_\Gamma(v) = \{\gamma \in \Gamma : \gamma \cdot v = v\}, \\ T_x\mathbf{X} &= \mathrm{Ker}\big(\mathrm{d}s|_v : T_v V \to E|_v\big), \\ O_x\mathbf{X} &= \mathrm{Coker}\big(\mathrm{d}s|_v : T_v V \to E|_v\big). \end{aligned} \tag{9.1}$$

For d-manifolds, $T_x\mathbf{X}, O_x\mathbf{X}$ are unique up to canonical isomorphism, so we can treat them as unique. For d-orbifolds (and classical orbifolds), things are more subtle: $G_x\mathbf{X}, T_x\mathbf{X}, O_x\mathbf{X}$ are unique *up to isomorphism*, but *not up to canonical isomorphism*. That is, to define $G_x\mathbf{X}, T_x\mathbf{X}, O_x\mathbf{X}$ in (9.1) we had to *choose* $v \in s^{-1}(0)$ with $\psi(v) = x$.

If v' is an alternative choice yielding $G_x\mathbf{X}', T_x\mathbf{X}', O_x\mathbf{X}'$, then $v' = \delta \cdot v$ for some $\delta \in \Gamma$, and we have isomorphisms

$$\begin{aligned} G_x\mathbf{X} &\longrightarrow G_x\mathbf{X}', & \gamma &\longmapsto \delta\gamma\delta^{-1}, \\ T_x\mathbf{X} &\longrightarrow T_x\mathbf{X}', & t &\longmapsto T_v\delta(t), \\ O_x\mathbf{X} &\longrightarrow O_x\mathbf{X}', & o &\longmapsto \hat{T}_v\delta(o), \end{aligned}$$

where $T_v\delta : T_vV \to T_{\delta\cdot v}V$, $\hat{T}_v\delta : E|_v \to E|_{\delta\cdot v}$ are induced by the Γ-actions on V, E. But these isomorphisms are not unique, as we could replace δ by $\delta\epsilon$ for any $\epsilon \in G_x\mathbf{X}$.

Because of this, $G_x\mathbf{X}$ is canonical up to conjugation, i.e. up to automorphisms $G_x\mathbf{X} \to G_x\mathbf{X}$ of the form $\gamma \mapsto \epsilon\gamma\epsilon^{-1}$ for $\epsilon \in G_x\mathbf{X}$, and similarly $T_x\mathbf{X}, O_x\mathbf{X}$ are canonical up to the action of elements of $G_x\mathbf{X}$, i.e. up to automorphisms $T_x\mathbf{X} \to T_x\mathbf{X}$ mapping $t \mapsto \epsilon \cdot t$. Our solution is to use the Axiom of Choice to *choose* an allowed triple $(G_x\mathbf{X}, T_x\mathbf{X}, O_x\mathbf{X})$ for all derived orbifolds $\mathbf{X}$ and $x \in \mathbf{X}$. Similarly, for any 1-morphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ of derived orbifolds and $x \in \mathbf{X}$ with $\mathbf{f}(x) = y \in \mathbf{Y}$, we can define a group morphism $G_x\mathbf{f} : G_x\mathbf{X} \to G_y\mathbf{Y}$, and $G_x\mathbf{f}$-equivariant linear maps $T_x\mathbf{f} : T_x\mathbf{X} \to T_y\mathbf{Y}$, $O_x\mathbf{f} : O_x\mathbf{X} \to O_y\mathbf{Y}$. These $G_x\mathbf{f}, T_x\mathbf{f}, T_y\mathbf{f}$ are only unique up to the action of an element of $G_y\mathbf{Y}$, so again we use the Axiom of Choice. We may not have $G_x(\mathbf{g} \circ \mathbf{f}) = G_x\mathbf{g} \circ G_x\mathbf{f}$, etc. If $\boldsymbol{\eta} : \mathbf{f} \Rightarrow \mathbf{g}$ is a 2-morphism of derived orbifolds, there is a canonical element $G_x\boldsymbol{\eta} \in G_y\mathbf{Y}$ such that

$$G_x\mathbf{g}(\gamma) = (G_x\boldsymbol{\eta})(G_x\mathbf{f}(\gamma))(G_x\boldsymbol{\eta})^{-1},$$

$$T_x\mathbf{g}(t) = G_x\boldsymbol{\eta} \cdot T_x\mathbf{f}(t), \quad O_x\mathbf{g}(t) = G_x\boldsymbol{\eta} \cdot O_x\mathbf{f}(t).$$

You can mostly ignore this issue about $G_x\mathbf{X}$, $T_x\mathbf{X}$, $O_x\mathbf{X}$ only being unique up to conjugation by an element of $G_x\mathbf{X}$, it is not very important in practice.

What *is* important is that $T_x\mathbf{X}$, $O_x\mathbf{X}$ are representations of the orbifold group $G_x\mathbf{X}$, so we can think about them using representation theory. For example, we have natural splittings

$$T_x\mathbf{X} = (T_x\mathbf{X})^{\rm tr} \oplus (T_x\mathbf{X})^{\rm nt}, \quad O_x\mathbf{X} = (O_x\mathbf{X})^{\rm tr} \oplus (O_x\mathbf{X})^{\rm nt}$$

into trivial $(\cdots)^{\rm tr}$ and nontrivial $(\cdots)^{\rm nt}$ subrepresentations. Then it is easy to prove:

Lemma 9.1

Let $\mathbf{X}$ be a derived orbifold and $x \in \mathbf{X}$, and suppose $(O_x\mathbf{X})^{\rm tr} = 0$ and $(O_x\mathbf{X})^{\rm nt}$ is not isomorphic to a $G_x\mathbf{X}$-subrepresentation of $(T_x\mathbf{X})^{\rm nt}$. Then it is not possible to make a small deformation of $\mathbf{X}$ near x so that it becomes a classical orbifold.

In contrast, derived manifolds can always be perturbed to manifolds.

9.2. Immersions, embeddings and d-submanifolds

A smooth map of manifolds $f : X \to Y$ is an *immersion* if $T_xf : T_xX \to T_yY$ is injective for all $x \in X$. It is an *embedding* if also f is a homeomorphism with its image $f(X)$.

Definition

Let $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ be a 1-morphism of derived manifolds. We call $\mathbf{f}$ a *weak immersion*, or *w-immersion*, if $T_x\mathbf{f} : T_x\mathbf{X} \to T_{\mathbf{f}(x)}\mathbf{Y}$ is injective for all $x \in \mathbf{X}$.

We call $\mathbf{f}$ an *immersion* if $T_x\mathbf{f} : T_x\mathbf{X} \to T_{\mathbf{f}(x)}\mathbf{Y}$ is injective and $O_x\mathbf{f} : O_x\mathbf{X} \to O_{\mathbf{f}(x)}\mathbf{Y}$ is surjective for all $x \in \mathbf{X}$.

We call $\mathbf{f}$ a (*w-*)*embedding* if it is a (w-)immersion and $f : X \to f(X)$ is a homeomorphism.

If instead $\mathbf{X}, \mathbf{Y}$ are derived orbifolds, we also require that $G_x\mathbf{f} : G_x\mathbf{X} \to G_{\mathbf{f}(x)}\mathbf{Y}$ is injective for all $x \in \mathbf{X}$ for (w-)immersions, and $G_x\mathbf{f} : G_x\mathbf{X} \to G_{\mathbf{f}(x)}\mathbf{Y}$ is an isomorphism for (w-)embeddings.

Proposition 9.2

Let $\mathbf{S}_{V',f,\hat f}:\mathbf{S}_{V,E,s}\to\mathbf{S}_{W,F,t}$ be a standard model 1-morphism in **dMan**. *Then $\mathbf{S}_{V',f,\hat f}$ is a w-immersion (or an immersion) if and only if for all $x\in s^{-1}(0)\subseteq V$ with $f(x)=y\in t^{-1}(0)\subseteq W$, the following sequence is exact at the second term (or the second and fourth terms, respectively):*

$$0 \longrightarrow T_xV \xrightarrow{\mathrm{d}s|_x\oplus T_xf} E|_x\oplus T_yW \xrightarrow{\hat f|_x\oplus -\mathrm{d}t|_y} F|_y \longrightarrow 0.$$

Proof.

This follows from the diagram with exact rows, (5.4) in §5:

$$\begin{array}{ccccccccccc} 0 & \to & T_x\mathbf{S}_{V,E,s} & \to & T_xV & \xrightarrow{\mathrm{d}s|_x} & E|_x & \to & O_x\mathbf{S}_{V,E,s} & \to & 0 \\ & & \downarrow{\scriptstyle T_x\mathbf{S}_{V',f,\hat f}} & & \downarrow{\scriptstyle T_xf} & & \downarrow{\scriptstyle \hat f|_x} & & \downarrow{\scriptstyle O_x\mathbf{S}_{V',f,\hat f}} & & \\ 0 & \to & T_y\mathbf{S}_{W,F,t} & \to & T_yW & \xrightarrow{\mathrm{d}t|_y} & F|_y & \to & O_y\mathbf{S}_{W,F,t} & \to & 0. \end{array}$$

□

Local models for (w-)immersions

Locally, (w-)immersions are modelled on standard model $\mathbf{S}_{V,f,\hat f}:\mathbf{S}_{V,E,s}\to\mathbf{S}_{W,F,t}$ with $f:V\to W$ an immersion. We can also take $\hat f:E\to f^*(F)$ to be injective/an isomorphism.

Theorem 9.3

Suppose $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$ is a (w-)immersion of derived manifolds, and $x\in\mathbf{X}$ with $\mathbf{f}(x)=y\in\mathbf{Y}$. Then there exists a standard model 1-morphism $\mathbf{S}_{V,f,\hat f}:\mathbf{S}_{V,E,s}\to\mathbf{S}_{W,F,t}$ in a 2-commutative diagram

$$\begin{array}{ccc} \mathbf{S}_{V,E,s} & \xrightarrow{\mathbf{S}_{V,f,\hat f}} & \mathbf{S}_{W,F,t} \\ \downarrow{\scriptstyle \mathbf{i}} & \Downarrow & \downarrow{\scriptstyle \mathbf{j}} \\ \mathbf{X} & \xrightarrow{\mathbf{f}} & \mathbf{Y}, \end{array}$$

where $\mathbf{i},\mathbf{j}$ are equivalences with open $x\in\mathbf{U}\subseteq\mathbf{X}$, $y\in\mathbf{V}\subseteq\mathbf{Y}$, and $f:V\to W$ is an immersion, and $\hat f:E\to f^*(F)$ is injective (or an isomorphism) if $\mathbf{f}$ is a w-immersion (or an immersion).

Derived submanifolds

An (immersed or embedded) submanifold $X \hookrightarrow Y$ is just an immersion or embedding $i : X \to Y$. For embedded submanifolds we can identify X with its image $i(X) \subseteq Y$, and regard X as a special subset of Y.

To define *derived submanifolds*, we just say that a (w-)immersion or (w-)embedding $\mathbf{i} : \mathbf{X} \to \mathbf{Y}$ is a (w-)immersed or (w-)embedded derived submanifold of $\mathbf{Y}$. We cannot identify $\mathbf{X}$ with a subset of $\mathbf{Y}$ in the (w-)embedded case, though we can think of it as a derived C^∞-subscheme.

We can regard an embedded submanifold $X \subset Y$ as either (i) the image of an embedding $i : X \to Y$, or (ii) locally the solutions of $g_1 = \cdots = g_n = 0$ on Y, for $g_j : Y \to \mathbb{R}$ smooth and transverse. For an immersion or embedding $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$, we can also write $\mathbf{X}$ locally as the zeroes of $\mathbf{g} : \mathbf{Y} \to \mathbb{R}^n$, but with no transversality.

Theorem 9.4

Suppose $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ *is an immersion of derived manifolds, and* $x \in \mathbf{X}$ *with* $\mathbf{f}(x) = y \in \mathbf{Y}$. *Then there exist open* $x \in \mathbf{U} \subseteq \mathbf{X}$, $y \in \mathbf{V} \subseteq \mathbf{Y}$ *with* $\mathbf{f}(\mathbf{U}) \subseteq \mathbf{V} \subseteq \mathbf{Y}$, *and a 1-morphism* $\mathbf{g} : \mathbf{V} \to \mathbb{R}^n$ *for* $n = \mathrm{vdim}\,\mathbf{Y} - \mathrm{vdim}\,\mathbf{X} \geqslant 0$, *in a 2-Cartesian diagram:*

$$\begin{array}{ccc} \mathbf{U} & \longrightarrow & * \\ \Big\downarrow{\scriptstyle \mathbf{f}|_{\mathbf{U}}} & \Downarrow & \Big\downarrow{\scriptstyle 0} \\ \mathbf{V} & \xrightarrow{\ \mathbf{g}\ } & \mathbb{R}^n. \end{array}$$

If $\mathbf{f}$ *is an embedding we can take* $\mathbf{U} = \mathbf{f}^{-1}(\mathbf{V})$.

Here $\mathbf{U}$ is a *fibre product* $\mathbf{V} \times_{\mathbf{g},\mathbb{R}^n,0} *$, of which more in §10.

9.3. Embedding derived manifolds into manifolds

Theorem 9.5

Suppose $\mathbf{f}:\mathbf{X}\to Y$ *is an embedding of d-manifolds, with* Y *an ordinary manifold. Then there is an equivalence* $\mathbf{X}\simeq\mathbf{S}_{V,E,s}$ *in* **dMan**, *where* V *is an open neighbourhood of* $f(X)$ *in* Y, *and* $E\to V$ *a vector bundle, and* $s\in C^\infty(E)$ *with* $s^{-1}(0)=f(X)$.

Sketch proof. (First version was due to David Spivak).

As $\mathbf{f}$ is an embedding, the C^∞-scheme morphism $\underline{f}:\underline{X}\to Y$ is an embedding, so that $\underline{X}$ is a C^∞-subscheme of Y. The relative cotangent complex $\mathbb{L}_{\mathbf{X}/Y}$ is a vector bundle $\underline{E}^*\to\underline{X}$ in degree -1. Take the dual and extend to a vector bundle $E\to V$ on an open neighbourhood V of $f(X)$ in Y. Then we show there exists $s\in C^\infty(E)$ defined near $f(X)=s^{-1}(0)$ such that $\mathbf{X}\simeq V\times_{0,E,s}V$. □

Whitney-style Embedding Theorems

Theorem 9.6 (Whitney 1936)

Let X *be a manifold with* $\dim X=m$. *Then generic smooth maps* $f:X\to\mathbb{R}^n$ *are immersions if* $n\geqslant 2m$, *and embeddings if* $n\geqslant 2m+1$.

If $\mathbf{X}$ is a derived manifold then 1-morphisms $\mathbf{f}:\mathbf{X}\to\mathbb{R}^n$ form a vector space, and we can take $\mathbf{f}$ to be generic in this.

Theorem 9.7

Let $\mathbf{X}$ *be a derived manifold. Then generic* 1*-morphisms* $\mathbf{f}:\mathbf{X}\to\mathbb{R}^n$ *are immersions if* $n\geqslant 2\dim T_x\mathbf{X}$ *for all* $x\in\mathbf{X}$, *and embeddings if* $n\geqslant 2\dim T_x\mathbf{X}+1$ *for all* $x\in\mathbf{X}$.

Sketch proof.

Near $x\in\mathbf{X}$ we can write $\mathbf{X}\simeq\mathbf{S}_{V,E,s}$ with $\dim V=\dim T_x\mathbf{X}$. Then generic $\mathbf{f}:\mathbf{X}\to\mathbb{R}^n$ factors through generic $g:V\to\mathbb{R}^n$. Apply Theorem 9.6 to see g is an immersion/embedding. □

Combining Theorems 9.5 and 9.7 yields:

Corollary 9.8

A d-manifold $\mathbf{X}$ *is equivalent in* **dMan** *to a standard model d-manifold* $\mathbf{S}_{V,E,s}$ *if and only if* $\dim T_x\mathbf{X}$ *is bounded above for all* $x \in \mathbf{X}$*. This always holds if* $\mathbf{X}$ *is compact.*

Proof.

If the $\dim T_x\mathbf{X}$ are bounded above we can choose $n \geqslant 0$ with $n \geqslant 2\dim T_x\mathbf{X} + 1$ for all $x \in \mathbf{X}$. Then a generic $\mathbf{f} : \mathbf{X} \to \mathbb{R}^n$ is an embedding, and $\mathbf{X} \simeq \mathbf{S}_{V,E,s}$ for V an open neighbourhood of $f(X)$ in $\mathbb{R}^n$. Conversely, if $\mathbf{X} \simeq \mathbf{S}_{V,E,s}$ then $\dim T_x\mathbf{X} \leqslant \dim V$ for $x \in X$. If $\mathbf{X}$ is compact then as $x \mapsto \dim T_x\mathbf{X}$ is upper semicontinuous, $\dim T_x\mathbf{X}$ is bounded above. □

This means that most interesting d-manifolds are principal d-manifolds (i.e. equivalent in **dMan** to some $\mathbf{S}_{V,E,s}$).

For d-orbifolds $\mathbf{X}$, as for Theorem 9.5 we can prove that if $\mathbf{f} : \mathbf{X} \to \mathfrak{Y}$ is an embedding for $\mathfrak{Y}$ an orbifold, then there is an equivalence $\mathbf{X} \simeq \mathfrak{V} \times_{0,\mathfrak{E},s} \mathfrak{V}$ in **dOrb**, where $\mathfrak{V} \subseteq \mathfrak{Y}$ is an open neighbourhood of $\mathbf{f}(\mathbf{X})$ in $\mathfrak{Y}$, and $\mathfrak{E} \to \mathfrak{V}$ an orbifold vector bundle with $s \in C^\infty(\mathfrak{E})$. This uses the condition on embeddings that $G_x\mathbf{f} : G_x\mathbf{X} \to G_{\mathbf{f}(x)}\mathfrak{Y}$ is an isomorphism for all $x \in \mathbf{X}$.
However, we have no good orbifold analogues of Theorem 9.7 or Corollary 9.8. If $\mathbf{X}$ has nontrivial orbifold groups $G_x\mathbf{X} \neq \{1\}$ it cannot have embeddings $\mathbf{f} : \mathbf{X} \to \mathbb{R}^n$, as $G_x\mathbf{f} : G_x\mathbf{X} \to G_{\mathbf{f}(x)}\mathbb{R}^n = \{1\}$ is not an isomorphism.
So we do not have useful criteria for when a d-orbifold can be covered by a single chart $(\mathfrak{V}, \mathfrak{E}, s)$ or (V, E, Γ, s, ψ).

9.4. Submersions

A smooth map of manifolds $f : X \to Y$ is a *submersion* if $T_x f : T_x X \to T_y Y$ is surjective for all $x \in X$. As for (w-)immersions, we have two derived analogues:

Definition

Let $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ be a 1-morphism of derived manifolds or derived orbifolds. We call $\mathbf{f}$ a *weak submersion*, or *w-submersion*, if $O_x\mathbf{f} : O_x\mathbf{X} \to O_{\mathbf{f}(x)}\mathbf{Y}$ is surjective for all $x \in \mathbf{X}$.
We call $\mathbf{f}$ a *submersion* if $T_x\mathbf{f} : T_x\mathbf{X} \to T_{\mathbf{f}(x)}\mathbf{Y}$ is surjective and $O_x\mathbf{f} : O_x\mathbf{X} \to O_{\mathbf{f}(x)}\mathbf{Y}$ is an isomorphism for all $x \in \mathbf{X}$.

Differential Geometry of derived manifolds and orbifolds
Fibre products of derived manifolds and orbifolds
Orbifold groups, tangent and obstruction spaces
Immersions, embeddings and d-submanifolds
Embedding derived manifolds into manifolds
Submersions
Orientations

Here is the analogue of Proposition 9.2, with the same proof:

Proposition 9.9

Let $\mathbf{S}_{V',f,\hat f} : \mathbf{S}_{V,E,s} \to \mathbf{S}_{W,F,t}$ be a standard model 1-morphism in **dMan**. *Then $\mathbf{S}_{V',f,\hat f}$ is a w-submersion (or a submersion) if and only if for all $x \in s^{-1}(0) \subseteq V$ with $f(x) = y \in t^{-1}(0) \subseteq W$, the following sequence is exact at the fourth term (or the third and fourth terms, respectively):*

$$0 \longrightarrow T_x V \xrightarrow{\mathrm{d}s|_x \oplus T_x f} E|_x \oplus T_y W \xrightarrow{\hat f|_x \oplus -\mathrm{d}t|_y} F|_y \longrightarrow 0.$$

Local models for (w-)submersions

Here is the analogue of Theorem 9.3:

Theorem 9.10

Suppose $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$ *is a (w-)submersion of derived manifolds, and* $x\in\mathbf{X}$ *with* $\mathbf{f}(x)=y\in\mathbf{Y}$. *Then there exists a standard model 1-morphism* $\mathbf{S}_{V,f,\hat f}:\mathbf{S}_{V,E,s}\to\mathbf{S}_{W,F,t}$ *in a 2-commutative diagram*

$$\begin{array}{ccc}\mathbf{S}_{V,E,s} & \xrightarrow[\mathbf{S}_{V,f,\hat f}]{} & \mathbf{S}_{W,F,t}\\ \downarrow{\scriptstyle\mathbf{i}} & \Downarrow & {\scriptstyle\mathbf{j}}\downarrow\\ \mathbf{X} & \xrightarrow{\mathbf{f}} & \mathbf{Y},\end{array}$$

where $\mathbf{i},\mathbf{j}$ *are equivalences with open* $x\in\mathbf{U}\subseteq\mathbf{X}$, $y\in\mathbf{V}\subseteq\mathbf{Y}$, *and* $f:V\to W$ *is a submersion, and* $\hat f:E\to f^*(F)$ *is surjective (or an isomorphism) if* $\mathbf{f}$ *is a w-submersion (or a submersion).*

If $g:X\to Z$ is a submersion of manifolds, then for any smooth map $h:Y\to Z$ the (transverse) fibre product $X\times_{g,Z,h}Y$ exists in **Man**. Here are two derived analogues, explained in §10:

Theorem 9.11

Suppose $\mathbf{g}:\mathbf{X}\to\mathbf{Z}$ *is a w-submersion in* **dMan**. *Then for any 1-morphism* $\mathbf{h}:\mathbf{Y}\to\mathbf{Z}$ *in* **dMan**, *the fibre product* $\mathbf{X}\times_{\mathbf{g},\mathbf{Z},\mathbf{h}}\mathbf{Y}$ *exists in* **dMan**.

Theorem 9.12

Suppose $\mathbf{g}:\mathbf{X}\to\mathbf{Z}$ *is a submersion in* **dMan**. *Then for any 1-morphism* $\mathbf{h}:Y\to\mathbf{Z}$ *in* **dMan** *with* Y *a manifold, the fibre product* $\mathbf{X}\times_{\mathbf{g},\mathbf{Z},\mathbf{h}}Y$ *exists in* **dMan** *and is a manifold.*
In particular, the fibres $\mathbf{X}_z=\mathbf{X}\times_{\mathbf{g},\mathbf{Z},z}*$ *of* $\mathbf{g}$ *for* $z\in\mathbf{Z}$ *are manifolds.*

Submersions as local projections

If $f : X \to Y$ is a submersion of manifolds and $x \in X$, we can find open $x \in U \subseteq X$ and $f(x) \in V \subseteq Y$ with $f(U) \subseteq V$ and a diffeomorphism $U \cong V \times W$ for some manifold W which identifies $f|_U : U \to V$ with the projection $\pi_V : V \times W \to V$. Here is a derived analogue, which can be deduced from Theorem 9.10:

Theorem 9.13

Let $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ *be a submersion of d-manifolds, and* $x \in \mathbf{X}$ *with* $\mathbf{f}(x) = y \in \mathbf{Y}$. *Then there exist open* $x \in \mathbf{U} \subseteq \mathbf{X}$, $y \in \mathbf{V} \subseteq \mathbf{Y}$ *with* $\mathbf{f}(\mathbf{U}) \subseteq \mathbf{V} \subseteq \mathbf{Y}$, *and an equivalence* $\mathbf{U} \simeq \mathbf{V} \times W$ *for* W *a manifold with* $\dim W = \operatorname{vdim} \mathbf{X} - \operatorname{vdim} \mathbf{Y} \geqslant 0$, *in a 2-commutative diagram*

$$\begin{array}{ccc} \mathbf{U} & \xrightarrow{\simeq} & \mathbf{V} \times W \\ & \Downarrow & \downarrow \pi_{\mathbf{V}} \\ & \xrightarrow{\mathbf{f}|_{\mathbf{U}}} & \mathbf{V}. \end{array}$$

9.5. Orientations

Here is one way to define orientations on ordinary manifolds. Let X be a manifold of dimension n. The *canonical bundle* K_X is $\Lambda^n T^*X$. It is a real line bundle over X. An *orientation* o on X is an orientation on the fibres of K_X. That is, o is an equivalence class $[\iota]$ of isomorphisms $\iota : \mathcal{O}_X \to K_X$, where $\mathcal{O}_X = X \times \mathbb{R}$ is the trivial line bundle on X, and two isomorphisms ι, ι' are equivalent if $\iota' = c \cdot \iota$ for $c : X \to (0, \infty)$ a smooth positive function on X. Isomorphisms $\iota : \mathcal{O}_X \to K_X$ are equivalent to non-vanishing n-forms $\omega = \iota(1)$ on X.
The *opposite orientation* is $-o = [-\iota]$.
An *oriented manifold* (X, o) is a manifold X with orientation o. Usually we just say X is an oriented manifold, and write $-X$ for $(X, -o)$ with the opposite orientation.

There is a natural analogue of canonical bundles for derived manifolds and orbifolds.

Theorem 9.14

(a) *Every d-manifold* $\mathbf{X}$ *has a* **canonical bundle** $K_{\mathbf{X}}$*, a* C^∞ *real line bundle over the underlying* C^∞*-scheme* $\underline{X}$*, natural up to canonical isomorphism, with* $K_{\mathbf{X}}|_x \cong \Lambda^{\rm top} T_x^*\mathbf{X} \otimes \Lambda^{\rm top} O_x\mathbf{X}$ *for* $x \in \mathbf{X}$*.*
(b) *If* $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ *is an étale 1-morphism (e.g. an equivalence), there is a canonical, functorial isomorphism* $K_{\mathbf{f}} : K_{\mathbf{X}} \to \underline{f}^*(K_{\mathbf{Y}})$*. If* $\mathbf{f}, \mathbf{g} : \mathbf{X} \to \mathbf{Y}$ *are 2-isomorphic then* $K_{\mathbf{f}} = K_{\mathbf{g}}$*.*
(c) *If* $\mathbf{X} \simeq \mathbf{S}_{V,E,s}$*, there is a canonical isomorphism*

$$K_{\mathbf{X}} \cong \big(\Lambda^{\dim V} T^*V \otimes \Lambda^{\operatorname{rank} E} E\big)|_{s^{-1}(0)}.$$

Analogues of **(a)–(c)** *hold for d-orbifolds and Kuranishi spaces, with* $K_{\mathbf{X}}$ *an orbifold line bundle over the underlying Deligne–Mumford* C^∞*-stack* $\mathcal{X}$*. In particular, the orbifold groups* $G_x\mathcal{X}$ *can act nontrivially on* $K_{\mathbf{X}}$*, so* $K_{\mathbf{X}}$ *may not be locally trivial.*

To prove Theorem 9.14 for Kuranishi spaces, we show that the line bundles $\big(\Lambda^{\dim V_i} T^*V_i \otimes \Lambda^{\operatorname{rank} E_i} E_i\big)|_{s_i^{-1}(0)}$ on $\operatorname{Im}\psi_i \subseteq X$ can be glued by canonical isomorphisms on overlaps $\operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$.

Definition

An *orientation* o on a d-manifold $\mathbf{X}$ is an equivalence class $[\iota]$ of isomorphisms $\iota : \mathcal{O}_{\underline{X}} \to K_{\mathbf{X}}$, where $\mathcal{O}_{\underline{X}}$ is the trivial line bundle on the C^∞-scheme $\underline{X}$, and two isomorphisms ι, ι' are equivalent if $\iota' = c \cdot \iota$ for $c : \underline{X} \to (0, \infty)$ a smooth positive function on $\underline{X}$.
An *oriented d-manifold* $(\mathbf{X}, o)$ is a d-manifold $\mathbf{X}$ with orientation o. Usually we just say $\mathbf{X}$ is an oriented d-manifold, and write $-\mathbf{X}$ for $(\mathbf{X}, -o)$ with the opposite orientation.
We make similar definitions for d-orbifolds and Kuranishi spaces.

An orientation on $\mathbf{S}_{V,E,s}$ is equivalent to an orientation (near $s^{-1}(0)$) on the total space of E.

Derived Differential Geometry

Lecture 10 of 14: Fibre products of derived manifolds and orbifolds

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These slides, and references, etc., available at
http://people.maths.ox.ac.uk/∼joyce/DDG2015

Plan of talk:

10. Fibre products of derived manifolds and orbifolds
10.1. D-transverse fibre products of derived manifolds

Fibre products of derived manifolds and orbifolds are very important. Standard models $\mathbf{S}_{V,E,s}$ are fibre products $V \times_{0,E,s} V$ in **dMan**. Theorems 9.4, 9.11, and 9.12 in §9 involved fibre products. Applications often involve fibre products, for example moduli spaces $\bar{\mathcal{M}}_k(\beta)$ of prestable J-holomorphic discs Σ in a symplectic manifold (M,ω) with boundary in a Lagrangian L, relative homology class $[\Sigma]=\beta\in H_2(M,L;\mathbb{Z})$, and k boundary marked points, are Kuranishi spaces with corners satisfying

$$\partial\bar{\mathcal{M}}_k(\beta)=\coprod_{i+j=k}\coprod_{\beta_1+\beta_2=\beta}\bar{\mathcal{M}}_{i+1}(\beta_1)\times_{\mathbf{ev}_{i+1},L,\mathbf{ev}_{j+1}}\bar{\mathcal{M}}_{j+1}(\beta_2). \quad (10.1)$$

Recall that smooth maps of manifolds $g:X\rightarrow Z$, $h:Y\rightarrow Z$ are *transverse* if for all $x\in X$, $y\in Y$ with $g(x)=h(y)=z\in Z$, then $T_xg\oplus T_yh:T_xX\oplus T_yY\rightarrow T_zZ$ is surjective. If g,h are transverse then a fibre product $W=X\times_{g,Z,h}Y$ exists in **Man**, with $\dim W=\dim X+\dim Y-\dim Z$.
We give two derived analogues of transversality, weak and strong:

Definition 10.1

Let $\mathbf{g}:\mathbf{X}\rightarrow\mathbf{Z}$, $\mathbf{h}:\mathbf{Y}\rightarrow\mathbf{Z}$ be 1-morphisms of d-manifolds. We call $\mathbf{g},\mathbf{h}$ *d-transverse* if for all $x\in\mathbf{X}$, $y\in\mathbf{Y}$ with $\mathbf{g}(x)=\mathbf{h}(y)=z$ in $\mathbf{Z}$, then $O_x\mathbf{g}\oplus O_y\mathbf{h}:O_x\mathbf{X}\oplus O_y\mathbf{Y}\rightarrow O_z\mathbf{Z}$ is surjective.

We call $\mathbf{g},\mathbf{h}$ *strongly d-transverse* if for all $x\in X$, $y\in Y$ with $g(x)=h(y)=z\in Z$, then $T_x\mathbf{g}\oplus T_y\mathbf{h}:T_x\mathbf{X}\oplus T_y\mathbf{Y}\rightarrow T_z\mathbf{Z}$ is surjective, and $O_x\mathbf{g}\oplus O_y\mathbf{h}:O_x\mathbf{X}\oplus O_y\mathbf{Y}\rightarrow O_z\mathbf{Z}$ is an isomorphism.

Here is the main result:

Theorem 10.2

Suppose $\mathbf{g}:\mathbf{X}\to\mathbf{Z}$, $\mathbf{h}:\mathbf{Y}\to\mathbf{Z}$ *are d-transverse 1-morphisms in* $\mathbf{dMan}$. *Then a fibre product* $\mathbf{W}=\mathbf{X}\times_{\mathbf{g},\mathbf{Z},\mathbf{h}}\mathbf{Y}$ *exists in the 2-category* $\mathbf{dMan}$, *with* $\mathrm{vdim}\,\mathbf{W}=\mathrm{vdim}\,\mathbf{X}+\mathrm{vdim}\,\mathbf{Y}-\mathrm{vdim}\,\mathbf{Z}$. *This* $\mathbf{W}$ *is a manifold if and only if* $\mathbf{g},\mathbf{h}$ *are strongly d-transverse. The topological space* W *of* $\mathbf{W}$ *is given by*

$$W=\{(x,y)\in X\times Y:\mathbf{g}(x)=\mathbf{h}(y)\text{ in }Z\}. \qquad (10.2)$$

For all $(x,y)\in W$ *with* $\mathbf{g}(x)=\mathbf{h}(y)=z$ *in* $\mathbf{Z}$, *there is a natural long exact sequence*

$$\begin{array}{l} 0\longrightarrow T_{(x,y)}\mathbf{W}\xrightarrow{T_{(x,y)}\mathbf{e}\oplus -T_{(x,y)}\mathbf{f}} T_x\mathbf{X}\oplus T_y\mathbf{Y}\xrightarrow{T_x\mathbf{g}\oplus T_y\mathbf{h}} T_z\mathbf{Z} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\downarrow \\ 0\longleftarrow O_z\mathbf{Z}\xleftarrow{O_x\mathbf{g}\oplus O_y\mathbf{h}} O_x\mathbf{X}\oplus O_y\mathbf{Y}\xleftarrow{O_{(x,y)}\mathbf{e}\oplus -O_{(x,y)}\mathbf{f}} O_{(x,y)}\mathbf{W}, \end{array} \qquad (10.3)$$

where $\mathbf{e}:\mathbf{W}\to\mathbf{X}$, $\mathbf{f}:\mathbf{W}\to\mathbf{Y}$ *are the projections.*

Note that quite a lot of the theorem can be seen from the exact sequence (10.3). D-transversality of $\mathbf{g},\mathbf{h}$ is equivalent to exactness at $O_z\mathbf{Z}$. The rest of the sequence determines $T_{(x,y)}\mathbf{W}$, $O_{(x,y)}\mathbf{W}$. In particular, $O_{(x,y)}\mathbf{W}$ is the direct sum of the cokernel of $T_x\mathbf{g}\oplus T_y\mathbf{h}:T_x\mathbf{X}\oplus T_y\mathbf{Y}\to T_z\mathbf{Z}$ and the kernel of $O_x\mathbf{g}\oplus O_y\mathbf{h}:O_x\mathbf{X}\oplus O_y\mathbf{Y}\to O_z\mathbf{Z}$. Therefore $O_{(x,y)}\mathbf{W}=0$ for all $(x,y)\in\mathbf{W}$ if and only if $\mathbf{g},\mathbf{h}$ are strongly d-transverse. But a d-manifold $\mathbf{W}$ is a manifold if and only if $O_w\mathbf{W}=0$ for all $w\in\mathbf{W}$. The equation $\mathrm{vdim}\,\mathbf{W}=\mathrm{vdim}\,\mathbf{X}+\mathrm{vdim}\,\mathbf{Y}-\mathrm{vdim}\,\mathbf{Z}$ holds by taking alternating sums of dimensions in (10.3) and using $\mathrm{vdim}\,\mathbf{X}=\dim T_x\mathbf{X}-\dim O_x\mathbf{X}$, etc.

Here is a way to think about d-transversality. Work in a suitable ∞-category $\mathbf{DerC^\infty Sch}$ of derived C^∞-schemes, such as that defined by Spivak. Then derived manifolds $\mathbf{X}$ are objects in $\mathbf{DerC^\infty Sch}$, which are *quasi-smooth*, that is, the cotangent complex $\mathbb{L}_{\mathbf{X}}$ lives in degrees $[-1, 0]$. If $\mathbf{X}$ is a manifold (it is *smooth*), its cotangent complex lives in degree 0 only.

For any 1-morphisms $\mathbf{g} : \mathbf{X} \to \mathbf{Z}$, $\mathbf{h} : \mathbf{Y} \to \mathbf{Z}$ of derived manifolds, a fibre product $\mathbf{W} = \mathbf{X} \times_{\mathbf{g},\mathbf{Z},\mathbf{h}} \mathbf{Y}$ exists in $\mathbf{DerC^\infty Sch}$ (as all fibre products do), with topological space W as in (10.2). The cotangent complexes form a distinguished triangle of $\mathcal{O}^\bullet_{\mathbf{W}}$-modules:

$$\cdots \longrightarrow (\mathbf{g} \circ \mathbf{e})^*(\mathbb{L}_{\mathbf{Z}}) \longrightarrow \mathbf{e}^*(\mathbb{L}_{\mathbf{X}}) \oplus \mathbf{f}^*(\mathbb{L}_{\mathbf{Y}}) \longrightarrow \mathbb{L}_{\mathbf{W}} \xrightarrow{[1]} \cdots \qquad (10.4)$$

Therefore $\mathbb{L}_{\mathbf{W}}$ lives in degrees $[-2, 0]$, so in general $\mathbf{W}$ is not quasi-smooth, and not a derived manifold. D-transversality is the necessary and sufficient condition for $H^{-2}(\mathbb{L}_{\mathbf{W}}|_{(x,y)}) = 0$ for all $(x, y) \in \mathbf{W}$, so that $\mathbb{L}_{\mathbf{W}}$ lives in degrees $[-1, 0]$. Then (10.3) is the dual of the cohomology exact sequence of (10.4) restricted to (x, y).

Fibre products over manifolds

If $\mathbf{Z}$ is a manifold then $O_z\mathbf{Z} = 0$ for all $z \in \mathbf{Z}$, so any $\mathbf{g} : \mathbf{X} \to \mathbf{Z}$, $\mathbf{h} : \mathbf{Y} \to \mathbf{Z}$ are d-transverse. Thus Theorem 10.2 gives:

Corollary 10.3

Suppose $\mathbf{g} : \mathbf{X} \to \mathbf{Z}$, $\mathbf{h} : \mathbf{Y} \to \mathbf{Z}$ are 1-morphisms in $\mathbf{dMan}$, with $\mathbf{Z}$ a manifold. Then a fibre product $\mathbf{W} = \mathbf{X} \times_{\mathbf{g},\mathbf{Z},\mathbf{h}} \mathbf{Y}$ exists in $\mathbf{dMan}$, with $\mathrm{vdim}\, \mathbf{W} = \mathrm{vdim}\, \mathbf{X} + \mathrm{vdim}\, \mathbf{Y} - \dim \mathbf{Z}$.

This is very useful. For example, the symplectic geometry equation (10.1) involves fibre products over a manifold (in the d-orbifold case), which automatically exist.

W-submersions and submersions

Recall from §9.4 that $\mathbf{g}:\mathbf{X}\to\mathbf{Z}$ in **dMan** is a *w-submersion* if $O_x\mathbf{g}:O_x\mathbf{X}\to O_z\mathbf{Z}$ is surjective for all $x\in\mathbf{X}$. For any other 1-morphism $\mathbf{h}:\mathbf{Y}\to\mathbf{Z}$ in **dMan**, this implies that $O_x\mathbf{g}\oplus O_y\mathbf{h}:O_x\mathbf{X}\oplus O_y\mathbf{Y}\to O_z\mathbf{Z}$ is surjective, so $\mathbf{g},\mathbf{h}$ are d-transverse, and thus a fibre product $\mathbf{W}=\mathbf{X}\times_{\mathbf{g},\mathbf{Z},\mathbf{h}}\mathbf{Y}$ exists in **dMan** by Theorem 10.2. This proves Theorem 9.11.

Also $\mathbf{g}:\mathbf{X}\to\mathbf{Z}$ is a *submersion* if $T_x\mathbf{g}:T_x\mathbf{X}\to T_z\mathbf{Z}$ is surjective and $O_x\mathbf{g}:O_x\mathbf{X}\to O_z\mathbf{Z}$ is an isomorphism for all $x\in\mathbf{X}$. For any other 1-morphism $\mathbf{h}:\mathbf{Y}\to\mathbf{Z}$ in **dMan** with $\mathbf{Y}$ a manifold, this implies that $T_x\mathbf{g}\oplus T_y\mathbf{h}:T_x\mathbf{X}\oplus T_y\mathbf{Y}\to T_z\mathbf{Z}$ is surjective, and $O_x\mathbf{g}\oplus O_y\mathbf{h}:O_x\mathbf{X}\oplus O_y\mathbf{Y}\to O_z\mathbf{Z}$ is an isomorphism, as $O_y\mathbf{Y}=0$. So $\mathbf{g},\mathbf{h}$ are strongly d-transverse, and $\mathbf{W}=\mathbf{X}\times_{\mathbf{g},\mathbf{Z},\mathbf{h}}\mathbf{Y}$ exists and is a manifold by Theorem 10.2. This proves Theorem 9.12.

Non d-transverse fibre products

If $\mathbf{g}:\mathbf{X}\to\mathbf{Z}$ and $\mathbf{h}:\mathbf{Y}\to\mathbf{Z}$ are 1-morphisms in **dMan** which are not d-transverse, then a fibre product $\mathbf{W}=\mathbf{X}\times_{\mathbf{g},\mathbf{Z},\mathbf{h}}\mathbf{Y}$ may or may not exist in **dMan**. If it does exist, then generally we have $\mathrm{vdim}\,\mathbf{W}<\mathrm{vdim}\,\mathbf{X}+\mathrm{vdim}\,\mathbf{Y}-\mathrm{vdim}\,\mathbf{Z}$.

Example 10.4

Let $\mathbf{X}=\mathbf{Y}=*$ be a point, and $\mathbf{Z}=\mathbf{S}_{*,\mathbb{R}^n,0}$ be the standard model d-manifold which is a point $*$ with obstruction space $O_*\mathbf{Z}=\mathbb{R}^n$ for $n>0$. Let $\mathbf{g}:\mathbf{X}\to\mathbf{Z}$, $\mathbf{h}:\mathbf{Y}\to\mathbf{Z}$ be the unique 1-morphisms, the standard model 1-morphism $\mathbf{S}_{*,\mathrm{id}_*,0}$.
Then $O_*\mathbf{g}$, $O_*\mathbf{h}$ map $0\to\mathbb{R}^n$, so $\mathbf{g},\mathbf{h}$ are not d-transverse. The fibre product $\mathbf{W}=\mathbf{X}\times_{\mathbf{g},\mathbf{Z},\mathbf{h}}\mathbf{Y}$ exists in **dMan** and is the point $*$. So $\mathrm{vdim}\,\mathbf{W}=0<\mathrm{vdim}\,\mathbf{X}+\mathrm{vdim}\,\mathbf{Y}-\mathrm{vdim}\,\mathbf{Z}=0+0-(-n)=n$.

10.2. Fibre products of derived orbifolds

For fibre products in the 2-categories of d-orbifolds **dOrb** or (equivalently) Kuranishi spaces **Kur**, an analogue of Theorem 10.2 holds. We have to be careful of two points:

(a) As in §9.1, if $\mathbf{g}:\mathbf{X}\to\mathbf{Z}$ is a 1-morphism in **dOrb** or **Kur** and $x\in\mathbf{X}$ with $\mathbf{g}(x)=z\in\mathbf{Z}$, then $T_x\mathbf{g}:T_x\mathbf{X}\to T_z\mathbf{Z}$ and $O_x\mathbf{g}:O_x\mathbf{X}\to O_z\mathbf{Z}$ are only naturally defined up to the action of an element of the orbifold group $G_z\mathbf{Z}$ on $T_z\mathbf{Z},O_z\mathbf{Z}$. We must include this in the definition of (strong) d-transversality.

(b) When a d-transverse fibre product $\mathbf{W}=\mathbf{X}\times_{\mathbf{g},\mathbf{Z},\mathbf{h}}\mathbf{Y}$ exists, the underlying topological space is generally not $W=\{(x,y)\in X\times Y:\mathbf{g}(x)=\mathbf{h}(y)\}$, as in (10.2). Instead, the continuous map $W\to X\times Y$ is finite, but not injective, as the fibre over $(x,y)\in X\times Y$ with $\mathbf{g}(x)=\mathbf{h}(y)=z\in\mathbf{Z}$ is $G_x\mathbf{g}(G_x\mathbf{X})\backslash G_z\mathbf{Z}/G_y\mathbf{h}(G_y\mathbf{Y})$.

For (a), here is the orbifold analogue of Definition 10.1:

Definition 10.5

Let $\mathbf{g}:\mathbf{X}\to\mathbf{Z}$, $\mathbf{h}:\mathbf{Y}\to\mathbf{Z}$ be 1-morphisms of derived orbifolds. We call $\mathbf{g},\mathbf{h}$ *d-transverse* if for all $x\in\mathbf{X}$, $y\in\mathbf{Y}$ with $\mathbf{g}(x)=\mathbf{h}(y)=z$ in $\mathbf{Z}$, and all $\gamma\in G_z\mathbf{Z}$, then $O_x\mathbf{g}\oplus(\gamma\cdot O_y\mathbf{h}):O_x\mathbf{X}\oplus O_y\mathbf{Y}\to O_z\mathbf{Z}$ is surjective.

We call $\mathbf{g},\mathbf{h}$ *strongly d-transverse* if for all $x\in X$, $y\in Y$ with $g(x)=h(y)=z\in Z$, and all $\gamma\in G_z\mathbf{Z}$, then $T_x\mathbf{g}\oplus(\gamma\cdot T_y\mathbf{h}):T_x\mathbf{X}\oplus T_y\mathbf{Y}\to T_z\mathbf{Z}$ is surjective, and $O_x\mathbf{g}\oplus(\gamma\cdot O_y\mathbf{h}):O_x\mathbf{X}\oplus O_y\mathbf{Y}\to O_z\mathbf{Z}$ is an isomorphism.

Here is the orbifold analogue of Theorem 10.2:

Theorem 10.6

Suppose $\mathbf{g}:\mathbf{X}\to\mathbf{Z}$, $\mathbf{h}:\mathbf{Y}\to\mathbf{Z}$ *are d-transverse* 1*-morphisms in* **dOrb** *or* **Kur**. *Then a fibre product* $\mathbf{W}=\mathbf{X}\times_{\mathbf{g},\mathbf{Z},\mathbf{h}}\mathbf{Y}$ *exists in* **dOrb** *or* **Kur**, *with* $\mathrm{vdim}\,\mathbf{W}=\mathrm{vdim}\,\mathbf{X}+\mathrm{vdim}\,\mathbf{Y}-\mathrm{vdim}\,\mathbf{Z}$.
This $\mathbf{W}$ *is an orbifold if and only if* $\mathbf{g},\mathbf{h}$ *are strongly d-transverse.*
The topological space W *of* $\mathbf{W}$ *is given as a set by*

$$\begin{aligned} W=\big\{(x,y,C):x\in X,\;\; y\in Y,\;\; \mathbf{g}(x)=\mathbf{h}(y)=z\in Z,\\ C\in G_x\mathbf{g}(G_x\mathbf{X})\backslash G_z\mathbf{Z}/G_y\mathbf{h}(G_y\mathbf{Y})\big\}. \end{aligned}\tag{10.5}$$

For $(x,y,C)\in W$ *with* $\gamma\in C\subseteq G_z\mathbf{Z}$, *there is an exact sequence*

$$\begin{array}{ccccccc} 0\to T_{(x,y,C)}\mathbf{W} & \longrightarrow & T_x\mathbf{X}\oplus T_y\mathbf{Y} & \xrightarrow[T_x\mathbf{g}\oplus(\gamma\cdot T_y\mathbf{h})]{} & T_z\mathbf{Z} \\ & & & & \downarrow \\ 0\longleftarrow O_z\mathbf{Z} & \xleftarrow{O_x\mathbf{g}\oplus(\gamma\cdot O_y\mathbf{h})} & O_x\mathbf{X}\oplus O_y\mathbf{Y} & \longleftarrow & O_{(x,y,C)}\mathbf{W}. \end{array}\tag{10.6}$$

Transverse fibre products of quotient orbifolds

Example 10.7

Suppose U,V,W are manifolds, Γ,Δ,K are finite groups acting smoothly on U,V,W, $\rho:\Gamma\to K$, $\sigma:\Delta\to K$ are group morphisms, and $g:U\to W$, $h:V\to W$ are ρ-, σ-equivariant smooth maps. Then $[g,\rho]:[U/\Gamma]\to[W/K]$, $[h,\sigma]:[V/\Delta]\to[W/K]$ are smooth maps of orbifolds.
We say that $[g,\rho],[h,\sigma]$ are *transverse* if $g:U\to W$ and $\kappa\cdot h:V\to W$ are transverse maps of manifolds for all $\kappa\in K$.
If $[g,\rho],[h,\sigma]$ are transverse then the fibre product in **Orb** is

$$[U/\Gamma]\times_{[g,\rho],[W/K],[h,\sigma]}[V/\Delta]\simeq\big[(U\times_{g,W,K\cdot h}(V\times K))/(\Gamma\times\Delta)\big].$$

Here $K\cdot h:V\times K\to W$ maps $(v,\kappa)\mapsto\kappa\cdot h(v)$, and $\Gamma\times\Delta$ acts on $U\times_{g,W,K\cdot h}(V\times K)$ by $(\gamma,\delta):(u,v,\kappa)\mapsto\big(\gamma\cdot u,\delta\cdot v,\rho(\gamma)\kappa\sigma(\delta)^{-1}\big)$.

In particular, if $U = V = W = *$ then we have

$$[*/\Gamma] \times_{[*/K]} [*/\Delta] = [\Gamma\backslash K/\Delta].$$

This explains the double quotient $G_x\mathbf{g}(G_x\mathbf{X})\backslash G_z\mathbf{Z}/G_y\mathbf{h}(G_y\mathbf{Y})$ in (10.5). If $\Gamma = \Delta = \{1\}$ then we have

$$* \times_{[*/K]} * = K,$$

considered as a 0-manifold with the discrete topology. So although the topological spaces X, Y, Z are all single points $*$, the topological space W is $|K|$ points, and (10.2) fails if $|K| > 1$.

10.3. Sketch proof of Theorem 10.2

Recall that d-manifolds **dMan** are a full 2-subcategory of d-spaces **dSpa**, a kind of derived C^∞-scheme. We first prove that all fibre products exist in **dSpa**, and that the forgetful functor to topological spaces **dSpa** $\rightarrow$ **Top** preserves fibre products. We do this by writing down an explicit fibre product $\mathbf{W} = \mathbf{X} \times_{\mathbf{g},\mathbf{Z},\mathbf{h}} \mathbf{Y}$ for any 1-morphisms $\mathbf{g} : \mathbf{X} \rightarrow \mathbf{Z}$, $\mathbf{h} : \mathbf{Y} \rightarrow \mathbf{Z}$ in **dSpa**, and verifying it satisfies the universal property.
Thus, if $\mathbf{g} : \mathbf{X} \rightarrow \mathbf{Z}$, $\mathbf{h} : \mathbf{Y} \rightarrow \mathbf{Z}$ are d-transverse 1-morphisms in **dMan**, then a fibre product $\mathbf{W} = \mathbf{X} \times_{\mathbf{g},\mathbf{Z},\mathbf{h}} \mathbf{Y}$ exists in **dSpa**, with topological space

$$W = \{(x, y) \in X \times Y : \mathbf{g}(x) = \mathbf{h}(y) \text{ in } Z\}.$$

If we can show $\mathbf{W}$ is a d-manifold, with $\mathrm{vdim}\,\mathbf{W} = \mathrm{vdim}\,\mathbf{X} + \mathrm{vdim}\,\mathbf{Y} - \mathrm{vdim}\,\mathbf{Z}$, then $\mathbf{W}$ is also a fibre product in **dMan**, as the universal property in **dSpa** implies that in **dMan**.

For $\mathbf{W}$ to be a d-manifold with given dimension is a local property: we have to show an open neighbourhood of each (x,y) in $\mathbf{W}$ is a d-manifold of given dimension.
So let $x\in\mathbf{X}$, $y\in\mathbf{Y}$ with $\mathbf{g}(x)=\mathbf{h}(y)=z\in\mathbf{Z}$. We can choose small open neighbourhoods $x\in\mathbf{T}\subseteq\mathbf{X}$, $y\in\mathbf{U}\subseteq\mathbf{Y}$, $z\in\mathbf{V}\subseteq\mathbf{Z}$ with $\mathbf{g}(\mathbf{T}),\mathbf{h}(\mathbf{U})\subseteq\mathbf{V}$, and equivalences $\mathbf{T}\simeq\mathbf{S}_{T,E,t}$, $\mathbf{U}\simeq\mathbf{S}_{U,F,u}$, $\mathbf{V}\simeq\mathbf{S}_{V,G,v}$ with standard model d-manifolds.
By making $\mathbf{T},\mathbf{U},\mathbf{V}$ smaller we suppose $V\subseteq\mathbb{R}^n$ is open, and $G\to V$ is a trivial vector bundle $V\times\mathbb{R}^k\to V$.
Then as in §5.3, $\mathbf{g}|_{\mathbf{T}}:\mathbf{T}\to\mathbf{V}$, $\mathbf{h}|_{\mathbf{U}}:\mathbf{U}\to\mathbf{V}$ are 2-isomorphic to standard model 1-morphisms $\mathbf{S}_{T',g,\hat g}:\mathbf{S}_{T,E,t}\to\mathbf{S}_{V,G,v}$ and $\mathbf{S}_{U',h,\hat h}:\mathbf{S}_{U,F,u}\to\mathbf{S}_{V,G,v}$.
Thus, the open neighbourhood $\mathbf{T}\times_{\mathbf{g}|_{\mathbf{T}},\mathbf{Z},\mathbf{h}|_{\mathbf{U}}}\mathbf{U}$ of (x,y) in $\mathbf{W}$ is equivalent in **dSpa** to the fibre product $\mathbf{S}_{T,E,t}\times_{\mathbf{S}_{T',g,\hat g},\mathbf{S}_{V,G,v},\mathbf{S}_{U',h,\hat h}}\mathbf{S}_{U,F,u}$ in **dSpa**.

This reduces proving Theorem 10.2 to showing that fibre products $\mathbf{S}_{T,E,t}\times_{\mathbf{S}_{T',g,\hat g},\mathbf{S}_{V,G,v},\mathbf{S}_{U',h,\hat h}}\mathbf{S}_{U,F,u}$ of standard model d-manifolds by d-transverse standard model 1-morphisms exist in **dMan**, and have the expected dimension, and long exact sequence (10.3), where we make the simplifying assumptions that $V\subseteq\mathbb{R}^n$ is open and $G=V\times\mathbb{R}^k$ is a trivial vector bundle.
We prove this by defining a standard model d-manifold $\mathbf{S}_{S,D,s}$, and showing it is 1-isomorphic in **dSpa** to the explicit fibre product $\mathbf{S}_{T,E,t}\times_{\mathbf{S}_{T',g,\hat g},\mathbf{S}_{V,G,v},\mathbf{S}_{U',h,\hat h}}\mathbf{S}_{U,F,u}$ already constructed in **dSpa**.
Explicitly, we take S to be an open neighbourhood of $\{(x,y)\in t^{-1}(0)\times u^{-1}(0): g(x)=h(y)\text{ in } v^{-1}(0)\}$ in $T'\times U'$.
On S we have a morphism of vector bundles

$$\pi_T^*(\hat g)\oplus\pi_U^*(\hat h)\oplus A:\pi_T^*(E)\oplus\pi_U^*(F)\oplus\mathbb{R}^n\longrightarrow\mathbb{R}^k \tag{10.7}$$

where $A:S\times\mathbb{R}^n\to\mathbb{R}^k$ is constructed from $v:V\to\mathbb{R}^k$ using Hadamard's Lemma. For S small enough, d-transversality implies (10.7) is surjective, and we define $D\to S$ to be its kernel.

10.4. Orientations on fibre products

Suppose X, Y, Z are oriented smooth manifolds, and $g : X \to Z$, $h : Y \to Z$ are transverse smooth maps. Then on the fibre product $W = X \times_{g,Z,h} Y$ in **Man** we can define an orientation, depending on the orientations of X, Y, Z, which is natural, except that it depends on a choice of *orientation convention*. A different orientation convention would multiply the orientation of W by a sign depending on $\dim X, \dim Y, \dim Z$.

We will first explain how to define the orientation on W in the classical case, and then generalize all of this to d-transverse fibre products of derived manifolds and orbifolds.

Recall from §9.5 that if X is an n-manifold, the *canonical bundle* is the real line bundle $K_X = \Lambda^n T^*X$ over X, and an orientation on X is an orientation on the fibres of K_X. To orient fibre products, we first show that given a transverse Cartesian square in **Man**

$$\begin{array}{ccc} W & \xrightarrow{\quad f \quad} & Y \\ \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle h} \\ X & \xrightarrow{\quad g \quad} & Z, \end{array} \tag{10.8}$$

there is a natural isomorphism (depending on orientation convention)

$$K_W \cong e^*(K_X) \otimes f^*(K_Y) \otimes (g \circ e)^*(K_Z)^*. \tag{10.9}$$

Thus orientations on the fibres of K_X, K_Y, K_Z determine an orientation on the fibres of K_W, and hence an orientation on W.

Proposition 10.8

Suppose we are given an exact sequence of finite-dimensional $\mathbb{R}$-vector spaces, with $E^i = 0$ for $|i| \gg 0$

$$\cdots \xrightarrow{\mathrm{d}} E^{i-1} \xrightarrow{\mathrm{d}} E^i \xrightarrow{\mathrm{d}} E^{i+1} \xrightarrow{\mathrm{d}} \cdots .$$

Then there is an isomorphism, depending on orientation convention

$$\bigotimes_{i\in\mathbb{Z},\ i \text{ odd}} \Lambda^{\mathrm{top}} E^i \cong \bigotimes_{i\in\mathbb{Z},\ i \text{ even}} \Lambda^{\mathrm{top}} E^i .$$

Proof.

Using kernels and cokernels, we can choose vector spaces V^i for $i \in \mathbb{Z}$ and isomorphisms $E^i \cong V^i \oplus V^{i+1}$, such that $\mathrm{d} : E^i \to E^{i+1}$ is

$$\begin{pmatrix} 0 & \mathrm{id}_{V_{i+1}} \\ 0 & 0 \end{pmatrix} : \begin{matrix} V^i \oplus \\ V^{i+1} \end{matrix} \longrightarrow \begin{matrix} V^{i+1} \oplus \\ V^{i+2}. \end{matrix}$$

Then

$$\bigotimes_{i\in\mathbb{Z},\ i \text{ odd}} \Lambda^{\mathrm{top}} E^i \cong \bigotimes_{i\in\mathbb{Z}} \Lambda^{\mathrm{top}} V^i \cong \bigotimes_{i\in\mathbb{Z},\ i \text{ even}} \Lambda^{\mathrm{top}} E^i .$$

□

Given a transverse Cartesian square (10.8), to define the isomorphism (10.9), note that we have an exact sequence

$$0 \to \begin{matrix} (g\circ e)^* \\ (T^*Z) \end{matrix} \xrightarrow{e^*(T^*g)\oplus f^*(T^*h)} \begin{matrix} e^*(T^*X) \\ \oplus f^*(T^*Y) \end{matrix} \xrightarrow{T^*e\oplus -T^*f} T^*W \to 0. \quad (10.10)$$

Applying Proposition 10.8 to (10.10) gives an isomorphism

$$K_W \otimes (g\circ e)^*(K_Z) \cong e^*(K_X) \otimes f^*(K_Y),$$

and rearranging gives (10.9). The 'orientation convention' is the choice of where to put signs in (10.10), how to identify $\Lambda^{\mathrm{top}}(U \oplus V) \cong (\Lambda^{\mathrm{top}} U) \otimes_{\mathbb{R}} (\Lambda^{\mathrm{top}} V)$, whether to write $E^i = V^i \oplus V^{i+1}$ or $E^i = V^{i+1} \oplus V^i$ in Proposition 10.8, and so on.

Theorem 10.9

Suppose we are given a d-transverse 2-Cartesian square in **dMan**

$$\begin{array}{ccc} \mathbf{W} & \xrightarrow{\mathbf{f}} & \mathbf{Y} \\ \downarrow \mathbf{e} & \eta \Uparrow & \mathbf{h} \downarrow \\ \mathbf{X} & \xrightarrow{\mathbf{g}} & \mathbf{Z}. \end{array}$$

Then there is a canonical isomorphism of C^∞-line bundles on the C^∞-scheme $\underline{W}$, depending on an orientation convention:

$$K_{\mathbf{W}} \cong \underline{e}^*(K_{\mathbf{X}}) \otimes_{\mathcal{O}_W} \underline{f}^*(K_{\mathbf{Y}}) \otimes_{\mathcal{O}_W} (\underline{g} \circ \underline{e})^*(K_{\mathbf{Z}})^*. \qquad (10.11)$$

Hence orientations on $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ induce an orientation on $\mathbf{W}$.

Sketch proof.

We have $K_{\mathbf{X}}|_x \cong (\Lambda^{\mathrm{top}} T_x\mathbf{X})^* \otimes (\Lambda^{\mathrm{top}} O_x\mathbf{X})$. Applying Proposition 10.8 to the long exact sequence (10.3) gives an isomorphism

$$K_{\mathbf{W}}|_w \cong K_{\mathbf{X}}|_x \otimes K_{\mathbf{Y}}|_y \otimes K_{\mathbf{Z}}^*|_z$$

for all $w \in \mathbf{W}$ with $\mathbf{e}(w) = x \in \mathbf{X}$, $\mathbf{f}(w) = y \in \mathbf{Y}$ and $\mathbf{g}(x) = \mathbf{h}(y) = z \in \mathbf{Z}$. Doing this in families gives (10.11). □

Fibre products have commutativity and associativity properties, up to canonical equivalence. The corresponding orientations given by Theorem 10.9 differ by a sign depending on the dimensions, and the orientation convention. For example, with my orientation conventions, if $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ are oriented d-manifolds and $\mathbf{g} : \mathbf{X} \to \mathbf{Z}$, $\mathbf{h} : \mathbf{Y} \to \mathbf{Z}$ are d-transverse then in oriented d-manifolds we have

$$\mathbf{X} \times_{\mathbf{g},\mathbf{Z},\mathbf{h}} \mathbf{Y} \simeq (-1)^{(\mathrm{vdim}\,\mathbf{X} - \mathrm{vdim}\,\mathbf{Z})(\mathrm{vdim}\,\mathbf{Y} - \mathrm{vdim}\,\mathbf{Z})} \mathbf{Y} \times_{\mathbf{h},\mathbf{Z},\mathbf{g}} \mathbf{X}.$$

When $\mathbf{Z} = *$ so that $\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y} = \mathbf{X} \times \mathbf{Y}$ this gives

$$\mathbf{X} \times \mathbf{Y} \simeq (-1)^{\mathrm{vdim}\,\mathbf{X}\,\mathrm{vdim}\,\mathbf{Y}} \mathbf{Y} \times \mathbf{X}.$$

For $\mathbf{e} : \mathbf{V} \to \mathbf{Y}$, $\mathbf{f} : \mathbf{W} \to \mathbf{Y}$, $\mathbf{g} : \mathbf{W} \to \mathbf{Z}$, and $\mathbf{h} : \mathbf{X} \to \mathbf{Z}$ we have

$$\mathbf{V} \times_{\mathbf{e},\mathbf{Y},\mathbf{f}\circ\boldsymbol{\pi}_{\mathbf{W}}} (\mathbf{W} \times_{\mathbf{g},\mathbf{Z},\mathbf{h}} \mathbf{X}) \simeq (\mathbf{V} \times_{\mathbf{e},\mathbf{Y},\mathbf{f}} \mathbf{W}) \times_{\mathbf{g}\circ\boldsymbol{\pi}_{\mathbf{W}},\mathbf{Z},\mathbf{h}} \mathbf{X},$$

so associativity holds without signs.