Derived Differential Geometry Prof Joyce 14 lectures TT 2015

Overview

Derived Differential Geometry is the study of derived smooth manifolds and orbifolds, where "derived" is in the sense of the Derived Algebraic Geometry of Jacob Lurie [13] and Toën-Vezzosi [15,16]. There are by now several different models for (higher) categories of derived manifolds and derived orbifolds: the "derived manifolds" of Spivak [14], the "d-manifolds" and "d-orbifolds" of Joyce [2,3,6], the "derived manifolds" of Borisov-Noel [9,10], and the "M-Kuranishi spaces" and "Kuranishi spaces" of Joyce [4]. For derived manifolds without boundary, these are all known to be roughly equivalent, at least at the level of homotopy categories.

Actually, a prototype version of derived orbifolds has been used for many years in the work of Fukaya, Oh, Ohta and Ono [7,11,12] as their "Kuranishi spaces", a geometric structure on moduli spaces of J-holomorphic curves in symplectic geometry, but it was not understood until recently that these are part of the world of derived geometry – see [4].

Derived manifolds are a (higher) category of geometric spaces which include ordinary smooth manifolds, but also many more singular objects – for instance, if X,Y are embedded submanifolds of a manifold Z, then the intersection $X \cap Y$ has the structure of a derived manifold of dimension $\dim X + \dim Y - \dim Z$ (which can be negative), and any closed subset S of $\mathbb{R}^n$ can be given the structure of a derived manifold of dimension $n-1$. They have a "derived" geometric structure, which is difficult to define, but means that derived manifolds have their own differential geometry, and in many ways behave as well (and sometimes better) than smooth manifolds.

One reason derived manifolds and derived orbifolds are important is that many moduli spaces (families of isomorphism classes of geometric objects) in differential geometry, and in complex algebraic geometry, can be given the structure of derived manifolds and derived orbifolds. For example, any moduli space of solutions of a nonlinear elliptic partial differential equation on a compact manifold is a derived manifold. Also, compact, oriented derived manifolds and derived orbifolds have "virtual classes", generalizing the fact that a compact, oriented n-dimensional manifold X has a fundamental class $[X]$ in its top-dimensional homology group $H_n(X;\mathbb{Z})$. This means that derived manifolds and orbifolds have applications in enumerative invariant problems (e.g. Donaldson invariants, Gromov-Witten invariants, Seiberg-Witten invariants, Donaldson-Thomas invariants, . . .), and generalizations such as Floer homology theories and Fukaya categories.

This lecture course will define and discuss 2-categories of derived manifolds and derived orbifolds, and their applications to moduli spaces of solutions of nonlinear elliptic partial differential equations including J-holomorphic curves, and "counting" problems in differential geometry, and complex algebraic geometry.

We explain two different ways to define these 2-categories, firstly using C^∞-rings, C^∞-schemes and C^∞-algebraic geometry [1,5,8], following Spivak [14], Joyce [2,3,6] and Borisov-Noël [9,10], or secondly using an "atlas of charts" (Kuranishi neighbourhoods) approach in [4], which is based on Fukaya et al. [7,11,12]. (See reading list below.)

Synopsis

Lecture 1: Different kinds of spaces in algebraic geometry: schemes, stacks, higher stacks, derived stacks. Basics of category theory, categories, functors. The Yoneda Lemma. Schemes as functors $\mathbf{Alg}_{\mathbb{K}} \to$ **Sets**, and stacks as functors $\mathbf{Alg}_{\mathbb{K}} \to$ **Groupoids**. Grothendieck's approach to moduli spaces as 'representable functors'.

Lecture 2: What is derived geometry? Derived schemes and stacks. Commutative differential graded algebras, examples. Bézout's Theorem and derived Bézout's Theorem. Patching together local models in derived geometry. Fibre products in ordinary categories; why we need higher categories in derived geometry. Outlook on derived geometry. Versions of derived manifolds due to Spivak, Borisov-Noël, and myself. Sample properties of derived manifolds $\boldsymbol{X}$: tangent spaces $T_x\boldsymbol{X}$, obstruction spaces $O_x\boldsymbol{X}$, d-transversality, existence of d-transverse fibre products in the 2-category **dMan**. Application to moduli problems, existence of d-manifold and d-orbifold structures on many moduli spaces.

Lecture 3: C^∞-algebraic geometry. C^∞-rings and their modules, the cotangent module. Sheaves on topological spaces. C^∞-schemes. Differences with ordinary algebraic geometry.

Lecture 4: 2-categories, d-spaces and d-manifolds. Strict and weak 2-categories. Fibre products in 2-categories. Differential graded C^∞-rings, and the 2-category of square zero dg C^∞-rings. D-spaces, a kind of derived C^∞-scheme, which are essentially schemes over square zero dg C^∞-rings, and form a 2-category **dSpa**, which contains C^∞-schemes and manifolds as full (2-)subcategories. Existence of fibre products in **dSpa**, and gluing d-spaces by equivalences on open d-subspaces. D-manifolds, a kind of derived manifold, as d-spaces $\boldsymbol{X}$ locally of the form $U \times_W V$ for U,V,W manifolds.

Lecture 5: Differential-geometric description of d-manifolds. The $O(s)$ and $O(s^2)$ notation. Standard model d-manifolds $\boldsymbol{S}_{V,E,s}$, 1-morphisms $\boldsymbol{S}_{V',f,f'}$, and 2-morphisms $\boldsymbol{S}_{\Lambda}$. Tangent spaces and obstruction spaces of d-manifolds. Criteria for when a standard model 1-morphism $\boldsymbol{S}_{V',f,f'}$ is étale or an equivalence.

Lecture 6: M-Kuranishi spaces, a simple way to define an (ordinary) category of derived manifolds, using an 'atlas of charts' approach. M-Kuranishi neighbourhoods and their morphisms. M-coordinate changes. The sheaf property of morphisms of M-Kuranishi neighbourhoods. Definition of M-Kuranishi spaces, and their morphisms. Composition of morphisms, proof of existence using the sheaf property. Manifolds as M-Kuranishi spaces. Tangent and obstruction spaces.

Lecture 7: Orbifolds. Why orbifolds should form a 2-category. Examples. Hilsum-Skandalis morphisms of quotient orbifolds. Orbifold charts, 1-morphisms, 2-morphisms, coordinate changes. Stacks on topological spaces. The stack property of the 2-categories of orbifold charts on open sets S in X. Definition of the weak 2-category of orbifolds **Orb** by an 'atlas of charts approach'. Composition of 1-morphisms defined using the stack property.

Lecture 8: Kuranishi spaces, following my arXiv:1409.6908, §4. Kuranishi neighbourhoods, 1-morphisms coordinate changes, and 2-morphisms of coordinate changes on a topological space X. Criteria for a 1-morphism to be a coordinate change. The stack property of the 2-categories of Kuranishi neighbourhoods on open sets S in X. Definition of the weak 2-category of Kuranishi spaces **Kur** by an 'atlas of charts approach'. Composition of 1-morphisms defined using the stack property. Kuranishi spaces with trivial orbifold groups give a weak 2-category of derived manifolds $\mathbf{Kur}_{\mathbf{trG}}$ equivalent to **dMan**.

Lecture 9: Differential geometry of derived manifolds and orbifolds. Orbifold groups, tangent spaces and obstruction spaces of derived orbifolds. (Weak) immersions, (weak) embeddings and derived submanifolds of derived manifolds and orbifolds. Embedding derived manifolds into manifolds, the Whitney Embedding Theorem, necessary and sufficient conditions for a derived manifold to be principal (covered by a single Kuranishi neighbourhood). (Weak) submersions. Orientations on derived manifolds.

Lecture 10: D-transversality (or strong d-transversality) for 1-morphisms $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$, $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ in **dMan** or **dOrb**, as conditions for existence of a fibre product $\boldsymbol{W} = \boldsymbol{X}_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ (or for $\boldsymbol{W}$ to exist and be a manifold). $\boldsymbol{g}$ a weak submersion (or a submersion) ensures $\boldsymbol{g},\boldsymbol{h}$ d-transverse (or strongly d-transverse). Sketch proof of the existence of d-transverse fibre products. Orientations on fibre products.

Lecture 11: Derived manifolds and orbifolds with boundary, and with corners. Manifolds with corners X, boundaries ∂X, k-corners $C_k(X)$, and the corner functor. Tangent bundles TX and b-tangent bundles ${}^bTX$. Conditions for existence of transverse fibre products; manifolds with generalized corners. Kuranishi spaces with corners as clean way to define derived orbifolds with corners. Differential geometry of Kuranishi spaces with corners.

Lecture 12: Bordism groups $B_n(Y)$ and derived bordism groups $dB_n(Y)$ for a manifold Y, and the isomorphism $B_n(Y) \cong dB_n(Y)$. Virtual classes in homology for compact oriented derived manifolds and orbifolds (without boundary). Discussion of the problem of forming virtual chains for compact oriented derived orbifolds with corners, and its application in Lagrangian Floer cohomology.

Lecture 13: Existence of derived manifold and orbifold structures on moduli spaces. Moduli spaces of solutions of nonlinear elliptic p.d.e.s. 'Truncation functors' from other geometric structures: FOOO Kuranishi spaces, polyfolds, $\mathbb{C}$-schemes with perfect obstruction theories, quasi-smooth derived $\mathbb{C}$-schemes, derived $\mathbb{C}$-schemes with –2-shifted symplectic structures. A conjectural approach to moduli spaces in differential geometry using Grothendieck-style 'representable moduli 2-functors'.

Lecture 14: Moduli spaces of J-holomorphic curves in symplectic geometry. J-holomorphic curves with marked points, Deligne-Mumford stable curves. Moduli spaces are compact oriented Kuranishi spaces. Virtual classes for these define Gromov-Witten invariants.

Course website

You will be able to download lecture notes (once they are written) from http://people.maths.ox.ac.uk/~joyce/DDG2015. Students may wish to print out the lecture notes and bring them to the lectures.

Reading:

[1] D. Joyce, '*An introduction to C^∞-schemes and C^∞-algebraic geometry*', arXiv:1104.4951. (Survey.)

[2] D. Joyce, '*An introduction to d-manifolds and derived differential geometry*', pages 230-281 in L. Brambila-Paz et al. '*Moduli spaces'*, L.M.S. Lecture Notes 411, C.U.P., 2014. arXiv:1206.4207. (Survey.)

[3] D. Joyce, '*D-manifolds, d-orbifolds and derived differential geometry: a detailed summary*', arXiv:1208.4948.

[4] D. Joyce, '*A new definition of Kuranishi space*', arXiv:1409.6908.

Further reading:

[5] D. Joyce, '*Algebraic Geometry over C^∞-rings*', arXiv:1001.0023, 2010.

[6] D. Joyce, *'D-manifolds and d-orbifolds: a theory of derived differential geometry'*, preliminary version of book (2012), 768 pages, available from http://people.maths.ox.ac.uk/~joyce/dmanifolds.html

[7] K. Fukaya, '*Floer homology of Lagrangian submanifolds*', arXiv:1106.4882. (Survey.)

[8] E.J. Dubuc, '*C^∞-schemes*', Amer. J. Math. 103 (1981), 683-690.

Even further reading:

[9] D. Borisov, '*Derived manifolds and Kuranishi models*', arXiv:1212.1153.

[10] D. Borisov and J. Noël, '*Simplicial approach to derived differential manifolds*', arXiv:1112.0033.

[11] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, '*Lagrangian intersection Floer theory - anomaly and obstruction*', Parts I and II. A.M.S./International Press, 2009.

[12] K. Fukaya and K. Ono, '*Arnold Conjecture and Gromov-Witten invariant*', Topology 38 (1999), 933-1048.

[13] J. Lurie, '*Derived Algebraic Geometry V: Structured spaces*', arXiv:0905.0459.

[14] D.I. Spivak, '*Derived smooth manifolds*', Duke Mathematical Journal 153 (2010), 55-128. arXiv:0810.5174.

[15] B. Toën, '*Derived Algebraic Geometry*', arXiv:1401.1044.

[16] B. Toën and G. Vezzosi, '*Homotopical Algebraic Geometry II: Geometric Stacks and Applications*', Mem. Amer. Math. Soc. 193 (2008), no. 902. math.AG/0404373.