Lecture 2: Enumerative invariants and wall-crossing formulae via vertex algebras

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These slides available at
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1. Introduction

An *enumerative invariant theory* in Algebraic or Differential Geometry is the study of invariants $I_\alpha(\tau)$ which 'count' τ-semistable objects E with fixed topological invariants $[\![E]\!]=\alpha$ in some geometric problem, usually by means of a virtual class $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ for the moduli space $\mathcal{M}_\alpha^{\rm ss}(\tau)$ of τ-semistable objects in some homology theory, with $I_\alpha(\tau)=\int_{[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}}\mu_\alpha$ for some natural cohomology class μ_α. We call the theory $\mathbb{C}$-*linear* if the objects E live in a $\mathbb{C}$-linear additive category $\mathcal{A}$. For example:

- Invariants counting semistable vector bundles on curves.
- Mochizuki-style invariants counting coherent sheaves on surfaces. (Think of as algebraic Donaldson invariants.)
- Donaldson–Thomas invariants of Calabi–Yau or Fano 3-folds.
- Donaldson–Thomas type invariants of Calabi–Yau 4-folds.
- Invariants counting representations of quivers Q.
- $\mathrm{U}(m)$ Donaldson invariants of 4-manifolds.

I have proved that many such theories in Algebraic Geometry, in which either the moduli spaces are automatically smooth (e.g. coherent sheaves on curves, quiver representations), or the invariants are defined using Behrend–Fantechi obstruction theories and virtual classes, share a common universal structure.

I expect this universal structure also to extend to Calabi–Yau 4-fold invariants defined using Borisov–Joyce / Oh–Thomas virtual classes, and to Donaldson invariants in Differential Geometry.

Here is an outline of this structure:

(a) As in Lecture 1, we form two moduli stacks $\mathcal{M}, \mathcal{M}^{\rm pl}$ of all objects E in $\mathcal{A}$, where $\mathcal{M}$ is the usual moduli stack, and $\mathcal{M}^{\rm pl}$ the 'projective linear' moduli stack of objects E modulo 'projective isomorphisms', i.e. quotient by $\lambda\,\mathrm{id}_E$ for $\lambda \in \mathbb{G}_m$.

(b) We are given a quotient $K_0(\mathcal{A}) \twoheadrightarrow K(\mathcal{A})$, where $K(\mathcal{A})$ is the lattice of topological invariants $[\![E]\!]$ of E (e.g. fixed Chern classes). We split $\mathcal{M} = \coprod_{\alpha \in K(\mathcal{A})} \mathcal{M}_\alpha$, $\mathcal{M}^{\rm pl} = \coprod_{\alpha \in K(\mathcal{A})} \mathcal{M}^{\rm pl}_\alpha$.

(c) There is a symmetric biadditive *Euler form* $\chi : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$.

(d) We can form the homology $H_*(\mathcal{M}), H_*(\mathcal{M}^{\rm pl})$ over $\mathbb{Q}$, with $H_*(\mathcal{M})=\bigoplus_{\alpha\in K(\mathcal{A})}H_*(\mathcal{M}_\alpha)$, $H_*(\mathcal{M}^{\rm pl})=\bigoplus_{\alpha\in K(\mathcal{A})}H_*(\mathcal{M}_\alpha^{\rm pl})$. Define shifted versions $\hat H_*(\mathcal{M}), \check H_*(\mathcal{M}^{\rm pl})$ by $\hat H_n(\mathcal{M}_\alpha)=H_{n-\chi(\alpha,\alpha)}(\mathcal{M}_\alpha)$, $\check H_n(\mathcal{M}_\alpha^{\rm pl})=H_{n+2-\chi(\alpha,\alpha)}(\mathcal{M}_\alpha^{\rm pl})$. As in Lecture 1, we make $\hat H_*(\mathcal{M})$ into a *graded vertex algebra*, and $\check H_*(\mathcal{M}^{\rm pl})$ into a *graded Lie algebra*.

(e) There is a notion of *stability condition* τ on $\mathcal{A}$. When $\mathcal{A}=\mathrm{coh}(X)$, this can be Gieseker stability for a polarization on X. For each $\alpha\in K(\mathcal{A})$ we can form moduli spaces $\mathcal{M}_\alpha^{\rm st}(\tau)\subseteq\mathcal{M}_\alpha^{\rm ss}(\tau)$ of τ-(semi)stable objects in class α. Here $\mathcal{M}_\alpha^{\rm st}(\tau)$ is a substack of $\mathcal{M}_\alpha^{\rm pl}$, and is a $\mathbb{C}$-scheme with perfect obstruction theory. Also $\mathcal{M}_\alpha^{\rm ss}(\tau)$ is proper. Thus, if $\mathcal{M}_\alpha^{\rm st}(\tau)=\mathcal{M}_\alpha^{\rm ss}(\tau)$ we have a virtual class $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$, which we regard as an element of $H_*(\mathcal{M}_\alpha^{\rm pl})$. The virtual dimension is $\mathrm{vdim}_{\mathbb{R}}[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}=2-\chi(\alpha,\alpha)$, so $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ lies in $\check H_0(\mathcal{M}_\alpha^{\rm pl})\subset\check H_0(\mathcal{M}^{\rm pl})$, which is a Lie algebra by (d).

(f) For many theories, there is a problem defining the invariants $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ when $\mathcal{M}_\alpha^{\rm st}(\tau)\neq\mathcal{M}_\alpha^{\rm ss}(\tau)$, i.e. when the moduli spaces $\mathcal{M}_\alpha^{\rm ss}(\tau)$ contain *strictly τ-semistable points*.
I give a systematic way to define invariants $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm inv}$ in homology over $\mathbb{Q}$ (not $\mathbb{Z}$) in these cases, using auxiliary pair invariants, with $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm inv}=[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ if $\mathcal{M}_\alpha^{\rm st}(\tau)=\mathcal{M}_\alpha^{\rm ss}(\tau)$. (This method is well known, e.g. in Joyce–Song D–T theory.)
I prove the $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm inv}$ are independent of the choices used in the pair invariant method.

(g) If $\tau,\tilde\tau$ are stability conditions and $\alpha\in K(\mathcal{A})$, I prove a wall crossing formula

$$[\mathcal{M}_\alpha^{\rm ss}(\tilde\tau)]_{\rm inv}=\sum_{\alpha_1+\cdots+\alpha_n=\alpha}\tilde U(\alpha_1,\ldots,\alpha_n;\tau,\tilde\tau)\cdot\bigl[\bigl[\ldots\bigl[[\mathcal{M}_{\alpha_1}^{\rm ss}(\tau)]_{\rm inv},[\mathcal{M}_{\alpha_2}^{\rm ss}(\tau)]_{\rm inv}\bigr],\ldots\bigr],[\mathcal{M}_{\alpha_n}^{\rm ss}(\tau)]_{\rm inv}\bigr],\qquad(1)$$

where $\tilde U(-)$ are combinatorial coefficients defined in my previous work on wall-crossing formulae for motivic invariants, and [,] is the Lie bracket on $\check H_0(\mathcal{M}^{\rm pl})$ from (d).

(h) In some theories the natural obstruction theory on $\mathcal{M}_\alpha^{\rm st}(\tau)=\mathcal{M}_\alpha^{\rm ss}(\tau)$ has a trivial summand $\mathbb{C}^{o_\alpha}$ in its obstruction sheaf for $o_\alpha>0$, and so the virtual class $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ is zero. In these cases one defines a *reduced* obstruction theory on $\mathcal{M}_\alpha^{\rm st}(\tau)$ by deleting the $\mathbb{C}^{o_\alpha}$ factor, and obtains *reduced* virtual classes $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm red}$. For example, this holds for coherent sheaves on surfaces X with geometric genus $p_g>0$, with $o_\alpha=p_g$ when $\operatorname{rank}\alpha>0$.
My theory extends to 'reduced' invariants, allowing o_α to depend on $\alpha\in K(\mathcal{A})$ with $o_\alpha+o_\beta\geqslant o_{\alpha+\beta}$, giving invariants $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm red}$ in $\check H_{2o_\alpha}(\mathcal{M}_\alpha^{\rm pl})$. Generalizing (1), they satisfy the wall crossing formula

$$[\mathcal{M}_\alpha^{\rm ss}(\tilde\tau)]_{\rm red}=\sum_{\substack{\alpha_1+\cdots+\alpha_n=\alpha:\\ o_{\alpha_1}+\cdots+o_{\alpha_n}=o_\alpha}}\tilde U(\alpha_1,\ldots,\alpha_n;\tau,\tilde\tau)\cdot\bigl[\bigl[\ldots\bigl[[\mathcal{M}_{\alpha_1}^{\rm ss}(\tau)]_{\rm red},[\mathcal{M}_{\alpha_2}^{\rm ss}(\tau)]_{\rm red}\bigr],\ldots\bigr],[\mathcal{M}_{\alpha_n}^{\rm ss}(\tau)]_{\rm red}\bigr]. \quad (2)$$

If $o_\alpha=o>0$ for all α this reduces to $[\mathcal{M}_\alpha^{\rm ss}(\tilde\tau)]_{\rm red}=[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm red}$, that is, the invariants are independent of stability condition.

(i) When $\mathcal{A} = \mathrm{coh}(X)$ or $D^b \mathrm{coh}(X)$ for X a Calabi–Yau 3-fold, the natural obstruction theory on $\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$ has terms in degree -2 from $\mathrm{Ext}^3(E, E)$. We can remove these by taking trace-free Ext to define Donaldson–Thomas invariants, changing the real virtual dimension by 2.
To include these in the theory, as in Lecture 1, for $\mathcal{A}$ odd Calabi–Yau we can modify (d) above to make $\hat{H}_*(\mathcal{M})$ into a *graded vertex Lie algebra* (with grading changed by 2) and $\check{H}_*(\mathcal{M}^{\mathrm{pl}})$ into a *graded Lie algebra* (with grading changed by 2). So we can include Donaldson–Thomas theory in our picture.
For ordinary D–T invariants this does not add much to the Joyce–Song / Kontsevich–Soibelman picture. However, for a local Calabi–Yau 3-fold with an action of a group G (e.g. $\mathbb{G}_m$ acting on K_X for X a surface) we can do Donaldson–Thomas theory in G-equivariant homology, giving non-motivic invariants, with applications to Thomas' equivariant Vafa–Witten theory.

Note that the vertex algebra / Lie algebra picture of Lecture 1 is very general, e.g. it works for $\mathcal{A} = \mathrm{coh}(X)$ or $\mathcal{A} = D^b \mathrm{coh}(X)$ for any smooth projective $\mathbb{C}$-scheme X. However, to get an enumerative invariant theory is much more restrictive, we basically need X to be a curve, surface, Calabi–Yau 3- or 4-fold, or Fano 3-fold.

The proofs of the WCFs (1), (2) are very long and complicated. The rough idea is that for 'simple' wall-crossings, involving splittings $E = E_1 \oplus \cdots \oplus E_n$ in $\mathcal{A}$ for n at most 2, I can prove the WCF (with splittings $\alpha = \alpha_1 + \cdots + \alpha_n$ for $n \leqslant 2$) by $\mathbb{G}_m$-localization on a master space. Then I show that complicated wall-crossings in $\mathcal{A}$ can be reduced to a sequence of simple wall-crossings in an auxiliary category $\mathcal{B}$ in an exact sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \text{mod-}\mathbb{C}Q \longrightarrow 0.$$

2. Invariants counting sheaves on surfaces

Let X be a complex projective surface, with geometric genus $p_g = \dim H^0(K_X)$. We usually restrict to $p_g > 0$, that is, $b^2_+(X) > 1$. Let $\kappa \in K^0_{\rm top}(X)$ be a topological K-theory class on X. We often write $\kappa = (r, \alpha, k)$ for $r = \operatorname{rank}\kappa$, $\alpha = c_1(\kappa) \in H^2(X, \mathbb{Z})$ and $k = \operatorname{ch}_2(\kappa) \in \frac{1}{2}\mathbb{Z}$ with $\int_X \alpha^2 + 2k \in 2\mathbb{Z}$, and usually restrict to $r > 0$. Choose a Kähler class ω on X. Then we can define *Gieseker (semi)stability* τ of coherent sheaves on X using ω, and can form moduli stacks $\mathcal{M}^{\rm st}_\kappa(\tau) \subseteq \mathcal{M}^{\rm ss}_\kappa(\tau)$ of τ-(semi)stable coherent sheaves on X with class κ. Here $\mathcal{M}^{\rm st}_\kappa(\tau)$ has a Behrend–Fantechi obstruction theory (which is *reduced* if $p_g > 0$) and $\mathcal{M}^{\rm ss}_\kappa(\tau)$ has a projective coarse moduli scheme. Thus, if $\mathcal{M}^{\rm st}_\kappa(\tau) = \mathcal{M}^{\rm ss}_\kappa(\tau)$ (if there are no strictly τ-semistable sheaves in class κ) then $\mathcal{M}^{\rm ss}_\kappa(\tau)$ is proper with a B–F obstruction theory, and so has a *virtual class* $[\mathcal{M}^{\rm ss}_\kappa(\tau)]_{\rm virt}$ in $H_*(\mathcal{M}^{\rm ss}_\kappa(\tau), \mathbb{Z})$. In nice cases (e.g. Hilbert schemes) $\mathcal{M}^{\rm ss}_\kappa(\tau)$ is smooth and $[\mathcal{M}^{\rm ss}_\kappa(\tau)]_{\rm virt} = [\mathcal{M}^{\rm ss}_\kappa(\tau)]_{\rm fund}$ is the fundamental class of $\mathcal{M}^{\rm ss}_\kappa(\tau)$ as a compact complex manifold.

We can construct many *universal cohomology classes* S_{jkl} on $\mathcal{M}^{\rm ss}_\kappa(\tau)$ — in the case when $\mathcal{M}^{\rm ss}_\kappa(\tau)$ is a fine moduli space, by $S_{jkl}=\mathrm{ch}_l(\mathcal{U})\backslash e_{jk}$ for $\mathcal{U}\to X\times\mathcal{M}^{\rm ss}_\kappa(\tau)$ the universal sheaf and e_{jk} a basis element for $H_k(X,\mathbb{Q})$. Then we can form *enumerative invariants* $I_P=\int_{[\mathcal{M}^{\rm ss}_\kappa(\tau)]_{\rm virt}}P(S_{jkl})$ for any polynomial $P(S_{jkl})$ in these universal classes homogeneous of the correct dimension. There is a huge literature by many authors studying invariants of this kind for particular κ (e.g. rank $r=2$) and $P(S_{jkl})$. They include *Donaldson invariants* of the underlying oriented 4-manifold X, *K-theoretic Donaldson invariants*, *Vafa–Witten invariants* (instanton branch), *Segre integrals*, *Verlinde integrals*, *virtual Euler characteristics* and χ_y*-genera* of $\mathcal{M}^{\rm ss}_\kappa(\tau)$, and so on. Often people show that these invariants I_P can be encoded in generating functions of a nice form. There are also many open conjectures like this by Göttsche, Kool and others. In fact, for rank $r>1$ and $c_1(X)\neq 0$ there are lots of conjectures and few theorems.

I will report on a project which in some sense determines *all possible* invariants $I_P=\int_{[\mathcal{M}_\kappa^{\rm ss}(\tau)]_{\rm virt}}P(S_{jkl})$, as it determines the virtual classes $[\mathcal{M}_\kappa^{\rm ss}(\tau)]_{\rm virt}$, and more generally invariants $[\mathcal{M}_\kappa^{\rm ss}(\tau)]_{\rm inv}$ also defined if $\mathcal{M}_\kappa^{\rm st}(\tau)\neq\mathcal{M}_\kappa^{\rm ss}(\tau)$, as elements of a polynomial algebra $H_*(\mathcal{M}_\kappa^{\rm pl})=e^\kappa\otimes\mathbb{Q}[s_{jkl}]$ dual to $H^*(\mathcal{M}_\kappa^{\rm pl})=\mathbb{Q}[S_{jkl}]$, as in Lecture 1.
We give an expression for $[\mathcal{M}_\kappa^{\rm ss}(\tau)]_{\rm inv}$ in terms of non-explicit universal functions in infinitely many variables $r_0,r_1,\ldots,$ depending on the rank r of κ, with coefficients in a number field $\mathbb{F}_r\subset\mathbb{C}$.
This proves the structural part of many conjectures in the literature.
There is an important difference between $p_g=0$ and $p_g>0$. If $p_g=0$ (i.e. $b^2_+=1$) then $[\mathcal{M}_\kappa^{\rm ss}(\tau)]_{\rm inv}$ depends on the Kähler form ω used to define τ, but if $p_g>0$ (i.e. $b^2_+>1$) it is independent.
For $p_g>0$ we define $[\mathcal{M}_\kappa^{\rm ss}(\tau)]_{\rm inv}$ using *reduced* obstruction theories. The WCFs (1) for $p_g=0$ and (2) for $p_g>0$ are different (there are more terms when $p_g=0$). Today I discuss only $p_g>0$.
$p_g=0$ is more difficult, as it involves (naïvely) non-convergent sums.

3. The main results. 3.1. Normalizing $c_1(\kappa)$

Let $L \to X$ be a line bundle with $c_1(L) = \lambda \in H^2(X, \mathbb{Z})$. Then $-\otimes L : D^b \operatorname{coh}(X) \to D^b \operatorname{coh}(X)$ is an equivalence inducing an isomorphism $\mathcal{M}_\kappa \to \mathcal{M}_{\kappa\otimes[\![L]\!]}$. Under the isomorphism $H_*(\mathcal{M}_\kappa, \mathbb{Q}) \cong \mathbb{Q}[s_{jkl}]$, this is identified with an algebra isomorphism $\Omega_\lambda : \mathbb{Q}[s_{jkl}] \to \mathbb{Q}[s_{jkl}]$ acting on generators by

$$\Omega_\lambda : s_{jkl} \longmapsto \sum_{j',k',l':2l-k=2l'-k'} A_{jk}^{j'k'} s_{j'k'l'},$$

where $(A_{jk}^{j'k'})$ is the matrix of $-\otimes L$ on $K^0_{\rm top}(X)$, and is polynomial in λ. Thus Ω_λ makes sense for $\lambda \in H^2(X, \mathbb{Q})$, as well as $\lambda \in H^2(X, \mathbb{Z})$. We have $\Omega_\lambda([\mathcal{M}^{\rm ss}_\kappa(\tau)]_{\rm inv}) = [\mathcal{M}^{\rm ss}_{\kappa\otimes[\![L]\!]}(\tau)]_{\rm inv}$. So for $\kappa = (r, \alpha, k)$ with $r > 0$, we find it helpful to consider $\Omega_{-\alpha/r}([\mathcal{M}^{\rm ss}_{(r,\alpha,k)}(\tau)]_{\rm inv})$. Effectively, we are tensoring by a 'fractional line bundle' L with $c_1(L) = -\alpha/r$, to modify $\kappa = (r, \alpha, k)$ so that it has $c_1(\kappa) = 0$. The advantage is that formulae for $\Omega_{-\alpha/r}([\mathcal{M}^{\rm ss}_{(r,\alpha,k)}(\tau)]_{\rm inv})$ are nearly independent of α (they depend on $\int_X \alpha \cup \beta \mod r$ for $\beta \in \mathrm{SW}(X)$).

3.2. The universal variables r_l. The number field $\mathbb{F}_r$

We want to give an expression for $\Omega_\lambda([\mathcal{M}^{\rm ss}_\kappa(\tau)]_{\rm inv})$ involving universal functions independent of X, and of the bases (e_{jk}) for $H_k(X,\mathbb{Q})$ and (ϵ_{jk}) for $H^k(X,\mathbb{Q})$ which determine the (co)homology variables s_{jkl}, S_{jkl}. To do this we will use 'universal variables' r_l where $r_l \in H^*(X,\mathbb{Q}) \otimes \mathbb{Q}[s_{jkl}]$ for $0,1,\ldots$ are given by

$$r_l = \sum_{j,k,j',k':\, l\geqslant k/2} \lambda^{jk}_{j'k'} \epsilon_{j'k'} \boxtimes s_{jkl}, \quad l=1,2,\ldots, \tag{3}$$

with $(\lambda^{j'k'}_{jk})$ the inverse matrix of $(\alpha,\beta)\mapsto \int_X \alpha\cup\beta$ on $H^*(X)$.

We write $\boldsymbol{r}=(r_0,r_1,r_2,\ldots)$.

For $r\geqslant 1$ (the rank of κ) define a number field $\mathbb{F}_r\subset\mathbb{C}$ by

$$\mathbb{F}_r = \begin{cases} \mathbb{Q}, & r=1 \text{ or } 2, \\ \mathbb{Q}[e^{\frac{\pi i}{2r}}], & r\geqslant 3 \text{ is odd}, \\ \mathbb{Q}[e^{\frac{\pi i}{r}}], & r\geqslant 3 \text{ is even}. \end{cases}$$

3.3. The main theorem

Theorem 1

When $p_g > 0$, for $r \geqslant 1$ and $(r,\alpha,k) \in K^0_{\rm sst}(X)$ there is a formula

$$\Omega_{-\alpha/r}\bigl([\mathcal{M}^{\rm ss}_{(r,\alpha,k)}(\tau)]_{\rm fd}\bigr) = \bigl[q^{\mathrm{vdim}\,\mathcal{M}^{\rm ss}_{(r,\alpha,k)}(\tau)_{\rm fd}}\bigr] \tag{4}$$

$$\left(\sum_{\substack{\beta_1,\ldots,\beta_{r-1}\\ \in H^2(X,\mathbb{Z})^{1,1}:\\ \mathfrak{s}_{\beta_a}\in \mathrm{SW}(X),\\ a=1,\ldots,r-1}} \begin{array}{c} r^2 \cdot \rho_r^{\int_X \mathrm{td}_2(X)} \cdot \eta_r^{\int_X c_1(X)^2} \cdot \prod_{1\leqslant a\leqslant b\leqslant r-1} \zeta_{r,ab}^{\int_X \beta_a\cup\beta_b} \cdot \\ \phi_r^{\int_X \alpha\cup c_1(X)} \cdot \prod_{a=1}^{r-1}\bigl(\mathrm{SW}([\mathfrak{s}_{\beta_a}])\theta_{r,a}^{\int_X \alpha\cup\beta_a}\bigr) \cdot \\ \exp\Bigl[\int_X A_r(\beta_1,\ldots,\beta_{r-1},c_1(X),\mathrm{td}_2(X),q,\boldsymbol{r})\Bigr] \end{array}\right).$$

Here $[\mathcal{M}^{\rm ss}_{(r,\alpha,k)}(\tau)]_{\rm fd}$ is the 'fixed determinant' invariant, equal to $[\mathcal{M}^{\rm ss}_{(r,\alpha,k)}(\tau)]_{\rm inv}$ when $b^1(X)=0$, and $\rho_r,\eta_r,\zeta_{r,ab},\phi_r,\theta_{r,a}\in\mathbb{F}_r\setminus\{0\}$, and A_r is a universal function independent of X, and $\mathrm{SW}(\mathfrak{s}_{\beta_a})\in\mathbb{Z}$ are Seiberg–Witten invariants of X. Furthermore:

Theorem 1 (Continued)

(i) $\rho_r = \pm\frac{1}{r}$.

(ii) $\theta_{r,a} \in \{e^{\frac{2\pi ib}{r}} : 1 \leqslant b < r\}$ *is a nontrivial* r^{th} *root of unity.*

(iii) $\phi_r \in \{e^{\frac{2\pi ib}{r}} : 1 \leqslant b \leqslant r\}$ *is an* r^{th} *root of unity.*

(iv) η_r *and* $\zeta_{r,ab}$ *for* $1 \leqslant a \leqslant b < r$ *lie in* $\mathbb{F}_r \setminus \{0\}$.

(v) A_r *lies in the quotient of* $\mathbb{F}_r[\beta_1,\ldots,\beta_{r-1},c_1(X),\mathrm{td}_2(X), r_0,r_1,r_2,\ldots][[q]]_{q^{>0}}$ *by an ideal generated by things like* $c_1(X)^3, c_1(X)\cup\mathrm{td}_2(X),\ldots$. *Here to regard* A_r *as independent of* X, *we just consider* $\beta_a, c_1(X),\ldots$ *to be formal variables. But when we fix a surface* X, *then we regard* $A_r(\beta_1,\ldots,\boldsymbol{r})$ *as lying in* $H^*(X,\mathbb{Q})\otimes\mathbb{Q}[s_{jkl}][[q]]$, *where* $\beta_a, c_1(X),\mathrm{td}_2(X)\in H^*(X,\mathbb{Q})$ *are the given values, and* $r_l\in H^*(X,\mathbb{Q})\otimes\mathbb{Q}[s_{jkl}]$ *are as in* (3). *Then* $\int_X A_r(\cdots)$ *applies* $\int_X : H^*(X,\mathbb{Q})\to\mathbb{Q}$ *so that* $\int_X A_r(\cdots)\in\mathbb{Q}[s_{jkl}][[q]]_{q^{>0}}$.

Note that α appears in (4) only through $[q^{\mathrm{vdim}\,\mathcal{M}^{\mathrm{ss}}_{(r,\alpha,k)}(\tau)_{\mathrm{fd}}}]$ and $\phi_r^{\int_X\alpha\cup c_1(X)}, \theta_{r,a}^{\int_X\alpha\cup\beta_a}$, and so via $\int_X\alpha\cup c_1(X), \int_X\alpha\cup\beta_a \mod r$.

3.4. Example: Hilbert schemes

For rank $r = 1$, fixed determinant moduli spaces $\mathcal{M}^{\mathrm{ss}}_{(r,\alpha,k)}(\tau)_{\mathrm{fd}}$ are basically Hilbert schemes $\mathrm{Hilb}^n(X)$. Also there are no Seiberg–Witten terms in (4). In this case we can rewrite and strengthen Theorem 1 to give:

Theorem 2

Writing $\boldsymbol{u} = (u_2, u_3, \ldots)$, there exists a formal function $H(c_1, c_2, \boldsymbol{u})$ in $\mathbb{Q}[u_3, u_4, \ldots][[e^{-u_2}, c_1, c_2]]$, defined uniquely as the solution to a p.d.e., such that for any complex projective surface X we have

$$\sum_{n \geqslant 0} q^n [\mathrm{Hilb}^n(X)]_{\mathrm{fund}} = \exp\left[\int_X \Big(r_0 + H\big(c_1(X), c_2(X), r_2 - \log q, r_3, r_4, \ldots\big)\Big)\right]. \tag{5}$$

We can compute $H(c_1, c_2, \boldsymbol{u})$ up to some order in e^{-u_2}, c_1, c_2 using Mathematica. If an algebraic group G acts on X, equation (5) also holds in **equivariant** *homology $H_*^G(\mathcal{M})$.*

3.5. An application: Virasoro constraints

The following is a minor extension of work by Arkadij Bojko, Woonam Lim, and Miguel Moreira.

Theorem 3

Hilbert schemes $[\mathrm{Hilb}^n(X)]_{\mathrm{fund}}$ satisfy 'Virasoro constraints' (some complicated identities) for all complex projective surfaces X.

Previously this was known for X with $b^1(X) = 0$ (Moreira 2021).

Sketch proof.

By MOOP 2020, Virasoro constraints hold for $[\mathrm{Hilb}^n(X)]_{\mathrm{fund}}$ for X projective toric. When $X = \mathbb{CP}^2$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$, this implies $H(c_1, c_2, \boldsymbol{u})$ in Theorem 3 satisfies a large family of p.d.e.s. These p.d.e.s then imply Virasoro for all X. This works when $b^1(X) > 0$ as the odd variables s_{jkl} are packaged inside even variables r_l. □

I expect to deduce Virasoro constraints for sheaf counting invariants for all projective surfaces X, following Bojko–Lim–Moreira 2022.

3.6. Example: Donaldson invariants in arbitrary rank

Let $L \in H^2(X,\mathbb{Q})$, and write $L=\sum_{j=1}^{b^2} L_j\epsilon_{j2}$. The *rank r Donaldson invariants* of X are

$$D^X_{(r,\alpha,k)}(L+\mathrm{upt})=\int_{[\mathcal{M}^{\rm ss}_{(r,\alpha,k)}(\tau)]_{\rm fd}}\exp\Bigl(\sum\nolimits_{j=1}^{b^2} L_jS_{j22}+S_{102}u\Bigr).$$

Theorem 4

$$D^X_{(r,\alpha,k)}(L+upt)=\bigl[q^{\mathrm{vdim}\,\mathcal{M}^{\rm ss}_{(r,\alpha,k)}(\tau)_{\rm fd}}\bigr] \tag{6}$$

$$\left(\sum_{\substack{\beta_1,\ldots,\beta_{r-1}\\ \in H^2(X,\mathbb{Z})^{1,1}:\\ \mathfrak{s}_{\beta_a}\in\mathrm{SW}(X),\\ a=1,\ldots,r-1}} \begin{aligned}&r^2\rho_r^{\int_X \mathrm{td}_2(X)}\eta_r^{\int_X c_1(X)^2}\phi_r^{\int_X \alpha\cup c_1(X)}\prod_{1\leqslant a\leqslant b\leqslant r-1}\zeta_{r,ab}^{\int_X\beta_a\cup\beta_b}\\ &\prod_{a=1}^{r-1}\bigl(\mathrm{SW}([\mathfrak{s}_{\beta_a}])\theta_{r,a}^{\int_X\alpha\cup\beta_a}\bigr)\cdot\exp\Bigl[q^2\Bigl(\tfrac12\int_X L^2+ru\Bigr)\\ &+q\Bigl(\int_X L\cup\Bigl(C_rc_1(x)+\sum_{a=1}^{r-1}C_{r,a}\beta_a\Bigr)\Bigr)\Bigr]\end{aligned}\right).$$

Here $C_r, C_{r,a}\in\mathbb{F}_r$. The $\exp[\cdots]$ term comes from the terms in $q^2r_2^2, q^2r_2, qc_1(x)\cup r_2, q\beta_a\cup r_2$ in A_r, just $r+2$ coefficients.

3.7. Symmetries of the generating function

Here is (4) again:

$$\Omega_{-\alpha/r}\bigl([\mathcal{M}^{\rm ss}_{(r,\alpha,k)}(\tau)]_{\rm fd}\bigr)=\Bigl[q^{\mathrm{vdim}\,\mathcal{M}^{\rm ss}_{(r,\alpha,k)}(\tau)_{\rm fd}}\Bigr]$$

$$\left(\sum_{\substack{\beta_1,\ldots,\beta_{r-1}\\ \in H^2(X,\mathbb{Z})^{1,1}:\\ \mathfrak{s}_{\beta_a}\in\mathrm{SW}(X),\\ a=1,\ldots,r-1}}\begin{array}{l} r^2\cdot\rho_r^{\int_X\mathrm{td}_2(X)}\cdot\eta_r^{\int_X c_1(X)^2}\cdot\prod_{1\leqslant a\leqslant b\leqslant r-1}\zeta_{r,ab}^{\int_X\beta_a\cup\beta_b}\cdot\\ \phi_r^{\int_X\alpha\cup c_1(X)}\cdot\prod_{a=1}^{r-1}\bigl(\mathrm{SW}([\mathfrak{s}_{\beta_a}])\theta_{r,a}^{\int_X\alpha\cup\beta_a}\bigr)\cdot\\ \exp\Bigl[\int_X A_r(\beta_1,\ldots,\beta_{r-1},c_1(X),\mathrm{td}_2(X),q,\boldsymbol{r})\Bigr]\end{array}\right).$$

This has an obvious symmetry group S_{r-1} by permutation of $\beta_1,\ldots,\beta_{r-1}$. Less obvious, if β is a Seiberg–Witten class then so is $-c_1(X)-\beta$, with $\mathrm{SW}([\mathfrak{s}_{-c_1(X)-\beta}])=(-1)^{\int_X\mathrm{td}_2(X)}\,\mathrm{SW}([\mathfrak{s}_\beta])$. So replacing β_a by $-c_1(X)-\beta_a$, and ρ_r by $-\rho_r$, gives a $\mathbb{Z}_2$-symmetry for $a=1,\ldots,r-1$. This gives a symmetry group $\Gamma_r=S_{r-1}\ltimes\mathbb{Z}_2^{r-1}$ acting on choices of $\rho_r,\eta_r,\phi_r,\theta_{r,a},\zeta_{r,ab},A_r$.

Symmetries of the generating function

(a) It turns out that the data $\rho_r, \eta_r, \phi_r, \theta_{r,a}, \zeta_{r,ab}, A_r$ is unique up to this action of $\Gamma_r = S_{r-1} \ltimes \mathbb{Z}_2^{r-1}$. We can conjugate everything by an element of the Galois group $\mathrm{Gal}(\mathbb{F}_r)$; this is equivalent to the action of an element of Γ_r, giving a morphism $\mathrm{Gal}(\mathbb{F}_r) \to \Gamma_r$.
(b) We can use the Γ_r-action to standardize the constants $\rho_r, \eta_r, \phi_r, \theta_{r,a}, \zeta_{r,ab}$: after applying an element of Γ_r we can take

$$\rho_r = \tfrac{1}{r}, \quad \phi_r = 1, \quad \theta_{r,a} = e^{\frac{2\pi i a}{r}}, \quad a = 1, \ldots, r-1.$$

There are also conjectural values for $\eta_r, \zeta_{r,ab}$ due to Göttsche 2021, but I haven't proved these yet, except for small r.
(c) If r is odd then $\mathrm{vdim}\, \mathcal{M}^{\mathrm{ss}}_{(r,\alpha,k)}(\tau)_{\mathrm{fd}}$ is always even. Then all q^{odd} terms in the whole sum (4) are zero, even though individual terms in the sum can have nonzero q^{odd} terms.
(d) $\mathrm{vdim}\, \mathcal{M}^{\mathrm{ss}}_{(r,\alpha,k)}(\mu^\omega)_{\mathrm{fd}} \equiv \int_X \alpha \cup c_1(X) + \int_X \mathrm{td}_2(X) \mod 2$ if r is even. If $n \not\equiv \int_X \alpha \cup c_1(X) + \int_X \mathrm{td}_2(X) \mod 2$ then q^n terms in the whole sum (4) are zero.
(e) Parts (c),(d) give an extra $\mathbb{Z}_2$ symmetry of (4) under $q \mapsto -q$.

3.8. Sketch of the proof: rank 1 case

First I prove the rank 1 case, Theorem 2 on Hilbert schemes. Define $\mathrm{Hilb}(X,q)=\sum_{n\geqslant 0}q^n[\mathrm{Hilb}^n(X)]_{\mathrm{fund}}\in\mathbb{Q}[s_{jkl}][[q]]$. Using Ellingsrud–Göttsche–Lehn 2001 I show that

$$\mathrm{Hilb}(X,q)=1+q(\cdots),\tag{7}$$

$$\begin{aligned}&\frac{\partial}{\partial q}\mathrm{Hilb}(X,q)=\\ &\int_X \mathrm{Res}_z\Big\{z^{-1}\exp\Big[-\sum_{\substack{j,k,j',k',\\ l'>k'/2:\ l'\geqslant(k+k')/2}}\frac{z^{(k+k')/2-l'}}{(l'-(k+k')/2)!}\mu_{jk}^{j'k'}\epsilon_{jk}\boxtimes s_{j'k'l'}\Big]\\ &\circ\exp\Big[-z^2\,\epsilon_{14}\boxtimes q\frac{\partial}{\partial q}+\sum_{j,k,\ l>k/2}(l-1)!z^l\epsilon_{jk}\boxtimes\frac{\partial}{\partial s_{jkl}}\Big]\cdot\mathrm{Hilb}(X,q)\Big\},\end{aligned}\tag{8}$$

where $(\mu_{jk}^{j'k'})$ is the inverse Mukai pairing. Then I show that (5) is the unique solution to (7)–(8), where $H(c_1,c_2,\boldsymbol{u})$ is the solution to a p.d.e. derived from (7)–(8).

3.9. Constructing invariants by induction on rank

There is a method to compute invariants $[\mathcal{M}^{\rm ss}_{(r,\alpha,k)}(\tau)]_{\rm inv}$ by induction on the rank $r=1,2,\ldots$ starting from rank 1 data. This is due to Mochizuki 2009 in the algebraic case, and is the analogue of the construction of Donaldson invariants from Seiberg–Witten invariants. Fix a line bundle $L\to X$, and define an auxiliary abelian category $\mathcal{A}$ with objects (V,E,ϕ), where V is a finite-dimensional $\mathbb{C}$-vector space, $E\in\mathrm{coh}(X)$, and $\phi:V\otimes_{\mathbb{C}}L\to E$ is a morphism. Write the class of (E,V,ϕ) as $[\![E,V,\phi]\!]=((r,\alpha,k),d)$ where $[\![E]\!]=(r,\alpha,k)$ and $\dim_{\mathbb{C}}V=d$. Starting from τ on $\mathrm{coh}(X)$ we define a 1-parameter family of stability conditions $\acute\tau_t$ on $\mathcal{A}$ for $t\in[0,\infty)$. Thus we get semistable moduli stacks $\mathcal{M}^{\rm ss}_{((r,\alpha,k),d)}(\acute\tau_t)$ of objects in $\mathcal{A}$. My theory defines 'pair invariants' $[\mathcal{M}^{\rm ss}_{((r,\alpha,k),d)}(\acute\tau_t)]_{\rm inv}$ (at least when $r>0$ and $d=0,1$) satisfying a wall-crossing formula under change of stability condition $\acute\tau_t$.

It turns out that:

- When $d=0$, $\mathcal{M}^{\rm ss}_{((r,\alpha,k),0)}(\acute\tau_t)=\mathcal{M}^{\rm ss}_{(r,\alpha,k)}(\tau)$. Thus the sheaf invariants $[\mathcal{M}^{\rm ss}_{(r,\alpha,k)}(\tau)]_{\rm inv}$ are pair invariants with $d=0$.
- If $r=1$, $\mathcal{M}^{\rm ss}_{((1,\alpha,k),1)}(\acute\tau_t)$ is independent of t and may be written using Seiberg–Witten invariants and Hilbert schemes.
- If $r>1$, $d=1$ and $t\gg 0$ then $\mathcal{M}^{\rm ss}_{((r,\alpha,k),1)}(\acute\tau_t)=\emptyset$, so $[\mathcal{M}^{\rm ss}_{((r,\alpha,k),d)}(\acute\tau_t)]_{\rm inv}=0$. Thus wall-crossing from $t\gg 0$ to $t=0$ gives a WCF of the general form

  $[\mathcal{M}^{\rm ss}_{((r,\alpha,k),1)}(\acute\tau_0)]_{\rm inv}=$ sum of repeated Lie brackets of $[\mathcal{M}^{\rm ss}_{((1,\alpha',k'),1)}(\acute\tau_0)]_{\rm inv}$ and $[\mathcal{M}^{\rm ss}_{(r'',\alpha'',k'')}(\tau)]_{\rm inv}$ for $r''<r$,

  using the Lie bracket on $H_*(\mathcal{M}^{\rm pl}_{\mathcal{A}})$ from Lecture 1.
- If $L=\mathcal{O}_X(-N)$ for $N\gg 0$ we can recover $[\mathcal{M}^{\rm ss}_{(r,\alpha,k)}(\tau)]_{\rm inv}$ from $[\mathcal{M}^{\rm ss}_{((r,\alpha,k),1)}(\acute\tau_0)]_{\rm inv}$.
- By induction we may now compute $[\mathcal{M}^{\rm ss}_{(r,\alpha,k)}(\tau)]_{\rm inv}\Rightarrow[\mathcal{M}^{\rm ss}_{((r+1,\alpha,k),1)}(\acute\tau_0)]_{\rm inv}\Rightarrow[\mathcal{M}^{\rm ss}_{(r+1,\alpha,k)}(\tau)]_{\rm inv}\Rightarrow\cdots$.
- Thus, we can compute $[\mathcal{M}^{\rm ss}_{(r,\alpha,k)}(\tau)]_{\rm inv}$ for $r>1$ in terms of classes of $\mathrm{Hilb}^n(X)$, $\mathrm{Pic}^0(X)$ and Seiberg–Witten invariants.

As in Lecture 1, with $(\mathrm{N}_{jk}^{j'k'})$ the matrix of the symmetrized Mukai pairing, we may write the Lie bracket on $H_*(\mathcal{M}^{\mathrm{pl}}_{\mathrm{rk}>0})$ as

$$\begin{aligned}
&\bigl[e^\alpha u(s_{jkl}), e^\beta v(s'_{j'k'l'})\bigr]_{\mathrm{rk}>0} = \mathrm{Res}_z\Bigl[(-1)^{\zeta(\alpha,\beta)} z^{\zeta(\alpha,\beta)+\zeta(\beta,\alpha)}\cdot \\
&\Bigl\{\exp\Bigl(z\frac{\mathrm{rk}\,\beta}{\mathrm{rk}(\alpha+\beta)}\Bigl(\sum_{j,k,l} s_{jk(l+1)}\frac{\partial}{\partial s_{jkl}}\Bigr)\Bigr)\circ \\
&\exp\Bigl(-z\frac{\mathrm{rk}\,\alpha}{\mathrm{rk}(\alpha+\beta)}\Bigl(\sum_{j',k',l'} s'_{j'k'(l'+1)}\frac{\partial}{\partial s'_{j'k'l'}}\Bigr)\Bigr)\circ \\
&\exp\Bigl(-\sum_{\substack{j,k,j',k',\\ l\geqslant k/2,\; l'\geqslant k'/2}} (-1)^{l}(l+l'-(k+k')/2-1)!\, z^{(k+k')/2-l-l'}\cdot \\
&\qquad \mathrm{N}_{jk}^{j'k'}\frac{\partial^2}{\partial s_{jkl}\partial s'_{j'k'l'}}\Bigr)\bigl(e^\alpha u(s_{jkl})\cdot e^{\beta'} v(s'_{j'k'l'})\bigr)\Bigr\}\Bigl|_{s'_{jkl}=s_{jkl}}\Bigr],
\end{aligned} \tag{9}$$

where $\zeta(\alpha,\beta)=\mathrm{rank}\,\mathcal{E}xt^\bullet_{\alpha,\beta}$.

3.10. Changing the generating function to the right form

Equation (9) is a complicated mess. What this means in practice: if you suppose (4) holds in rank r, and you use this to compute the generating function of invariants in rank $r+1$ using the inductive method, computing the Lie brackets using (9), and you get to the end without dying, the result does not look like (4) in rank $r+1$. Instead, it gives you a really complicated residue in an extra formal variable z, which depends on the line bundle $L \to X$, even though the answer $[\mathcal{M}^{\rm ss}_{(r+1,\alpha,k)}(\tau)]_{\rm fd}$ is independent of L. Worse, you can't use one L for the whole generating function, L must be more and more negative as the power of q increases.
The most difficult part of the proof is to show this residue can actually be written in the form (4) for rank $r+1$.

To do this we change variables in the residue from z to another formal variable y. Then it turns out that there exists a smooth projective curve Σ, meromorphic functions $x_1, \ldots, x_r, y : \Sigma \to \mathbb{C} \cup \{\infty\}$, and points $\sigma_0, \sigma_\infty \in \Sigma$ with $y(\sigma_i) = i$, such that:

- The group Γ_{r+1} acts on Σ, and y is Γ_{r+1}-invariant and gives an isomorphism $\Sigma/\Gamma_{r+1} \cong \mathbb{C} \cup \{\infty\}$. Thus, any Γ_{r+1}-invariant meromorphic function on Σ is actually a rational function of $y \in \mathbb{C} \cup \{\infty\}$.
- Every part of the residue $\mathrm{Res}_y(y^{-1}W)$ which will define the generating function (4) in rank $r+1$ lifts to the curve Σ, as the Laurent expansion at $\sigma_\infty \in \Sigma$ of a $\mathbb{Q}$-rational function in $x_1, \ldots, x_r, y$, in the local coordinate y.
- The entire sum $y^{-1}W$ inside $\mathrm{Res}_y(y^{-1}W)$ is Γ_{r+1}-invariant, although the components are not. Thus, the entire sum is a rational function of $y \in \mathbb{C} \cup \{\infty\}$. It turns out to have a simple pole at $y = 0$, and no other poles in $\mathbb{C}$. Thus $\mathrm{Res}_y(y^{-1}W) = W|_{y=0}$, or equivalently, $W|_{\sigma_0}$.

- Thus, we are dealing with meromorphic functions on Σ, which are presented initially as formal Laurent series in y near $\sigma_\infty \in \Sigma$. We want instead to evaluate these meromorphic functions at $\sigma_0 \in \Sigma$, and this evaluation gives (4) and the data $\rho_{r+1}, \eta_{r+1}, \phi_{r+1}, \theta_{r+1,a}, \zeta_{r+1,ab}, A_{r+1}$.
- $y^{-1}(0)$ is a free Γ_{r+1}-orbit in Σ, and $\sigma_0 \in y^{-1}(0)$ is chosen arbitrarily. Different choices give different data $\rho_{r+1}, \ldots, A_{r+1}$, differing by the action of Γ_{r+1}.
- All terms in (4) come from $\mathbb{Q}$-rational functions in $x_1, \ldots, x_r, y$ in Σ. But when we evaluate these at $\sigma_0 \in \Sigma$, which is not a $\mathbb{Q}$-point for $r+1>2$, we get coefficients in $\mathbb{F}_{r+1}$.
- The curve Σ can be written completely explicitly, though in a complicated way. This enables me to compute $\mathbb{F}_{r+1}, \rho_{r+1}, \phi_{r+1}, \theta_{r+1,a}$ explicitly.