Derived symplectic geometry and categorification

Dominic Joyce, Oxford University

July 2014

Based on: arXiv:1304.4508, arXiv:1305.6302, arXiv:1211.3259, arXiv:1305.6428, arXiv:1312.0090, arXiv:1403.2403, arXiv:1404.1329, and work in progress.

Joint work with Oren Ben-Bassat, Chris Brav, Vittoria Bussi, Dennis Borisov, Delphine Dupont, Sven Meinhardt, and Balázs Szendrői. Funded by the EPSRC.

These slides available at
http://people.maths.ox.ac.uk/∼joyce/

Plan of talk:

1. Classical symplectic geometry

Let M be a smooth manifold. Then M has a tangent bundle and cotangent bundle T^*M. We have k-forms $\omega \in C^\infty(\Lambda^k T^*M)$, and the de Rham differential $\mathrm{d}_{dR} : C^\infty(\Lambda^k T^*M) \to C^\infty(\Lambda^{k+1} T^*M)$. A k-form ω is *closed* if $\mathrm{d}_{dR}\omega = 0$.
A 2-form ω on M is *nondegenerate* if $\omega\cdot : TM \to T^*M$ is an isomorphism. This is possible only if $\dim M = 2n$ for $n \geqslant 0$. A *symplectic structure* is a closed, nondenerate 2-form ω on M.
Symplectic geometry is the study of symplectic manifolds (M, ω).

A *Lagrangian* in (M, ω) is a submanifold $i : L \to M$ such that $\dim L = n$ and $i^*(\omega) = 0$.
By a deep, difficult construction, one can (under extra assumptions) construct a *Fukaya category* $D^b\mathcal{F}(M, \omega)$, a triangulated category whose objects are compact Lagrangians L in M, and whose morphisms $\mathrm{Hom}^*(L_1, L_2)$ are the Lagrangian Floer cohomology groups $HF^*(L_1, L_2)$. Kontsevich's *Homological Mirror Symmetry Conjecture* concerns equivalences of categories

$$D^b\mathcal{F}(M, \omega) \simeq D^b \operatorname{coh}(\check{M}, J),$$

where $(\check{M}, J)$ is a 'mirror' complex manifold.
We will discuss analogues of these ideas in derived algebraic geometry (not the HMS Conjecture though).

2. Derived algebraic geometry

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, e.g. $\mathbb{K} = \mathbb{C}$. Work in the context of Toën and Vezzosi's theory of *derived algebraic geometry* (*DAG*). This gives ∞-categories of *derived* $\mathbb{K}$*-schemes* $\mathbf{dSch}_{\mathbb{K}}$ and *derived stacks* $\mathbf{dSt}_{\mathbb{K}}$. In this talk, for simplicity, we are mostly interested in derived schemes.

This is a very technical subject. It is not easy to motivate DAG, or even to say properly what a derived scheme is, in an elementary talk. So I will lie a little bit.

What is a derived scheme?

$\mathbb{K}$-schemes in classical algebraic geometry are geometric spaces X which can be covered by Zariski open sets $Y \subseteq X$ with $Y \cong \operatorname{Spec} A$ for A a commutative $\mathbb{K}$-algebra. General $\mathbb{K}$-schemes are very singular, but *smooth* $\mathbb{K}$*-schemes* X are very like smooth manifolds over $\mathbb{K}$, many differential geometric ideas like cotangent bundles TX, T^*X work nicely for them.

Think of a derived $\mathbb{K}$-scheme $\mathbf{X}$ as a geometric space which can be covered by Zariski open sets $\mathbf{Y} \subseteq \mathbf{X}$ with $\mathbf{Y} \simeq \operatorname{Spec} A^\bullet$ for $A^\bullet = (A, \mathrm{d})$ a commutative differential graded algebra (cdga) over $\mathbb{K}$, in degrees $\leqslant 0$.

We require $\mathbf{X}$ to be *locally finitely presented*, that is, we can take the $A^\bullet$ to be finitely presented, a strong condition.

Why derived algebraic geometry?

One reason derived algebraic geometry can be a powerful tool, is the combination of two facts:

(A) Many algebro-geometric spaces one wants to study, such as moduli spaces of coherent sheaves, or complexes, or representations, etc., which in classical algebraic geometry may be very singular, also have an incarnation as (locally finitely presented) derived schemes (or derived stacks).

(B) Within the framework of DAG, one can treat (locally finitely presented) derived schemes or stacks very much like smooth, nonsingular objects (Kontsevich's 'hidden smoothness philosophy'). Some nice things work in the derived world, which do not work in the classical world.

Tangent and cotangent complexes

In going from classical to derived geometry, we always replace vector bundles, sheaves, representations, . . . , by *complexes* of vector bundles, ... A classical smooth $\mathbb{K}$-scheme X has a tangent bundle TX and dual cotangent bundle T^*X, which are vector bundles on X, of rank the dimension $\dim X \in \mathbb{N}$.

Similarly, a derived $\mathbb{K}$-scheme $\boldsymbol{X}$ has a *tangent complex* $\mathbb{T}_{\boldsymbol{X}}$ and a dual *cotangent complex* $\mathbb{L}_{\boldsymbol{X}}$, which are perfect complexes of coherent sheaves on $\boldsymbol{X}$, of rank the virtual dimension $\mathrm{vdim}\,\boldsymbol{X} \in \mathbb{Z}$.

An example of nice behaviour in the derived world

Here is an example of the 'hidden smoothness philosophy'. Suppose we have a Cartesian square of smooth $\mathbb{K}$-schemes (or indeed, smooth manifolds)

$$\begin{array}{ccc} W & \xrightarrow{\quad f \quad} & Y \\ \downarrow{\scriptstyle e} & & {\scriptstyle h}\downarrow \\ X & \xrightarrow{\quad g \quad} & Z, \end{array}$$

with g, h transverse. Then we have an exact sequence of vector bundles on W, which we can use to compute TW:

$$0 \rightarrow TW \xrightarrow{Te \oplus Tf} e^*(TX) \oplus f^*(TY) \xrightarrow{e^*(Tg) \oplus -f^*(Th)} (g \circ e)^*(TZ) \rightarrow 0.$$

Classical symplectic geometry
Derived algebraic geometry
PTVV's shifted symplectic geometry
A Darboux theorem for shifted symplectic schemes
Categorification using perverse sheaves

Similarly, if we have a homotopy Cartesian square of derived $\mathbb{K}$-schemes

$$\begin{array}{ccc} \mathbf{W} & \xrightarrow{\quad \mathbf{f} \quad} & \mathbf{Y} \\ \downarrow{\scriptstyle \mathbf{e}} & & {\scriptstyle \mathbf{h}}\downarrow \\ \mathbf{X} & \xrightarrow{\quad \mathbf{g} \quad} & \mathbf{Z}, \end{array}$$

with no transversality, we have a distinguished triangle on $\mathbf{W}$

$$\mathbb{T}_{\mathbf{W}} \xrightarrow{\mathbb{T}_{\mathbf{e}} \oplus \mathbb{T}_{\mathbf{f}}} \mathbf{e}^*(\mathbb{T}_{\mathbf{X}}) \oplus \mathbf{f}^*(\mathbb{T}_{\mathbf{Y}}) \xrightarrow{\mathbf{e}^*(\mathbb{T}_{\mathbf{g}}) \oplus -\mathbf{f}^*(\mathbb{T}_{\mathbf{h}})} (\mathbf{g} \circ \mathbf{e})^*(\mathbb{T}_{\mathbf{Z}}) \rightarrow \mathbb{T}_{\mathbf{W}}[+1],$$

which we can use to compute $\mathbb{T}_{\mathbf{W}}$. This is false for classical schemes. So, derived schemes with arbitrary morphisms, have good behaviour analogous to smooth classical schemes with transverse morphisms, and are better behaved than classical schemes.

3. PTVV's shifted symplectic geometry

Pantev, Toën, Vaquié and Vezzosi (arXiv:1111.3209) defined a version of symplectic geometry in the derived world.
Let $\mathbf{X}$ be a derived $\mathbb{K}$-scheme. The cotangent complex $\mathbb{L}_{\mathbf{X}}$ has exterior powers $\Lambda^p\mathbb{L}_{\mathbf{X}}$. The *de Rham differential* $\mathrm{d}_{dR} : \Lambda^p\mathbb{L}_{\mathbf{X}} \to \Lambda^{p+1}\mathbb{L}_{\mathbf{X}}$ is a morphism of complexes. Each $\Lambda^p\mathbb{L}_{\mathbf{X}}$ is a complex, so has an internal differential $\mathrm{d} : (\Lambda^p\mathbb{L}_{\mathbf{X}})^k \to (\Lambda^p\mathbb{L}_{\mathbf{X}})^{k+1}$. We have $\mathrm{d}^2 = \mathrm{d}_{dR}^2 = \mathrm{d}\circ\mathrm{d}_{dR} + \mathrm{d}_{dR}\circ\mathrm{d} = 0$.
A *p-form of degree k* on $\mathbf{X}$ for $k \in \mathbb{Z}$ is an element $[\omega^0]$ of $H^k\bigl(\Lambda^p\mathbb{L}_{\mathbf{X}}, \mathrm{d}\bigr)$. A *closed p-form of degree k* on $\mathbf{X}$ is an element

$$[(\omega^0, \omega^1, \dots)] \in H^k\bigl(\textstyle\bigoplus_{i=0}^{\infty} \Lambda^{p+i}\mathbb{L}_{\mathbf{X}}[i], \mathrm{d}+\mathrm{d}_{dR}\bigr).$$

There is a projection $\pi : [(\omega^0, \omega^1, \dots)] \mapsto [\omega^0]$ from closed p-forms $[(\omega^0, \omega^1, \dots)]$ of degree k to p-forms $[\omega^0]$ of degree k.

Nondegenerate 2-forms and symplectic structures

Let $[\omega^0]$ be a 2-form of degree k on $\mathbf{X}$. Then $[\omega^0]$ induces a morphism $\omega^0 : \mathbb{T}_{\mathbf{X}} \to \mathbb{L}_{\mathbf{X}}[k]$, where $\mathbb{T}_{\mathbf{X}} = \mathbb{L}_{\mathbf{X}}^\vee$ is the tangent complex of $\mathbf{X}$. We call $[\omega^0]$ *nondegenerate* if $\omega^0 : \mathbb{T}_{\mathbf{X}} \to \mathbb{L}_{\mathbf{X}}[k]$ is a quasi-isomorphism.
If $\mathbf{X}$ is a derived scheme then the complex $\mathbb{L}_{\mathbf{X}}$ lives in degrees $(-\infty, 0]$ and $\mathbb{T}_{\mathbf{X}}$ in degrees $[0, \infty)$. So $\omega^0 : \mathbb{T}_{\mathbf{X}} \to \mathbb{L}_{\mathbf{X}}[k]$ can be a quasi-isomorphism only if $k \leqslant 0$, and then $\mathbb{L}_{\mathbf{X}}$ lives in degrees $[k, 0]$ and $\mathbb{T}_{\mathbf{X}}$ in degrees $[0, -k]$. If $k = 0$ then $\mathbf{X}$ is a smooth classical $\mathbb{K}$-scheme, and if $k = -1$ then $\mathbf{X}$ is quasi-smooth.
A closed 2-form $\omega = [(\omega^0, \omega^1, \dots)]$ of degree k on $\mathbf{X}$ is called a *k-shifted symplectic structure* if $[\omega^0] = \pi(\omega)$ is nondegenerate.

Although the details are complex, PTVV are following a simple recipe for translating some piece of geometry from smooth manifolds/smooth classical schemes to derived schemes:

(i) replace manifolds/smooth schemes X by derived schemes $\mathbf{X}$.

(ii) replace vector bundles $TX, T^*X, \Lambda^p T^*X, \ldots$ by complexes $\mathbb{T}_{\mathbf{X}}, \mathbb{L}_{\mathbf{X}}, \Lambda^p\mathbb{L}_{\mathbf{X}}, \ldots$.

(iii) replace sections of $TX, T^*X, \Lambda^p T^*X, \ldots$ by cohomology classes of the complexes $\mathbb{T}_{\mathbf{X}}, \mathbb{L}_{\mathbf{X}}, \Lambda^p\mathbb{L}_{\mathbf{X}}, \ldots$, in degree $k \in \mathbb{Z}$.

(iv) replace isomorphisms of vector bundles by quasi-isomorphisms of complexes.

Note that in (iii), we can specify the degree $k \in \mathbb{Z}$ of the cohomology class (e.g. $[\omega] \in H^k(\Lambda^p\mathbb{L}_{\mathbf{X}})$), which doesn't happen at the classical level.

Calabi–Yau moduli schemes and moduli stacks

PTVV prove that if Y is a Calabi–Yau m-fold over $\mathbb{K}$ and $\boldsymbol{\mathcal{M}}$ is a derived moduli scheme or stack of (complexes of) coherent sheaves on Y, then $\boldsymbol{\mathcal{M}}$ has a $(2-m)$-shifted symplectic structure ω.

This suggests applications — lots of interesting geometry concerns Calabi–Yau moduli schemes, e.g. Donaldson–Thomas theory.

We can understand the associated nondegenerate 2-form $[\omega^0]$ in terms of *Serre duality*. At a point $[E] \in \boldsymbol{\mathcal{M}}$, we have $h^i(\mathbb{T}_{\boldsymbol{\mathcal{M}}})|_{[E]} \cong \mathrm{Ext}^{i-1}(E,E)$ and $h^i(\mathbb{L}_{\boldsymbol{\mathcal{M}}})|_{[E]} \cong \mathrm{Ext}^{1-i}(E,E)^*$.

The Calabi–Yau condition gives $\mathrm{Ext}^i(E,E) \cong \mathrm{Ext}^{m-i}(E,E)^*$, which corresponds to $h^{i+1}(\mathbb{T}_{\boldsymbol{\mathcal{M}}})|_{[E]} \cong h^{i+1}(\mathbb{L}_{\boldsymbol{\mathcal{M}}}[2-m])|_{[E]}$. This is the cohomology at $[E]$ of the quasi-isomorphism $\omega^0 : \mathbb{T}_{\boldsymbol{\mathcal{M}}} \to \mathbb{L}_{\boldsymbol{\mathcal{M}}}[2-m]$.

Lagrangians and Lagrangian intersections

Let $(\mathbf{X},\omega)$ be a k-shifted symplectic derived scheme or stack. Then Pantev et al. define a notion of *Lagrangian* $\mathbf{L}$ in $(\mathbf{X},\omega)$, which is a morphism $\mathbf{i}:\mathbf{L}\to\mathbf{X}$ of derived schemes or stacks together with a homotopy $i^*(\omega)\sim 0$ satisfying a nondegeneracy condition, implying that $\mathbb{T}_{\mathbf{L}}\simeq\mathbb{L}_{\mathbf{L}/\mathbf{X}}[k-1]$.
If $\mathbf{L},\mathbf{M}$ are Lagrangians in $(\mathbf{X},\omega)$, then the fibre product $\mathbf{L}\times_{\mathbf{X}}\mathbf{M}$ has a natural $(k-1)$-shifted symplectic structure.
If (S,ω) is a classical smooth symplectic scheme, then it is a 0-shifted symplectic derived scheme in the sense of PTVV, and if $L,M\subset S$ are classical smooth Lagrangian subschemes, then they are Lagrangians in the sense of PTVV. Therefore the (derived) Lagrangian intersection $L\cap M=L\times_S M$ is a -1-shifted symplectic derived scheme.

Summary of the story so far

- Derived schemes behave better than classical schemes in some ways – they are analogous to smooth schemes, or manifolds. So, we can extend stories in smooth geometry to derived schemes. This introduces an extra degree $k\in\mathbb{Z}$.
- PTVV define a version of ('k-shifted') symplectic geometry for derived schemes. This is a new geometric structure.
- 0-shifted symplectic derived schemes are just classical smooth symplectic schemes.
- Calabi–Yau m-fold moduli schemes are $(2-m)$-shifted symplectic.
- One can go from k-shifted symplectic to $(k-1)$-shifted symplectic by taking intersections of Lagrangians.

4. A Darboux theorem for shifted symplectic schemes

Theorem (Brav, Bussi and Joyce arXiv:1305.6302)

Suppose $(\mathbf{X},\omega)$ is a k-shifted symplectic derived $\mathbb{K}$-scheme for $k<0$. If $k\not\equiv 2 \mod 4$, then each $x\in\mathbf{X}$ admits a Zariski open neighbourhood $\mathbf{Y}\subseteq\mathbf{X}$ with $\mathbf{Y}\simeq\operatorname{Spec}(A,\mathrm{d})$ for (A,d) an explicit cdga generated by graded variables x_j^{-i},y_j^{k+i} for $0\leqslant i\leqslant -k/2$, and $\omega|_{\mathbf{Y}}=[(\omega^0,0,0,\ldots)]$ where x_j^l,y_j^l have degree l, and

$$\omega^0=\textstyle\sum_{i=0}^{[-k/2]}\sum_{j=1}^{m_i}\mathrm{d}_{dR}y_j^{k+i}\mathrm{d}_{dR}x_j^{-i}.$$

Also the differential d in (A,d) is given by Poisson bracket with a Hamiltonian H in A of degree $k+1$.

If $k\equiv 2 \mod 4$, we have two statements, one étale local with ω^0 standard, and one Zariski local with the components of ω^0 in the degree $k/2$ variables depending on some invertible functions.

Sketch of the proof of the theorem

Suppose $(\mathbf{X},\omega)$ is a k-shifted symplectic derived $\mathbb{K}$-scheme for $k<0$, and $x\in\mathbf{X}$. Then $\mathbb{L}_{\mathbf{X}}$ lives in degrees $[k,0]$. We first show that we can build Zariski open $x\in\mathbf{Y}\subseteq\mathbf{X}$ with $\mathbf{Y}\simeq\operatorname{Spec}(A,\mathrm{d})$, for $A=\bigoplus_{i\leqslant 0}A^i,\mathrm{d}$ a cdga over $\mathbb{K}$ with A^0 a smooth $\mathbb{K}$-algebra, and such that A is freely generated over A^0 by graded variables x_j^{-i},y_j^{k+i} in degrees $-1,-2,\ldots,k$. We take $\dim A^0$ and the number of x_j^{-i},y_j^{k+i} to be minimal at x.
Using theorems about periodic cyclic cohomology, we show that on $Y\simeq\operatorname{Spec}(A,\mathrm{d})$ we can write $\omega|_Y=[(\omega^0,0,0,\ldots)]$, for ω^0 a 2-form of degree k with $\mathrm{d}\omega^0=\mathrm{d}_{dR}\omega^0=0$. Minimality at x implies ω^0 is strictly nondegenerate near x, so we can change variables to write $\omega^0=\sum_{i,j}\mathrm{d}_{dR}y_j^{k+i}\mathrm{d}_{dR}x_j^{-i}$. Finally, we show d in (A,d) is a symplectic vector field, which integrates to a Hamiltonian H.

The case of -1-shifted symplectic derived schemes

When $k=-1$ the Hamiltonian H in the theorem has degree 0. Then the theorem reduces to:

Corollary

Suppose $(\mathbf{X},\omega)$ is a -1-shifted symplectic derived $\mathbb{K}$-scheme. Then $(\mathbf{X},\omega)$ is Zariski locally equivalent to a derived critical locus $\mathbf{Crit}(H:U\to\mathbb{A}^1)$, for U a smooth classical $\mathbb{K}$-scheme and $H:U\to\mathbb{A}^1$ a regular function. Hence, the underlying classical $\mathbb{K}$-scheme $X=t_0(\mathbf{X})$ is Zariski locally isomorphic to a classical critical locus $\mathrm{Crit}(H:U\to\mathbb{A}^1)$.

Combining this with results of Pantev et al. from §3 gives interesting consequences in classical algebraic geometry:

Corollary

Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a classical moduli $\mathbb{K}$-scheme of coherent sheaves, or complexes of coherent sheaves, on Y. Then $\mathcal{M}$ is Zariski locally isomorphic to the critical locus $\mathrm{Crit}(H:U\to\mathbb{A}^1)$ of a regular function on a smooth $\mathbb{K}$-scheme.

Here we note that $\mathcal{M}=t_0(\boldsymbol{\mathcal{M}})$ for $\boldsymbol{\mathcal{M}}$ the corresponding derived moduli scheme, which is -1-shifted symplectic by PTVV.
A complex analytic analogue of this for moduli of coherent sheaves was proved using gauge theory by Joyce and Song arXiv:0810.5645, and for moduli of complexes was claimed by Behrend and Getzler. Note that the proof of the corollary is wholly algebro-geometric.

The case of -2-shifted symplectic derived schemes

Let $(\mathbf{X},\omega)$ be a -2-shifted symplectic derived $\mathbb{K}$-scheme. Then the Zariski local models for $(\mathbf{X},\omega)$ given by the 'Darboux Theorem' depend on the following data:

- A smooth $\mathbb{K}$-scheme U
- An algebraic vector bundle $E \to U$
- A section $s \in H^0(E)$
- A nondegenerate quadratic form Q on E with $Q(s,s)=0$.

The underlying classical $\mathbb{K}$-scheme X of $\mathbf{X}$ is locally $s^{-1}(0)\subset U$.
The virtual dimension of $\mathbf{X}$ is $\mathrm{vdim}_{\mathbb{K}}\mathbf{X} = 2\dim_{\mathbb{K}} U - \mathrm{rank}_{\mathbb{K}} E$.
The cotangent complex $\mathbb{L}_{\mathbf{X}}|_X$ of $\mathbf{X}$ is locally given by

$$\big[\underset{-2}{TU}|_{s^{-1}(0)} \xrightarrow{Q\circ \mathrm{d}s} \underset{-1}{E^*}|_{s^{-1}(0)} \xrightarrow{\mathrm{d}s} \underset{0}{T^*U}|_{s^{-1}(0)}\big].$$

Borisov–Joyce are using this to define Donaldson–Thomas style invariants 'counting' coherent sheaves on Calabi–Yau 4-folds.

What is PTVV shifted symplectic geometry good for?

We can apply all this to Calabi–Yau geometry. Combining PTVV theory and the BBJ Darboux Theorem gives new local models for Calabi–Yau m-fold moduli schemes and stacks — for instance, CY3 moduli schemes are Zariski locally isomorphic to critical loci. These local models are classical, and easy to understand. However, we still need derived geometry to tell us how overlapping local models should be glued together, which can be complicated.
We can use all this to build new structures on the moduli schemes, 'quantizing' or 'categorifying' them, defining 'motivic invariants', etc.
There are also applications to algebraic symplectic geometry. If L, M are Lagrangians in a smooth symplectic $\mathbb{K}$-scheme (S,ω) then PTVV say $L\cap M$ is -1-shifted symplectic. We can try to use our theory to define algebraic versions of Lagrangian Floer cohomology $HF^*(L,M)$ and the Fukaya category $D^b\mathcal{F}(S,\omega)$.

5. Categorification using perverse sheaves

Theorem (Brav, Bussi, Dupont, Joyce, Szendrői arXiv:1211.3259)

Let $(\mathbf{X},\omega)$ be a -1-shifted symplectic derived $\mathbb{K}$-scheme. Then the 'canonical bundle' $\det(\mathbb{L}_{\mathbf{X}})$ is a line bundle over the classical scheme $X=t_0(\mathbf{X})$. Suppose we are given a square root line bundle $\det(\mathbb{L}_{\mathbf{X}})^{1/2}$. Then we can construct a canonical perverse sheaf $P^\bullet_{\mathbf{X},\omega}$ on X, such that if $(\mathbf{X},\omega)$ is Zariski locally modelled on $\mathbf{Crit}(f:U\rightarrow\mathbb{A}^1)$, then $P^\bullet_{\mathbf{X},\omega}$ is locally modelled on the perverse sheaf of vanishing cycles $\mathcal{PV}^\bullet_{U,f}$ of (U,f).
Similarly, we can construct a natural $\mathscr{D}$-module $D^\bullet_{\mathbf{X},\omega}$ on X, and when $\mathbb{K}=\mathbb{C}$ a natural mixed Hodge module $M^\bullet_{\mathbf{X},\omega}$ on X.

Sketch of the proof of the theorem

Roughly, we prove the theorem by taking a Zariski open cover $\{\mathbf{R}_i:i\in I\}$ of $\mathbf{X}$ with $\mathbf{R}_i\cong\mathbf{Crit}(f_i:U_i\rightarrow\mathbb{A}^1)$, and showing that $\mathcal{PV}^\bullet_{U_i,f_i}$ and $\mathcal{PV}^\bullet_{U_j,f_j}$ are canonically isomorphic on $R_i\cap R_j$, so we can glue the $\mathcal{PV}^\bullet_{U_i,f_i}$ to get a global perverse sheaf $P^\bullet_{\mathbf{X},\omega}$ on X.
In fact things are more complicated: the (local) isomorphisms $\mathcal{PV}^\bullet_{U_i,f_i}\cong\mathcal{PV}^\bullet_{U_j,f_j}$ are only canonical *up to sign*. To make them canonical, we use the square root $\det(\mathbb{L}_{\mathbf{X}})^{1/2}$ to define natural principal $\mathbb{Z}_2$-bundles Q_i on R_i, such that $\mathcal{PV}^\bullet_{U_i,f_i}\otimes_{\mathbb{Z}_2}Q_i\cong\mathcal{PV}^\bullet_{U_j,f_j}\otimes_{\mathbb{Z}_2}Q_j$ is canonical, and then we glue the $\mathcal{PV}^\bullet_{U_i,f_i}\otimes_{\mathbb{Z}_2}Q_i$ to get $P^\bullet_{\mathbf{X},\omega}$.

Corollary

Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a classical moduli $\mathbb{K}$-scheme of coherent sheaves, or complexes of coherent sheaves, on Y, with (symmetric) obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$. Suppose we are given a square root $\det(\mathcal{E}^\bullet)^{1/2}$ for $\det(\mathcal{E}^\bullet)$ (i.e. **orientation data**, *K–S). Then we have a natural perverse sheaf $P^\bullet_{\mathcal{M},s}$ on $\mathcal{M}$.*

The *hypercohomology* $\mathbb{H}^*(P^\bullet_{\mathcal{M},s})$ is a finite-dimensional graded vector space. The pointwise Euler characteristic $\chi(P^\bullet_{\mathcal{M},s})$ is the *Behrend function* $\nu_{\mathcal{M}}$ of $\mathcal{M}$. Thus

$$\sum_{i\in\mathbb{Z}}(-1)^i \dim \mathbb{H}^i(P^\bullet_{\mathcal{M},s}) = \chi(\mathcal{M},\nu_{\mathcal{M}}).$$

Now by Behrend 2005, the Donaldson–Thomas invariant of $\mathcal{M}$ is $DT(\mathcal{M}) = \chi(\mathcal{M},\nu_{\mathcal{M}})$. So, $\mathbb{H}^*(P^\bullet_{\mathcal{M},s})$ is a graded vector space with dimension $DT(\mathcal{M})$, that is, a *categorification* of $DT(\mathcal{M})$.

Categorifying Lagrangian intersections

Corollary

Let (S,ω) be a classical smooth symplectic $\mathbb{K}$-scheme of dimension $2n$, and $L, M \subseteq S$ be smooth algebraic Lagrangians, with square roots $K_L^{1/2}, K_M^{1/2}$ of their canonical bundles. Then we have a natural perverse sheaf $P^\bullet_{L,M}$ on $X = L \cap M$.

This is related to Kashiwara and Schapira 2008, and Behrend and Fantechi 2009. We think of the hypercohomology $\mathbb{H}^*(P^\bullet_{L,M})$ as being morally related to the (undefined) *Lagrangian Floer cohomology* $HF^*(L,M)$ by $\mathbb{H}^i(P^\bullet_{L,M}) \approx HF^{i+n}(L,M)$.
We are working on defining 'Fukaya categories' for algebraic/complex symplectic manifolds using these ideas.

Algebraic structures on perverse sheaves

Let $(\mathbf{X},\omega)$ be a -1-shifted symplectic derived scheme, and $\mathbf{i}:\mathbf{L}\to\mathbf{X}$ a Lagrangian. Choose a square root $\det(\mathbb{L}_{\mathbf{X}})^{1/2}$. Then the previous theorem gives a perverse sheaf $P^\bullet_{\mathbf{X},\omega}$ on X.
The Lagrangian structure induces a natural isomorphism $\alpha:\mathcal{O}_L\xrightarrow{\cong}i^*(\det(\mathbb{L}_{\mathbf{X}}))$. Choose an isomorphism $\beta:\mathcal{O}_L\xrightarrow{\cong}i^*(\det(\mathbb{L}_{\mathbf{X}})^{1/2})$ with $\beta^2=\alpha$. We are working on:

Conjecture

With the choices above, there is a natural morphism in $D^b_c(\mathbf{L})$

$$\mu_{\mathbf{L}}:\mathbb{Q}_{\mathbf{L}}[\mathrm{vdim}\,\mathbf{L}]\longrightarrow i^!(P^\bullet_{\mathbf{X},\omega}),$$

with given local models in 'Darboux form' presentations for $\mathbf{X},\omega,\mathbf{L}$.

The conjecture has important consequences. For example, if L_1,L_2,L_3 are Lagrangians in (S,ω), then by applying the conjecture to

$$L_1\cap L_2\cap L_3\longrightarrow(L_1\cap L_2)\times(L_2\cap L_3)\times(L_3\cap L_1),$$

which is Lagrangian in -1-shifted symplectic, we get the morphism of perverse sheaves needed to build the multiplication map $HF^*(L_1,L_2)\times HF^*(L_2,L_3)\longrightarrow HF^*(L_1,L_3)$. The stack version of the conjecture would define the multiplication in a perverse sheaf 'Cohomological Hall Algebra' for Calabi–Yau 3-folds.
In fact the conjecture is only the first in a series of statements we need to prove associativity of multiplication, build the A_∞-structure in $D^b\mathcal{F}(S,\omega)$, and so on.