Universal structures in enumerative invariant theories

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These slides available at `http://people.maths.ox.ac.uk/~joyce/`.

1. Outline of the conjectural picture

An *enumerative invariant theory* in Algebraic or Differential Geometry is the study of invariants $I_\alpha(\tau)$ which 'count' τ-semistable objects E with fixed topological invariants $[\![E]\!]=\alpha$ in some geometric problem, usually by means of a virtual class $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ for the moduli space $\mathcal{M}_\alpha^{\rm ss}(\tau)$ of τ-semistable objects in some homology theory, with $I_\alpha(\tau)=\int_{[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}}\mu_\alpha$ for some natural cohomology class μ_α. We call the theory $\mathbb{C}$-*linear* if the objects E live in a $\mathbb{C}$-linear additive category $\mathcal{A}$. For example:

- Mochizuki-style invariants counting coherent sheaves on surfaces. (Think of as algebraic Donaldson invariants.)
- Donaldson–Thomas invariants of Calabi–Yau or Fano 3-folds.
- Donaldson–Thomas type invariants of Calabi–Yau 4-folds.
- $\mathrm{U}(m)$ Donaldson invariants of 4-manifolds (with $b^2_+=1$).

We conjecture that many such theories share a common universal structure. Here is an outline of this structure:

(a) We form two moduli stacks $\mathcal{M}, \mathcal{M}^{\rm pl}$ of all objects E in $\mathcal{A}$, where $\mathcal{M}$ is the usual moduli stack, and $\mathcal{M}^{\rm pl}$ the 'projective linear' moduli stack of objects E modulo 'projective isomorphisms', i.e. quotient by $\lambda\,{\rm id}_E$ for $\lambda \in \mathbb{G}_m$ or ${\rm U}(1)$.

(b) We are given a quotient $K_0(A) \twoheadrightarrow K(\mathcal{A})$, where $K(\mathcal{A})$ is the lattice of topological invariants $[\![E]\!]$ of E (e.g. fixed Chern classes). We split $\mathcal{M} = \bigoplus_{\alpha\in K(\mathcal{A})} \mathcal{M}_\alpha$, $\mathcal{M}^{\rm pl} = \bigoplus_{\alpha\in K(\mathcal{A})} \mathcal{M}^{\rm pl}_\alpha$.

(c) There is a symmetric biadditive *Euler form* $\chi : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$.

(d) We can form the homology $H_*(\mathcal{M}), H_*(\mathcal{M}^{\rm pl})$ over $\mathbb{Q}$, with $H_*(\mathcal{M}) = \bigoplus_{\alpha\in K(\mathcal{A})} H_*(\mathcal{M}_\alpha)$, $H_*(\mathcal{M}^{\rm pl}) = \bigoplus_{\alpha\in K(\mathcal{A})} H_*(\mathcal{M}^{\rm pl}_\alpha)$. Define shifted versions $\hat H_*(\mathcal{M}), \check H_*(\mathcal{M}^{\rm pl})$ by $\hat H_n(\mathcal{M}_\alpha) = H_{n-\chi(\alpha,\alpha)}(\mathcal{M}_\alpha)$, $\check H_n(\mathcal{M}^{\rm pl}_\alpha) = H_{n+2-\chi(\alpha,\alpha)}(\mathcal{M}^{\rm pl}_\alpha)$. Then previous work by me makes $\hat H_*(\mathcal{M})$ into a *graded vertex algebra*, and $\check H_*(\mathcal{M}^{\rm pl})$ into a *graded Lie algebra*.

(e) There is a notion of *stability condition* τ on $\mathcal{A}$. When $\mathcal{A} = {\rm coh}(X)$, this can be Gieseker stability for a polarization on X. For Donaldson theory for a compact oriented 4-manifold X with $b^2_+(X) = 1$, the stability condition is the splitting $H^2_{\rm dR}(X,\mathbb{R}) = H^2_+(X) \oplus H^2_-(X)$ induced by a metric g. For each $\alpha \in K(\mathcal{A})$ we can form moduli spaces $\mathcal{M}^{\rm st}_\alpha(\tau) \subseteq \mathcal{M}^{\rm ss}_\alpha(\tau)$ of τ-(semi)stable objects in class α. Here $\mathcal{M}^{\rm st}_\alpha(\tau)$ is a substack of $\mathcal{M}^{\rm pl}_\alpha$, and has the structure of a 'virtual oriented manifold' (in Algebraic Geometry, it may be a $\mathbb{C}$-scheme with perfect obstruction theory; in Differential Geometry, under genericness it may be an oriented manifold). Also $\mathcal{M}^{\rm ss}_\alpha(\tau)$ is compact (proper). Thus, if $\mathcal{M}^{\rm st}_\alpha(\tau) = \mathcal{M}^{\rm ss}_\alpha(\tau)$ we have a virtual class $[\mathcal{M}^{\rm ss}_\alpha(\tau)]_{\rm virt}$, which we regard as an element of $H_*(\mathcal{M}^{\rm pl}_\alpha)$. The virtual dimension is ${\rm vdim}_{\mathbb{R}}[\mathcal{M}^{\rm ss}_\alpha(\tau)]_{\rm virt} = 2 - \chi(\alpha,\alpha)$, so $[\mathcal{M}^{\rm ss}_\alpha(\tau)]_{\rm virt}$ lies in $\check H_0(\mathcal{M}^{\rm pl}_\alpha) \subset \check H_0(\mathcal{M}^{\rm pl})$, which is a Lie algebra by (b).

We can prove all of (a)–(e) already in the cases we care about.

Here is the conjectural part of the picture:

(f) For many theories, there is a problem defining the invariants $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ when $\mathcal{M}_\alpha^{\rm st}(\tau)\neq\mathcal{M}_\alpha^{\rm ss}(\tau)$, i.e. when the moduli spaces $\mathcal{M}_\alpha^{\rm ss}(\tau)$ contain *strictly τ-semistable points* (in gauge theory, these are *reducible connections*).
We conjecture there is a systematic way to define $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ in homology over $\mathbb{Q}$ (not $\mathbb{Z}$) in these cases. (In gauge theory, this requires a condition analogous to $b^2_+\geqslant 1$.)

(g) If $\tau,\tilde\tau$ are stability conditions and $\alpha\in K(\mathcal{A})$, we expect that

$$[\mathcal{M}_\alpha^{\rm ss}(\tilde\tau)]_{\rm virt}=\sum_{\alpha_1+\cdots+\alpha_n=\alpha}\tilde U(\alpha_1,\ldots,\alpha_n;\tau,\tilde\tau)\cdot\bigl[\bigl[\ldots\bigl[[\mathcal{M}_{\alpha_1}^{\rm ss}(\tau)]_{\rm virt},[\mathcal{M}_{\alpha_2}^{\rm ss}(\tau)]_{\rm virt}\bigr],\ldots\bigr],[\mathcal{M}_{\alpha_n}^{\rm ss}(\tau)]_{\rm virt}\bigr],\qquad(1)$$

where $\tilde U(-)$ are combinatorial coefficients defined in my previous work on wall-crossing formulae for motivic invariants, and $[\,,\,]$ is the Lie bracket on $\check H_0(\mathcal{M}^{\rm pl})$ from (b).

(h) We can often give an explicit, inductive definition of the $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ using (1) and the method of *pair invariants*.

We prove our conjectures completely when $\mathcal{A}=\text{mod-}\mathbb{C}Q$ is the category of representations of a quiver Q without oriented cycles, and stability conditions τ are slope stability. In this case, if $\mathcal{M}_\alpha^{\rm st}(\tau)=\mathcal{M}_\alpha^{\rm ss}(\tau)$ then $\mathcal{M}_\alpha^{\rm ss}(\tau)$ is a smooth projective $\mathbb{C}$-scheme (a compact complex manifold), given by a GIT quotient $\mathbb{A}^N/\!/_\tau\,{\rm PGL}_\alpha$, so it has a fundamental class $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm fund}$, and we set $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}=[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm fund}$. But we also define $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ if $\mathcal{M}_\alpha^{\rm st}(\tau)\neq\mathcal{M}_\alpha^{\rm ss}(\tau)$.
In a sequel by Bojko–Joyce–Upmeier, we will extend this to quivers with relations $\text{mod-}\mathbb{C}Q/I$, with Behrend–Fantechi virtual cycles when $\mathcal{M}_\alpha^{\rm st}(\tau)=\mathcal{M}_\alpha^{\rm ss}(\tau)$, and to 4-Calabi–Yau dg-quivers, with Borisov–Joyce virtual cycles. These are toy models for $\mathcal{A}=\text{coh}(X)$ when X is a curve, a surface, or a Calabi–Yau 4-fold.

Remarks on counting strictly τ-semistables

When $\mathcal{M}^{\rm st}_\alpha(\tau)=\mathcal{M}^{\rm ss}_\alpha(\tau)$, the virtual classes $[\mathcal{M}^{\rm ss}_\alpha(\tau)]_{\rm virt}$ are defined using a geometric structure on $\mathcal{M}^{\rm ss}_\alpha(\tau)$ (e.g. smooth $\mathbb{C}$-schemes, or $\mathbb{C}$-schemes with perfect obstruction theories, or -2-shifted symplectic derived schemes) by a known construction.

When $\mathcal{M}^{\rm st}_\alpha(\tau)\neq\mathcal{M}^{\rm ss}_\alpha(\tau)$, we currently have *no definition* of $[\mathcal{M}^{\rm ss}_\alpha(\tau)]_{\rm virt}$ in terms of a geometric structure on $\mathcal{M}^{\rm ss}_\alpha(\tau)$.

For quivers, our proof works by showing that there are unique $[\mathcal{M}^{\rm ss}_\alpha(\tau)]_{\rm virt}$ when $\mathcal{M}^{\rm st}_\alpha(\tau)\neq\mathcal{M}^{\rm ss}_\alpha(\tau)$, extending the given ones when $\mathcal{M}^{\rm st}_\alpha(\tau)=\mathcal{M}^{\rm ss}_\alpha(\tau)$, which also satisfy the wall-crossing formula (1). So the definition involves *all stability conditions*, not just one.

For Joyce–Song Donaldson–Thomas invariants, counting strictly τ-semistables is a complicated mess, and uses rational weights.

Motivic invariants versus homology

An invariant $I(X)$ of algebraic $\mathbb{K}$-varieties X in a commutative ring R is *motivic* if $I(X)=I(Y)+I(X\setminus Y)$ if $Y\subset X$ is closed subvariety, and $I(X\times Y)=I(X)I(Y)$. Examples are the Euler characteristic $\chi(X)$, with $R=\mathbb{Z}$, and virtual Poincaré polynomials.

Over 2003-8 I worked on invariants $I^{\rm ss}_\alpha(\tau)$ which 'counted' Algebro-Geometric moduli stacks $\mathcal{M}^{\rm st}_\alpha(\tau)\subseteq\mathcal{M}^{\rm ss}_\alpha(\tau)$ using motivic invariants, including wall-crossing formulae under change of stability condition. An important tool was Ringel–Hall algebras and Lie algebras of stack functions $\mathrm{SF}(\mathcal{M})$ on moduli spaces $\mathcal{M}$. (See 'Configurations in abelian categories I–IV', and Joyce–Song.)

Homology $H_*(\mathcal{M})$ and virtual classes $[\mathcal{M}^{\rm ss}_\alpha(\tau)]_{\rm virt}$ are *not motivic*, so this old work does not apply. But the new theory works by taking the old results on invariants and wall-crossing formulae in a Lie algebra of stack functions $\mathrm{SF}^{\rm ind}_{\rm al}(\mathcal{M})$, and replacing this by the Lie algebra $\check{H}_0(\mathcal{M}^{\rm pl})$ that comes out of my vertex algebra work.

Which invariant theories fit into our framework?

I expect the following to satisfy versions of our conjecture:

- Counting vector bundles or sheaves on projective curves X.
- Counting sheaves plus extra data (Higgs fields, ...) on curves.
- Counting sheaves on surfaces with $h^{2,0}(X)=0$, à la Mochizuki.
- Donaldson–Thomas invariants of Fano 3-folds.
- Donaldson–Thomas type invariants of Calabi–Yau 4-folds.
- $\mathrm{U}(m)$ Donaldson invariants of 4-manifolds with $b^2_+ = 1$.
- Quivers, quivers with relations, CY4 dg-quivers.

Donaldson–Thomas invariants of Calabi–Yau 3-folds are related, but don't fit the structure above exactly (the virtual dimension is not $2 - \chi(\alpha, \alpha)$). Similarly for Donaldson invariants with $b^2_+ > 1$, and surfaces with $h^{2,0}(X) > 0$. (Actually our theory works here (?), but the invariants are zero; fix determinants to make them nonzero.)

Interesting questions and future projects

- Do the invariants $[\mathcal{M}^{\mathrm{ss}}_\alpha(\tau)]_{\mathrm{virt}}$ in these theories have a common universal structure determined by a small amount of data? (Something like Seiberg–Witten $\Rightarrow$ Donaldson invariants, MNOP Conjecture, etc.)
- Does the vertex algebra structure relate to deep properties of enumerative invariants? (Modularity of generating functions, etc.)
- How should the picture be modified for theories like Donaldson theory for $b^2_+ > 1$, surfaces with $h^{2,0}(X) > 0$? (Now no wall-crossing, but counting strictly τ-semistables and pair invariants make sense, so we may have something to say.)
- Replace $H_*(-)$ by a complex oriented generalized homology theory $E_*(-)$? K-theory enumerative invariants already studied.
- Extension to triangulated categories, Bridgeland stability?
- Does our picture have an interpretation in String Theory?

2. More details

2.1. Vertex and Lie algebras on homology of moduli stacks

We will explain the Algebraic Geometry version of our theory. Let $\mathcal{A}$ be a $\mathbb{C}$-linear abelian or triangulated category from Algebraic Geometry or Representation Theory, e.g. $\mathcal{A} = \mathrm{coh}(X)$ or $D^b \mathrm{coh}(X)$ for X a smooth projective $\mathbb{C}$-scheme, or $\mathcal{A} = \text{mod-}\mathbb{C}Q$ or $D^b \text{mod-}\mathbb{C}Q$. Write $\mathcal{M}$ for the moduli stack of objects in $\mathcal{A}$, which is an Artin $\mathbb{C}$-stack in the abelian case, and a higher $\mathbb{C}$-stack in the triangulated case. There is a morphism $\Phi : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ acting by $([E],[F]) \to [E \oplus F]$ on $\mathbb{C}$-points.
Now $\mathbb{G}_m$ acts on objects E in $\mathcal{A}$ with $\lambda \in \mathbb{G}_m$ acting as $\lambda\,\mathrm{id}_E : E \to E$. This induces an action $\Psi : [*/\mathbb{G}_m] \times \mathcal{M} \to \mathcal{M}$ of the group stack $[*/\mathbb{G}_m]$ on $\mathcal{M}$. We write $\mathcal{M}^{\mathrm{pl}} = \mathcal{M}/[*/\mathbb{G}_m]$ for the quotient, called the 'projective linear' moduli stack. There is a morphism $\mathcal{M} \to \mathcal{M}^{\mathrm{pl}}$ which is a $[*/\mathbb{G}_m]$-fibration on $\mathcal{M} \setminus \{[0]\}$.

We need some extra data:

- A quotient $K_0(A) \twoheadrightarrow K(\mathcal{A})$ giving splittings $\mathcal{M} = \bigoplus_{\alpha \in K(\mathcal{A})} \mathcal{M}_\alpha$, $\mathcal{M}^{\mathrm{pl}} = \bigoplus_{\alpha \in K(\mathcal{A})} \mathcal{M}_\alpha^{\mathrm{pl}}$.
- A symmetric biadditive *Euler form* $\chi : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$.
- A perfect complex $\Theta^\bullet$ on $\mathcal{M} \times \mathcal{M}$ satisfying some assumptions, including $\mathrm{rank}\,\Theta|_{\mathcal{M}_\alpha \times \mathcal{M}_\beta} = \chi(\alpha, \beta)$.
  If $\mathcal{A}$ is a 4-Calabi–Yau category, and we will use Borisov–Joyce virtual classes, we take $\Theta^\bullet = (\mathcal{E}xt^\bullet)^\vee$, where $\mathcal{E}xt^\bullet \to \mathcal{M} \times \mathcal{M}$ is the *Ext complex*. Otherwise we take $\Theta^\bullet = (\mathcal{E}xt^\bullet)^\vee + \sigma^*(\mathcal{E}xt^\bullet)$, where $\sigma : \mathcal{M} \times \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ swaps the factors.
- Signs $\epsilon_{\alpha,\beta} \in \{\pm 1\}$ for $\alpha, \beta \in K(\mathcal{A})$ with $\epsilon_{\alpha,\beta} \cdot \epsilon_{\alpha+\beta,\gamma} = \epsilon_{\alpha,\beta+\gamma} \cdot \epsilon_{\beta,\gamma}$ and $\epsilon_{\alpha,\beta} \cdot \epsilon_{\beta,\alpha} = (-1)^{\chi(\alpha,\beta)+\chi(\alpha,\alpha)\chi(\beta,\beta)}$.
  (These compare orientations on $\mathcal{M}_\alpha, \mathcal{M}_\beta, \mathcal{M}_{\alpha+\beta}$.)

Then we can make the homology $H_*(\mathcal{M})$, with grading shifted to $\hat{H}_*(\mathcal{M})$ as above, into a *graded vertex algebra*.

Writing $H_*([*/\mathbb{G}_m])=\mathbb{Q}[t]$ with $\deg t=2$, the state-field correspondence $Y(z)$ is given by, for $u\in H_a(\mathcal{M}_\alpha)$, $v\in H_b(\mathcal{M}_\beta)$

$$Y(u,z)v=\epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)}z^{\chi(\alpha,\beta)}\cdot H_*\bigl(\Phi\circ(\Psi\times\mathrm{id})\bigr) \qquad (2)$$
$$\Bigl\{\bigl(\sum_{i\geqslant 0}z^it^i\bigr)\boxtimes\bigl[(u\boxtimes v)\cap\exp\bigl(\sum_{j\geqslant 1}(-1)^{j-1}(j-1)!z^{-j}\,\mathrm{ch}_j([\Theta^\bullet])\bigr)\bigr]\Bigr\}.$$

The identity $\mathbb{1}$ is $1\in H_0(\mathcal{M}_0)$. Define $e^{zD}:\check H_*(\mathcal{M})\to\check H_*(\mathcal{M})[[z]]$ by $Y(v,z)\mathbb{1}=e^{zD}v$. Then $(\check H_*(\mathcal{M}),\mathbb{1},e^{zD},Y)$ is a graded vertex algebra. By a standard construction in vertex algebra theory, $\check H_{*+2}(\mathcal{M})/D(\check H_*(\mathcal{M}))$ is a graded Lie algebra. In the abelian category case at least, there is a canonical isomorphism $\check H_*(\mathcal{M}^{\mathrm{pl}})\cong\check H_{*+2}(\mathcal{M})/D(\check H_*(\mathcal{M}))$. This makes $\check H_*(\mathcal{M}^{\mathrm{pl}})$ into a graded Lie algebra, and $\check H_0(\mathcal{M}^{\mathrm{pl}})$ into a Lie algebra.

Remarks

- One can often write down $\check H_*(\mathcal{M})$ and $\check H_*(\mathcal{M}^{\mathrm{pl}})$ with their algebraic structures explicitly. The answer is usually simpler in the derived category case. For example, Jacob Gross showed that if a smooth projective $\mathbb{C}$-scheme X is a curve, surface, or toric variety, and $\mathcal{M}$ is the moduli stack of $D^b\operatorname{coh}(X)$, then

$$\hat H_*(\mathcal{M},\mathbb{Q})\cong\mathbb{Q}[K^0_{\mathrm{sst}}(X)]\otimes_R\mathrm{Sym}^*\bigl(K^0(X^{\mathrm{an}})\otimes_{\mathbb{Z}}t^2\mathbb{Q}[t^2]\bigr)$$
$$\otimes_R\textstyle\bigwedge^*\bigl(K^1(X^{\mathrm{an}})\otimes_{\mathbb{Z}}t\mathbb{Q}[t^2]\bigr), \qquad (3)$$

with a super-lattice vertex algebra structure. Thus we can use this for explicit computations in examples, as well as for abstract theory.
- It helps to study $[\mathcal{M}^{\mathrm{ss}}_\alpha(\tau)]_{\mathrm{virt}}$ in $\operatorname{coh}(X)$ using $H_*(\mathcal{M})$, $H_*(\mathcal{M}^{\mathrm{pl}})$ for $D^b\operatorname{coh}(X)$, so we can use the presentation (3).
- Although Lie algebras are much simpler than vertex algebras, it is difficult to write down the Lie bracket on $\check H_*(\mathcal{M}^{\mathrm{pl}})$ explicitly: the best way seems to be via the vertex algebra structure on $\hat H_*(\mathcal{M})$.

2.2. Virtual classes of moduli spaces

The vertex and Lie algebras $\hat{H}_*(\mathcal{M})$, $\check{H}_*(\mathcal{M}^{\rm pl})$ above work for $\mathcal{M}$ the moduli stack of objects in $\mathrm{coh}(X)$ or $D^b\,\mathrm{coh}(X)$ for X a smooth projective $\mathbb{C}$-scheme of any dimension. However, defining virtual classes $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ when $\mathcal{M}_\alpha^{\rm st}(\tau)=\mathcal{M}_\alpha^{\rm ss}(\tau)$ is much more restrictive:

- If $\dim\mathcal{A}=1$, say if $\mathcal{A}=\text{mod-}\mathbb{C}Q$ or $\mathcal{A}=\mathrm{coh}(X)$ for X a curve, then $\mathcal{M}_\alpha^{\rm ss}(\tau)$ is a smooth projective $\mathbb{C}$-scheme, and has a fundamental class $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm fund}$.
- If $\dim\mathcal{A}=2$, say if $\mathcal{A}=\text{mod-}\mathbb{C}Q/I$ or $\mathcal{A}=\mathrm{coh}(X)$ for X a surface, then $\mathcal{M}_\alpha^{\rm ss}(\tau)$ is a projective $\mathbb{C}$-scheme with obstruction theory, and has a Behrend–Fantechi virtual class $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$.
- If $\mathcal{A}=\mathrm{coh}(X)$ for X a Calabi–Yau or Fano 3-fold, one can also define Behrend–Fantechi virtual classes $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$.
- If $\mathcal{A}=\mathrm{coh}(X)$ for X a Calabi–Yau 4-fold, Borisov–Joyce define a very different kind of virtual class $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$, with *half the expected dimension* of the Behrend–Fantechi class.

2.3 On moduli stacks and moduli schemes

There are two main ways of forming moduli spaces in Algebraic Geometry: as *schemes* or *stacks*. An important difference is that if $\mathcal{M}$ is a moduli stack of objects E, then automorphism groups are remembered in the isotropy groups of $\mathcal{M}$ by $\mathrm{Iso}_{\mathcal{M}}([E])=\mathrm{Aut}(E)$, but moduli schemes forget automorphism groups.
Our moduli stacks $\mathcal{M},\mathcal{M}^{\rm pl}$ differ in that their isotropy groups are $\mathrm{Iso}_{\mathcal{M}}([E])=\mathrm{Aut}(E)$, but $\mathrm{Iso}_{\mathcal{M}^{\rm pl}}([E])=\mathrm{Aut}(E)/(\mathbb{G}_m\cdot\mathrm{id}_E)$.
If E is τ-stable then $\mathrm{Aut}(E)=\mathbb{G}_m\cdot\mathrm{id}_E$, so $\mathrm{Iso}_{\mathcal{M}^{\rm pl}}([E])=\{1\}$.
Because of this, the τ-stable moduli scheme $\mathcal{M}_\alpha^{\rm st}(\tau)$ is actually an *open substack* in $\mathcal{M}^{\rm pl}$ (but not $\mathcal{M}$). This makes $\mathcal{M}^{\rm pl}$ useful for us.
The τ-semistable moduli scheme $\mathcal{M}_\alpha^{\rm ss}(\tau)$ has the *good property* that it is usually compact (proper). But it has the *bad properties* that it does not map to $\mathcal{M}^{\rm pl}$ or $\mathcal{M}$, and the obstruction theory (or other nice structure) on $\mathcal{M}_\alpha^{\rm st}(\tau)$ does not extend to $\mathcal{M}_\alpha^{\rm ss}(\tau)$, so we cannot define a virtual class $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ unless $\mathcal{M}_\alpha^{\rm st}(\tau)=\mathcal{M}_\alpha^{\rm ss}(\tau)$.

3. Proof in the case of quivers

Let $Q=(Q_0,Q_1,h,t)$ be a quiver, with finite sets Q_0 of vertices and Q_1 of edges, and head and tail maps $h,t:Q_1\rightarrow Q_0$. Then we have a $\mathbb{C}$-linear abelian category $\text{mod-}\mathbb{C}Q$ of *representations* (V_v,ρ_e) of Q, comprising a finite-dimensional $\mathbb{C}$-vector space V_v for each $v\in Q_0$ and a linear map $\rho_e:V_{t(e)}\rightarrow V_{h(e)}$ for each $e\in Q_1$. The *dimension vector* of (V_v,ρ_e) is $\boldsymbol{d}\in\mathbb{N}^{Q_0}$, where $\boldsymbol{d}(v)=\dim V_v$. We can work out our theory very explicitly for $\mathcal{A}=\text{mod-}\mathbb{C}Q$. We take $K(\mathcal{A})=\mathbb{Z}^{Q_0}$. Then $\mathcal{M}=\coprod_{\boldsymbol{d}\in\mathbb{N}^{Q_0}}\mathcal{M}_{\boldsymbol{d}}$, $\mathcal{M}^{\rm pl}=\coprod_{\boldsymbol{d}\in\mathbb{N}^{Q_0}}\mathcal{M}_{\boldsymbol{d}}^{\rm pl}$, where $\mathcal{M}_{\boldsymbol{d}}=[R_{\boldsymbol{d}}/\mathrm{GL}_{\boldsymbol{d}}]$, $\mathcal{M}_{\boldsymbol{d}}^{\rm pl}=[R_{\boldsymbol{d}}/\mathrm{PGL}_{\boldsymbol{d}}]$ with

$$R_{\boldsymbol{d}}=\textstyle\prod_{e\in Q_1}\mathrm{Hom}(\mathbb{C}^{t(\boldsymbol{d}(e))},\mathbb{C}^{h(\boldsymbol{d}(e))}),\ \ \mathrm{GL}_{\boldsymbol{d}}=\prod_{v\in Q_0}\mathrm{GL}(\boldsymbol{d}(v)),$$

and $\mathrm{PGL}_{\boldsymbol{d}}=\mathrm{GL}_{\boldsymbol{d}}/\mathbb{G}_m$. Hence $H_*(\mathcal{M}_{\boldsymbol{d}})\cong H_*(B\,\mathrm{GL}_{\boldsymbol{d}})$ and $H_*(\mathcal{M}_{\boldsymbol{d}}^{\rm pl})\cong H_*(B\,\mathrm{PGL}_{\boldsymbol{d}})$, which we can write explicitly.

Slope stability conditions

Fix $\mu_v\in\mathbb{R}$ for all $v\in Q_0$. Define $\mu:\mathbb{N}^{Q_0}\setminus\{0\}\rightarrow\mathbb{R}$ by

$$\mu(\boldsymbol{d})=\bigl(\textstyle\sum_{v\in Q_0}\mu_v\boldsymbol{d}(v)\bigr)/\bigl(\sum_{v\in Q_0}\boldsymbol{d}(v)\bigr).$$

We call μ a *slope function*. An object $0\neq E\in\text{mod-}\mathbb{C}Q$ is called *μ-semistable* (or *μ-stable*) if whenever $0\neq E'\subsetneq E$ is a subobject we have $\mu(\mathbf{dim}E')\geqslant\mu(\mathbf{dim}E)$ (or $\mu(\mathbf{dim}E')>\mu(\mathbf{dim}E)$). Recall that $\mathcal{M}_{\boldsymbol{d}}^{\rm pl}=[R_{\boldsymbol{d}}/\mathrm{PGL}_{\boldsymbol{d}}]$ as a quotient stack. King (1994) showed that there is a linearization θ of the action of $\mathrm{PGL}_{\boldsymbol{d}}$ on $R_{\boldsymbol{d}}$, such that a $\mathbb{C}$-point $[E]\in[R_{\boldsymbol{d}}/\mathrm{PGL}_{\boldsymbol{d}}]$ is μ-(semi)stable in $\text{mod-}\mathbb{C}Q$ iff the corresponding point in $R_{\boldsymbol{d}}$ is GIT (semi)stable. Hence there are moduli schemes $\mathcal{M}_{\boldsymbol{d}}^{\rm st}(\mu)\subseteq\mathcal{M}_{\boldsymbol{d}}^{\rm ss}(\mu)$ which are the GIT (semi)stable quotients $R_{\boldsymbol{d}}/\!/_\theta^{\rm st}\mathrm{PGL}_{\boldsymbol{d}}\subseteq R_{\boldsymbol{d}}/\!/_\theta^{\rm ss}\mathrm{PGL}_{\boldsymbol{d}}$.
If Q has *no oriented cycles* then a $\mathbb{G}_m$ subgroup of $\mathrm{PGL}_{\boldsymbol{d}}$ acts on $R_{\boldsymbol{d}}$ with positive weights, so $\mathcal{M}_{\boldsymbol{d}}^{\rm ss}(\mu)=R_{\boldsymbol{d}}/\!/_\theta^{\rm ss}\mathrm{PGL}_{\boldsymbol{d}}$ is a projective $\mathbb{C}$-scheme. Also $\mathcal{M}_{\boldsymbol{d}}^{\rm st}(\mu)=R_{\boldsymbol{d}}/\!/_\theta^{\rm st}\mathrm{PGL}_{\boldsymbol{d}}$ is a smooth quasi-projective $\mathbb{C}$-scheme, an open substack of $\mathcal{M}_{\boldsymbol{d}}^{\rm pl}=[R_{\boldsymbol{d}}/\mathrm{PGL}_{\boldsymbol{d}}]$.

Thus, if Q has no oriented cycles, and μ is a slope function on $\text{mod-}\mathbb{C}Q$, and $\boldsymbol{d} \in \mathbb{N}^{Q_0} \setminus \{0\}$ with $\mathcal{M}^{\rm st}_{\boldsymbol{d}}(\mu) = \mathcal{M}^{\rm ss}_{\boldsymbol{d}}(\mu)$, then $\mathcal{M}^{\rm ss}_{\boldsymbol{d}}(\mu)$ is a smooth projective $\mathbb{C}$-scheme and an open substack of $\mathcal{M}^{\rm pl}_{\boldsymbol{d}}$, and has a fundamental class $[\mathcal{M}^{\rm ss}_{\boldsymbol{d}}(\mu)]_{\rm fund}$ in $H_*(\mathcal{M}^{\rm pl}_{\boldsymbol{d}})$. It has dimension $2 - \chi(\boldsymbol{d}, \boldsymbol{d})$, where $\chi : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \to \mathbb{Z}$ is

$$\chi(\boldsymbol{d}, \boldsymbol{e}) = 2 \sum_{v \in Q_0} \boldsymbol{d}(v)\boldsymbol{e}(v) - \sum_{e \in Q_1} \big(\boldsymbol{d}(h(e))\boldsymbol{e}(t(e)) + \boldsymbol{d}(t(e))\boldsymbol{e}(h(e))\big).$$

Theorem 1

Let Q be a quiver with no oriented cycles. Then for all slope functions μ on $\text{mod-}\mathbb{C}Q$ and $\boldsymbol{d} \in \mathbb{N}^{Q_0} \setminus \{0\}$, there exist unique classes $[\mathcal{M}^{\rm ss}_{\boldsymbol{d}}(\mu)]_{\rm virt} \in H_{2-\chi(\boldsymbol{d},\boldsymbol{d})}(\mathcal{M}^{\rm pl}_{\boldsymbol{d}}) = \check{H}_0(\mathcal{M}^{\rm pl}_{\boldsymbol{d}})$ such that:

(i) *If $\mathcal{M}^{\rm st}_{\boldsymbol{d}}(\mu) = \mathcal{M}^{\rm ss}_{\boldsymbol{d}}(\mu)$ then $[\mathcal{M}^{\rm ss}_{\boldsymbol{d}}(\mu)]_{\rm virt} = [\mathcal{M}^{\rm ss}_{\boldsymbol{d}}(\mu)]_{\rm fund}$.*

(ii) *The $[\mathcal{M}^{\rm ss}_{\boldsymbol{d}}(\mu)]_{\rm virt}$ transform according to the wall-crossing formula (1) above in the Lie algebra $\check{H}_0(\mathcal{M}^{\rm pl})$ under change of stability condition.*

We also prove:

Theorem 2

There is a notion of **morphism of quivers** *$\lambda : Q \to Q'$, which induces a functor $\lambda_* : \text{mod-}\mathbb{C}Q \to \text{mod-}\mathbb{C}Q'$, and morphisms of vertex algebras $\Omega : \hat{H}_*(\mathcal{M}) \to \hat{H}_*(\mathcal{M}')$ and of Lie algebras $\Omega^{\rm pl} : \check{H}_*(\mathcal{M}^{\rm pl}) \to \check{H}_*(\mathcal{M}'^{\rm pl})$. If μ' is a slope function on $\text{mod-}\mathbb{C}Q'$ then $\mu = \mu \circ \lambda_*$ is a slope function on $\text{mod-}\mathbb{C}Q$. Then for each $\boldsymbol{d} \in \mathbb{N}^{Q_0} \setminus \{0\}$ with $\lambda_*(\boldsymbol{d}) = \boldsymbol{d}' \in \mathbb{N}^{Q_0'} \setminus \{0\}$, the virtual classes $[\mathcal{M}^{\rm ss}_{\boldsymbol{d}}(\mu)]_{\rm virt}$ of Theorem 1 satisfy*

$$\textstyle\prod_{v \in Q_0} \boldsymbol{d}(v)! \cdot \Omega^{\rm pl}\big([\mathcal{M}^{\rm ss}_{\boldsymbol{d}}(\mu)]_{\rm virt}\big) = \prod_{v' \in Q_0'} \boldsymbol{d}'(v')! \cdot [\mathcal{M}'^{\rm ss}_{\boldsymbol{d}'}(\mu')]_{\rm virt}.$$

Sketch proof of Theorems 1 and 2

We call a slope function μ *decreasing* if for all edges $\overset{v}{\bullet} \xrightarrow{e} \overset{w}{\bullet}$ in Q we have $\mu_v > \mu_w$. Such μ exist if and only if Q has no oriented cycles. If μ is decreasing, for each $\boldsymbol{d} \in \mathbb{N}^{Q_0} \setminus \{0\}$, either:

(a) $\boldsymbol{d} = \delta_v$ for some $v \in Q_0$, that is, $\boldsymbol{d}(v) = 1$ and $\boldsymbol{d}(w) = 0$ for $w \neq v$. Then $\mathcal{M}^{\rm st}_{\boldsymbol{d}}(\mu) = \mathcal{M}^{\rm ss}_{\boldsymbol{d}}(\mu)$ is a single point $*$.

(b) $\boldsymbol{d} = n\delta_v$ for some $v \in Q_0$ and $n > 1$. Then $\mathcal{M}^{\rm st}_{\boldsymbol{d}}(\mu) = \emptyset$ and $\mathcal{M}^{\rm ss}_{\boldsymbol{d}}(\mu) \cong [*/\mathrm{PGL}(n,\mathbb{C})]$. Note that $2-\chi(\boldsymbol{d},\boldsymbol{d}) = 2-2n^2 < 0$.

(c) $\boldsymbol{d} \neq n\delta_v$ for any $v \in Q_0$, $n \geqslant 1$. Then $\mathcal{M}^{\rm st}_{\boldsymbol{d}}(\mu) = \mathcal{M}^{\rm ss}_{\boldsymbol{d}}(\mu) = \emptyset$.

Hence the classes $[\mathcal{M}^{\rm ss}_{\boldsymbol{d}}(\mu)]_{\rm virt}$ in Theorem 1 must be

$$[\mathcal{M}^{\rm ss}_{\boldsymbol{d}}(\mu)]_{\rm virt} = \begin{cases} 1 \in H_0(\mathcal{M}^{\rm pl}_{\boldsymbol{d}}) \cong R, & \boldsymbol{d} = \delta_v,\ v \in Q_0, \\ 0, & \text{otherwise,} \end{cases} \tag{4}$$

as in case (b) $[\mathcal{M}^{\rm ss}_{\boldsymbol{d}}(\mu)]_{\rm virt} \in H_{<0}(\mathcal{M}^{\rm pl}_{\boldsymbol{d}}) = 0$.

Equation (4) for some fixed decreasing μ, and the wall-crossing formula in Theorem 1(ii) from μ to $\dot\mu$, then determine unique classes $[\mathcal{M}^{\rm ss}_{\boldsymbol{d}}(\dot\mu)]_{\rm virt}$ for all slope functions $\dot\mu$ on $\text{mod-}\mathbb{C}Q$. We prove these satisfy Theorem 1(ii) for wall-crossing from $\dot\mu$ to $\ddot\mu$, for any two slope functions $\dot\mu, \ddot\mu$, by an associativity property of the wall-crossing formula proved in my 2003 work on motivic invariants.

So far we have constructed classes $[\mathcal{M}^{\rm ss}_{\boldsymbol{d}}(\mu)]_{\rm virt}$ as in Theorem 1, satisfying Theorem 1(ii), but we do not yet know they satisfy (i).

Next we prove these classes $[\mathcal{M}^{\rm ss}_{\boldsymbol{d}}(\mu)]_{\rm virt}$ satisfy Theorem 2, using the fact that since $\Omega^{\rm pl} : \check H_*(\mathcal{M}^{\rm pl}) \to \check H_*(\mathcal{M}'^{\rm pl})$ is a Lie algebra morphisms, it takes the wall-crossing formula (1) in $\check H_*(\mathcal{M}^{\rm pl})$ used to define $[\mathcal{M}^{\rm ss}_{\boldsymbol{d}}(\mu)]_{\rm virt}$ to an identity in $\check H_*(\mathcal{M}'^{\rm pl})$. The factors $\prod_v \boldsymbol{d}(v)!$, $\prod_{v'} \boldsymbol{d}'(v')!$ arise because of a combinatorial identity relating the number of different ways of splitting $\boldsymbol{d} = \boldsymbol{d}_1 + \cdots + \boldsymbol{d}_n$ in $\mathbb{N}^{Q_0} \setminus \{0\}$, and $\boldsymbol{d}' = \boldsymbol{d}'_1 + \cdots + \boldsymbol{d}'_n$ in $\mathbb{N}^{Q'_0} \setminus \{0\}$.

Finally we show the $[\mathcal{M}^{\rm ss}_{\boldsymbol d}(\mu)]_{\rm virt}$ satisfy Theorem 1(i). This is the most difficult part. If $\boldsymbol d(v)\in\{0,1\}$ and Q is a tree, we deduce the result using results of Joyce–Song on Donaldson–Thomas type invariants for quivers. Then we build up to progressively more general $Q,\boldsymbol d$ using Theorem 2 in different ways.

The methods we use to prove Theorem 1 are very special to quivers. We currently don't have nice ways to generalize them to cases such as $\mathcal{A}=\mathrm{coh}(X)$. But I believe the conjectures anyway.