Complex manifolds and Kähler Geometry

Lecture 13 of 16: Riemannian holonomy groups

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These slides available at
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13.4 Principal bundles and G-structures

13. Riemannian holonomy groups

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And now a word from our sponsor:
why not buy my book *'Riemannian Holonomy Groups and Calibrated Geometry'*, O.U.P. 2007, for all your friends for Christmas?

13.1. Parallel transport and holonomy groups

Let ∇^E be a connection on a vector bundle $E \to X$. Let $\gamma : [0,1] \to X$ be a smooth curve with $\gamma(0) = x$ and $\gamma(1) = y$. Then $\gamma^*(\nabla^E)$ is a connection on $\gamma^*(E) \to [0,1]$.
For each $e \in E_x$ there is a unique section s of $\gamma^*(E)$ with $s(0) = e$ and $\gamma^*(\nabla^E)s \equiv 0$. Define $P_\gamma(e) = s(1)$. Then $P_\gamma : E_x \to E_y$ is the *parallel transport map*.
Think of a connection ∇^E on $E \to X$ as identifying nearby fibres E_x, $E_{x'}$ for x, x' close together in X.

Parallel transport identifies the fibres of E all along a curve γ, so we can drag vectors along γ.

Holonomy groups

Let ∇^E be a connection on a vector bundle $E \to X$. Fix $x \in X$. Let $\gamma : [0,1] \to X$ be a piecewise-smooth loop based at x, so that $\gamma(0) = \gamma(1) = x$. Then P_γ is an invertible linear map $E_x \to E_x$. The *holonomy group* $\mathrm{Hol}_x(\nabla^E)$ of ∇^E is the set of parallel transports P_γ for all piecewise-smooth loops γ based at x.

Some properties of $\mathrm{Hol}_x(\nabla^E)$:

- It's a *Lie subgroup* of $\mathrm{GL}(E_x)$.
- Identify $E_x \cong \mathbb{R}^n$, so $\mathrm{Hol}_x(\nabla^E) \subseteq \mathrm{GL}(n,\mathbb{R})$. Then $\mathrm{Hol}_x(\nabla^E)$ is *independent of basepoint* $x \in X$, *up to conjugation in* $\mathrm{GL}(n,\mathbb{R})$.
- If X is simply-connected, then $\mathrm{Hol}_x(\nabla^E)$ is connected.
- Let $\mathfrak{hol}_x(\nabla^E)$ be the *Lie algebra* of $\mathrm{Hol}_x(\nabla^E)$. Then $R(\nabla^E)_x \in \mathfrak{hol}_x(\nabla^E) \otimes \Lambda^2 T^*X$ in $\mathrm{End}(E_x) \otimes \Lambda^2 T^*X$.

Now let ∇ be a connection on TX. It also acts on $\bigotimes^k TX \otimes \bigotimes^l T^*X$. A *constant tensor* S satisfies $\nabla S = 0$. If S is constant then S_x is invariant under the action of $\mathrm{Hol}_x(\nabla)$ on $\bigotimes^k T_xX \otimes \bigotimes^l T_x^*X$. Any S_x in $\bigotimes^k T_xX \otimes \bigotimes^l T_x^*X$ invariant under $\mathrm{Hol}_x(\nabla)$ extends to a unique constant tensor S on X by parallel transport. So *the constant tensors on* X *are determined by* $\mathrm{Hol}_x(\nabla)$.

13.2. Riemannian holonomy

Let g be a Riemannian metric on X, and $x \in X$. The *holonomy group* $\mathrm{Hol}_x(g)$ of g is the holonomy group $\mathrm{Hol}_x(\nabla)$ of its Levi-Civita connection. It is a closed Lie subgroup of $\mathrm{O}(n)$, which up to conjugation in $\mathrm{O}(n)$ is independent of basepoint x.

Riemannian holonomy groups have stronger properties than the general case.

Regard the Lie algebra $\mathfrak{hol}_x(g)$ as a vector subspace of $\Lambda^2 T_x^* X$. Using symmetries of R_{abcd}, eqns (1)-(3) of 11.1, we find that R_{abcd} lies in the vector subspace $S^2\,\mathfrak{hol}_x(g)$ in $\Lambda^2 T_x^* X \otimes \Lambda^2 T_x^* X$ at each $x \in X$. Thus, the holonomy group imposes strong restrictions on the curvature tensor R_{abcd} of g. These are the basis of the classification of Riemannian holonomy groups.

Reducible metrics

Let (X, g) and (Y, h) be Riemannian manifolds with $\dim X, \dim Y > 0$. The *product metric* $g \times h$ on $X \times Y$ is given by $g \times h|_{(x,y)} = g|_x + h|_y$ for $x \in X$ and $y \in Y$.

Proposition 13.1

The holonomy groups satisfy $\mathrm{Hol}(g \times h) = \mathrm{Hol}(g) \times \mathrm{Hol}(h)$.

We call (X, g) *irreducible* if it is not locally isometric to a Riemannian product.

Theorem 13.2

Let (X, g) *be an irreducible Riemannian n-manifold. Then the representation of* $\mathrm{Hol}(g)$ *on* $\mathbb{R}^n$ *is irreducible.*

Proof.

If $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^l$ for $\mathbb{R}^k, \mathbb{R}^l$ subrepresentations of $\mathrm{Hol}(g)$, can define a local isometry $X \cong Y \times Z$ with $\dim Y = k$, $\dim Z = l$, so X is reducible. $\square$

Symmetric spaces

Definition

A Riemannian manifold (X, g) is a *symmetric space* if for each $p \in X$ there is an isometry $s_p : X \to X$ with $s_p^2 = 1$ such that p is an isolated fixed point of s_p.

Let G be the group of isometries of (X, g) generated by $s_q \circ s_r$ for all $q, r \in X$. Then G is a connected Lie group and $X = G/H$ for some closed Lie subgroup H of G.

Symmetric spaces can be classified completely using Lie groups.

Definition

We call (X, g) *locally symmetric* if it is locally isometric to a symmetric space, and *nonsymmetric* otherwise.

Theorem 13.3

Let (X, g) have Levi-Civita connection ∇ and Riemann curvature R. Then (X, g) is locally symmetric if and only if $\nabla R = 0$.

13.3. Berger's classification of holonomy groups

Theorem 13.4 (Berger's Theorem, 1955)

Let X be a simply-connected n-manifold and g an irreducible, nonsymmetric Riemannian metric on X. Then either:

(i) $\mathrm{Hol}(g) = \mathrm{SO}(n)$;
(ii) $n = 2m$ *and* $\mathrm{Hol}(g) = \mathrm{U}(m)$;
(iii) $n = 2m$ *and* $\mathrm{Hol}(g) = \mathrm{SU}(m)$;
(iv) $n = 4m$ *and* $\mathrm{Hol}(g) = \mathrm{Sp}(m)$;
(v) $n = 4m$ *and* $\mathrm{Hol}(g) = \mathrm{Sp}(m)\,\mathrm{Sp}(1)$;
(vi) $n = 7$ *and* $\mathrm{Hol}(g) = G_2$; *or* (vii) $n = 8$ *and* $\mathrm{Hol}(g) = \mathrm{Spin}(7)$.

There are three assumptions in Berger's Theorem:

- As X is simply-connected, $\mathrm{Hol}(g)$ is connected.
- As g is irreducible, $\mathrm{Hol}(g)$ acts irreducibly on $\mathbb{R}^n$.
- As g is nonsymmetric, $\nabla R \not\equiv 0$.

Each excludes some possible holonomy groups. Without them, the list of holonomy groups would be much longer.

A sketch proof of Berger's Theorem

Let X be simply-connected and g irreducible and nonsymmetric, and let $H = \mathrm{Hol}(g)$. Then H is a closed, connected Lie subgroup of $\mathrm{SO}(n)$ acting irreducibly on $\mathbb{R}^n$.

Berger made a list of all such subgroups up to conjugation, and applied two tests to see if each could be a holonomy group. Berger's list are the groups passing both tests.

Berger's first test

Let R_{abcd} be the Riemann curvature of g, and $\mathfrak{h}$ the Lie algebra of H. Then $R_{abcd} \in S^2\mathfrak{h}$. Also, as in §11.1 we have

$$R_{abcd} + R_{adbc} + R_{acdb} = 0, \tag{13.1}$$

the first Bianchi identity. Let $\mathfrak{R}^H$ be the subspace of $S^2\mathfrak{h}$ satisfying (13.1). Now $\mathfrak{R}^H$ must be big enough to generate $\mathfrak{h}$. That is, a generic element of $\mathfrak{R}^H$ cannot lie in $S^2\mathfrak{g}$ for $\mathfrak{g} \subset \mathfrak{h}$ a proper Lie subalgebra. If $\mathfrak{R}^H$ is too small, H fails the first test.

Berger's second test

Now $\nabla_e R_{abcd}$ lies in $(\mathbb{R}^n)^* \otimes \mathfrak{R}^H$, and also as in §11.1 satisfies

$$\nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec} = 0, \qquad (13.2)$$

the second Bianchi identity. If these two requirements force $\nabla R = 0$, then g is locally symmetric. This excludes such H, the second test.

Inner product algebras

The four *inner product algebras* are

$\mathbb{R}$ — *real numbers*.

$\mathbb{C}$ — *complex numbers*.

$\mathbb{H}$ — *quaternions*.

$\mathbb{O}$ — *octonions*, or *Cayley numbers*.

They are real vector spaces with a multiplication '$\cdot$' and a norm '$|\,.\,|$' with $|a \cdot b| = |a||b|$.

Here $\mathbb{C}$ is not ordered, $\mathbb{H}$ is not commutative, and $\mathbb{O}$ is not associative. Also we have $\mathbb{C} \cong \mathbb{R}^2$, $\mathbb{H} \cong \mathbb{R}^4$ and $\mathbb{O} \cong \mathbb{R}^8$, with $\operatorname{Im} \mathbb{O} \cong \mathbb{R}^7$.

Understanding Berger's list

Group	**Acts on**
$SO(m)$	$\mathbb{R}^m$
$O(m)$	$\mathbb{R}^m$
$SU(m)$	$\mathbb{C}^m$
$U(m)$	$\mathbb{C}^m$
$Sp(m)$	$\mathbb{H}^m$
$Sp(m)Sp(1)$	$\mathbb{H}^m$
G_2	$\mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$
$Spin(7)$	$\mathbb{O} \cong \mathbb{R}^8$

Thus there are two holonomy groups for each of $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.

Remarks on Berger's list

(i) $\mathrm{SO}(n)$ is the holonomy group of generic metrics.
(ii) Metrics g with $\mathrm{Hol}(g) \subseteq \mathrm{U}(m)$ are *Kähler metrics*.
(iii) Metrics g with $\mathrm{Hol}(g) \subseteq \mathrm{SU}(m)$ are *Calabi–Yau metrics*. They are Ricci-flat and Kähler.
(iv) Metrics g with $\mathrm{Hol}(g) \subseteq \mathrm{Sp}(m)$ are called *hyperkähler metrics*. They are also Ricci-flat and Kähler.
(v) Metrics g with holonomy group $\mathrm{Sp}(m)\,\mathrm{Sp}(1)$ for $m \geqslant 2$ are called *quaternionic Kähler metrics*. They are Einstein, but not Kähler.
(vi) and (vii) G_2 and $\mathrm{Spin}(7)$ are the *exceptional holonomy groups*. Metrics with these holonomy groups are Ricci-flat.

13.4. Principal bundles and G-structures

Definition

Let X be a manifold and G a Lie group. A *principal bundle* over X with fibre G is a manifold P with a free (left) G-action and a smooth, surjective map $\pi : P \to X$ whose fibres are G-orbits, such that each $x \in X$ has an open neighbourhood $U \subseteq X$ with a diffeomorphism $\pi^{-1}(U) \cong U \times G$ identifying π and the G-action with the projection $U \times G \to U$ and G and G-action on $U \times G$.

Example 13.5

Let X be a manifold of dimension n. The *frame bundle* F of X is a principal bundle over X with fibre $\mathrm{GL}(n,\mathbb{R})$. The points of F are $(n+1)$-tuples $(x, e_1, \ldots, e_n)$, for $x \in X$ and $e_1, \ldots, e_n$ a basis for T_xX. We have $\pi : (x, e_1, \ldots, e_n) \mapsto x$, and $\mathrm{GL}(n,\mathbb{R})$ fixes x and acts on $e_1, \ldots, e_n$ by change of basis,
$A : (x, e_1, \ldots, e_n) \mapsto (x, \tilde{e}_1, \ldots, \tilde{e}_n)$, where $\tilde{e}_i = \sum_{j=1}^n A_{ij} e_j$.

Definition

Let X be a manifold, P a principal bundle over X with fibre G and projection $\pi : P \to X$, and H a Lie subgroup of G. A *principal subbundle* Q of P with fibre H is a submanifold Q of P closed under the action of H on P, such that the H-action on Q and the restriction $\pi|_Q : Q \to X$ make Q into a principal bundle over X with fibre H.

Let X be a manifold of dimension n, and G be a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$. A *$G$-structure* on X is a principal subbundle P of the frame bundle F of X with fibre G.

Example 13.6

Let (X, g) be a Riemannian manifold, and P be the subset of $(x, e_1, \ldots, e_n)$ in F with $e_1, \ldots, e_n$ an *orthonormal* basis for T_xX w.r.t. $g|_x$. All such bases are related by matrices in $\mathrm{O}(n)$, so P is an $\mathrm{O}(n)$-*structure*.

G-structures and holonomy groups

Let X be an n-manifold and ∇ a connection on TX. Fix $x \in X$ and a basis $(e_1, \ldots, e_n)$ for T_xX. This identifies $T_xX \cong \mathbb{R}^n$, so the holonomy group $\mathrm{Hol}_x(\nabla)$ lies in $\mathrm{GL}(T_xX) \cong \mathrm{GL}(n, \mathbb{R})$. Let G be a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$ containing $\mathrm{Hol}_x(\nabla)$. Define Q to be the set of $(y, f_1, \ldots, f_n)$ in the frame bundle F of X, such that if $\gamma : [0, 1] \to X$ is a smooth path with $\gamma(0) = x$, $\gamma(1) = y$, then there exists $g \in G$ with $(P_\gamma \circ g)e_i = f_i$ for $i = 1, \ldots, n$.
As $\mathrm{Hol}_x(\nabla) \subseteq G$ this is independent of choice of γ, and P is a *G-structure* on X. Thus, a connection ∇ on TX with holonomy in G induces a G-structure on X. Can take $G = \mathrm{Hol}_x(\nabla)$.
Let (X, g) be a Riemannian manifold with $\mathrm{Hol}(g) = H \subseteq \mathrm{O}(n) \subset \mathrm{GL}(n, \mathbb{R})$. Then X has a natural H-structure Q, which is a principal subbundle of the $\mathrm{O}(n)$-structure P constructed before.

There is a notion of *connection* on principal bundles. A (vector bundle) connection on TX is equivalent to a (principal bundle) connection on the frame bundle F.

A connection ∇ on TX or F has holonomy contained in G iff there exists a G-structure on X preserved by (closed under) ∇.

A G-structure Q is called *torsion-free* if there exists a torsion-free connection ∇ on TX preserving Q. If $G \subseteq \mathrm{O}(n)$ this ∇ is unique, and is the Levi-Civita connection of the Riemannian metric associated to Q. Studying torsion-free G-structures for $G \subseteq \mathrm{O}(n)$ is equivalent to studying metrics g with $\mathrm{Hol}(g) \subseteq G$.

Complex manifolds and Kähler Geometry

Lecture 14 of 16: The Kähler holonomy groups

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Plan of talk:

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14.1 Kähler geometry and Riemannian holonomy

14.2 Calabi–Yau manifolds

14.3 Hyperkähler manifolds

14.4 Calabi–Yau 2-folds

14.1. Kähler geometry and Riemannian holonomy

Let (X, g) be a Riemannian n-manifold, and ∇ the Levi-Civita connection of g. As in §13.1, the holonomy group $\mathrm{Hol}(g) \subseteq \mathrm{O}(n)$ measures the *constant tensors* under ∇. That is, there is a 1-1 correspondence between $S \in C^\infty(\bigotimes^k TX \otimes \bigotimes^l T^*X)$ with $\nabla S \equiv 0$, and $S_0 \in \bigotimes^k \mathbb{R}^n \otimes \bigotimes^l (\mathbb{R}^n)^*$ invariant under $\mathrm{Hol}(g)$.
Let (X, J, g) be a Kähler manifold, with Kähler form ω, and let ∇ be the Levi-Civita connection of g. Then as in §4.1 we have

$$\nabla g = \nabla J = \nabla \omega = 0.$$

So g, J, ω are constant tensors, and $\mathrm{Hol}(g) \subseteq \mathrm{O}(2n)$ preserves tensors g_0, J_0, ω_0 on $\mathbb{R}^{2n}$. Hence $\mathrm{Hol}(g) \subseteq \mathrm{U}(n)$, the unitary group, the subgroup of $\mathrm{GL}(2n, \mathbb{R})$ preserving g_0, J_0, ω_0. A metric g on a $2n$-manifold X is Kähler w.r.t. some complex structure J on X iff $\mathrm{Hol}(g) \subseteq \mathrm{U}(n)$.

In fact, the theory of Riemannian holonomy groups can be seen as a generalization of the theory of Kähler manifolds. Features such as decomposition of forms into (p, q)-forms and of de Rham cohomology groups into subspaces $H^{p,q}(X)$ work for other holonomy groups as well.

The *Kähler holonomy groups* are $\mathrm{U}(n)$ (Kähler metrics), $\mathrm{SU}(n)$ (Calabi–Yau metrics), and $\mathrm{Sp}(m)$ (hyperkähler metrics), where

$$\mathrm{Sp}(m) \subset \mathrm{SU}(2m) \subset \mathrm{U}(2m) \subset \mathrm{O}(4m).$$

They are the groups on Berger's list that are subgroups of $\mathrm{U}(n)$, and so are holonomy groups of Kähler metrics. Generic Kähler metrics have holonomy $\mathrm{U}(n)$. They occur in infinite-dimensional families. Kähler metrics with holonomy $\mathrm{SU}(n), \mathrm{Sp}(m)$ are special: they have extra constant tensors, and more structure. They occur in finite-dimensional families on compact manifolds.

14.2. Calabi–Yau manifolds

Metrics g on a $2n$-manifold X with $\mathrm{Hol}(g) \subseteq \mathrm{SU}(n)$ are called *Calabi–Yau metrics*. Here $\mathrm{SU}(n)$ is the subgroup of $A \in \mathrm{U}(n)$ with $\det_{\mathbb{C}} A = 1$. It is the subgroup of $\mathrm{GL}(2n, \mathbb{R})$ preserving the standard metric g_0, complex structure J_0, Kähler form ω_0, and holomorphic volume form $\Omega_0 = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n$ on $\mathbb{R}^{2n} \cong \mathbb{C}^n$.

Thus, we get constant tensors J, ω, Ω on X, where J is a complex structure and g is Kähler w.r.t. J with Kähler form ω, and a constant $(n, 0)$-form Ω.

This Ω is a nonvanishing holomorphic section of the canonical bundle K_X of (X, J), so it induces an isomorphism $K_X \cong \mathcal{O}_X$, which implies that $c_1(X) = 0$ in $H^2(X; \mathbb{Z})$. As K_X has a constant section, the connection on K_X is flat. So its curvature, the Ricci form ρ, is zero, and g is Ricci flat. Conversely, if (X, J, g) is Ricci flat then X has a cover $\pi : \tilde{X} \to X$ (a finite cover if X is compact) such that $\tilde{g} = \pi^*(g)$ has $\mathrm{Hol}(\tilde{g}) \subseteq \mathrm{SU}(n)$.

Definition

A *Calabi–Yau manifold*, or *Calabi–Yau n-fold*, is a compact Kähler manifold (X, J, g) with $\mathrm{Hol}(g) = \mathrm{SU}(n)$, where $n = \dim_{\mathbb{C}} X$.

This is not quite the same as the definition in §11.4: that was equivalent to $\mathrm{Hol}(g) \subseteq \mathrm{SU}(n)$, not $\mathrm{Hol}(g) = \mathrm{SU}(n)$. But this is better from the point of view of Riemannian holonomy, so we use it from now on.

Topological properties of Calabi–Yau manifolds

Lemma 14.1

Let (X, J, g) be a Calabi–Yau n-fold. Then

$$H^{0,0}(X) \cong H^{n,0}(X) \cong H^{0,n}(X) \cong H^{n,n}(X) \cong \mathbb{C},$$

and if $p \neq 0, n$ then

$$H^{p,0}(X) = H^{0,p}(X) = H^{p,n}(X) = H^{n,p}(X) = 0.$$

Proof.

Suppose $\alpha \in H^{p,0}(X)$, so that α is a holomorphic $(p, 0)$-form. Corollary 12.6 shows that $\nabla\alpha = 0$. But constant tensors are determined by the holonomy group of g, which is $\mathrm{SU}(n)$. The fixed subspace of $\mathrm{SU}(n)$ on $\Lambda^{p,0}(\mathbb{C}^n)^*$ is $\mathbb{C}$ if $p = 0, n$, and 0 otherwise. The rest follows from

$$\overline{H^{q,p}(X)} \cong H^{p,q}(X) \cong H^{n-p,n-q}(X)^*.$$ □

In particular, if (X, J, g) is a Calabi–Yau n-fold for $n > 2$ then (X, J) is a compact complex manifold admitting Kähler metrics, and $H^{2,0}(X) = 0$. So Corollary 9.10 (from the Kodaira Embedding Theorem) gives:

Corollary 14.2

(X, J, g) be a Calabi–Yau n-fold for $n > 2$. Then (X, J) is projective.

Therefore we can study Calabi–Yau n-folds for $n > 2$ using complex algebraic geometry.

Let (X, J, g) be a Calabi–Yau n-fold for $n > 1$. As X is compact and g is Ricci flat, Theorem 12.3 shows that X has a finite cover $\tilde{X}$ isometric to $T^k \times N$, where T^k has a flat metric and N is simply connected. But then $\mathrm{Hol}(g)$ is a finite extension of $\mathrm{Hol}(g_N)$. If $k > 0$ this contradicts $\mathrm{Hol}(g) = \mathrm{SU}(n)$. So $\tilde{X} = N$, giving:

Corollary 14.3

Let (X, J, g) be a Calabi–Yau n-fold for $n > 1$. Then $\pi_1(X)$ is finite.

If n is even we can improve this. Consider the elliptic operator

$$\bar{\partial} + \bar{\partial}^* : \bigoplus_{q \text{ even}} C^\infty(\Lambda^{0,q} X) \to \bigoplus_{q \text{ odd}} C^\infty(\Lambda^{0,q} X).$$

It has kernel $\bigoplus_{q \text{ even}} H^{0,q}(X)$ and cokernel $\bigoplus_{q \text{ odd}} H^{0,q}(X)$. Lemma 14.1 shows $H^{0,q}(X)$ is $\mathbb{C}$ if $q = 0, n$ and 0 otherwise. Hence $\mathrm{ind}(\bar{\partial} + \bar{\partial}^*) = 2$ if n is even, and 0 if n is odd.

Let $\tilde{X}$ be the universal cover of X, with $\pi : \tilde{X} \to X$. Then $\tilde{X}$ is also a Calabi–Yau n-fold, and π is a $k : 1$ cover, where $k = |\pi_1(X)|$. By properties of characteristic classes, the index of $\bar{\partial} + \bar{\partial}^*$ on $\tilde{X}$ is k times the index of $\bar{\partial} + \bar{\partial}^*$ on X, since both are given by curvature integrals. If n is even, both indices are two, which forces $k = 1$. Hence $\tilde{X} = X$, giving:

Proposition 14.4

Let (X, J, g) be a Calabi–Yau $2n$-fold. Then X is simply-connected.

When $n > 2$ is odd, Calabi–Yau n-folds can have nontrivial finite fundamental groups.

14.3. Hyperkähler manifolds

The *quaternions* are the $\mathbb{R}$-algebra $\mathbb{H} = \langle 1, i_1, i_2, i_3 \rangle_{\mathbb{R}}$, where

$$i_1 i_2 = -i_2 i_1 = i_3, \qquad i_2 i_3 = -i_3 i_2 = i_1,$$
$$i_3 i_1 = -i_1 i_3 = i_2, \qquad \text{and} \qquad i_1^2 = i_2^2 = i_3^2 = -1.$$

If $x = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$, define $\bar{x} = x_0 - x_1 i_1 - x_2 i_2 - x_3 i_3$, and $|x|^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2$. Then $\overline{(pq)} = \bar{q}\,\bar{p}$ and $|pq| = |p||q|$.

The Lie group $\mathrm{Sp}(m)$ is the group of $m \times m$ matrices A over $\mathbb{H}$ satisfying $A\bar{A}^T = I$. It acts on $\mathbb{H}^m = \mathbb{C}^{2m} = \mathbb{R}^{4m}$ preserving the metric g and complex structures J_1, J_2, J_3, induced by right multiplication of $\mathbb{H}^m$ by i_1, i_2, i_3. If $a_1^2 + a_2^2 + a_3^2 = 1$ then $a_1 J_1 + a_2 J_2 + a_3 J_3$ is also a complex structure on $\mathbb{R}^{4m}$ preserved by $\mathrm{Sp}(m)$, and g is Hermitian with respect to it.

If (X, g) is a Riemannian $4m$-manifold and $\mathrm{Hol}(g) \subseteq \mathrm{Sp}(m)$, there are constant complex structures J_1, J_2, J_3 on X such that $a_1 J_1 + a_2 J_2 + a_3 J_3$ is also a complex structure for $a_1^2 + a_2^2 + a_3^2 = 1$, and g is Kähler with respect to it. So g is Kähler in many different ways, and is called *hyperkähler*. There are also constant Kähler forms $\omega_1, \omega_2, \omega_3$ for J_1, J_2, J_3. As $\mathrm{Sp}(m) \subset \mathrm{SU}(2m)$, hyperkähler metrics are special examples of Calabi–Yau metrics, and are Ricci flat. We have $\mathrm{SU}(2) = \mathrm{Sp}(1)$.
Often we pick one complex structure J_1, and regard (X, J_1, g) as a Kähler manifold. Then $\omega_2 + i\omega_3$ is a (2,0)-form, which is constant, and so holomorphic. Thus $[\omega_2 + i\omega_3] \in H^{2,0}(X)$. The top power $(\omega_2 + i\omega_3)^m$ is a nonvanishing holomorphic section of K_X.
Many examples of noncompact hyperkähler manifolds are known, constructed explicitly by algebraic methods. But few compact hyperkähler manifolds are known.

To obtain (compact) hyperkähler manifolds we can try to construct the holomorphic data (X, J_1) and $\omega_2 + i\omega_3$ using complex algebraic geometry, and then get the metric g using the Calabi Conjecture. However, different constructions often yield deformation-equivalent hyperkähler manifolds. In dimension $4m$ for $m \geqslant 2$, two families of compact hyperkähler manifolds are known (Beauville), with $b^2 = 7$ and $b^2 = 20$. O'Grady found examples in dimension 12 with $b^2 = 8$, and dimension 20 with $b^2 \geqslant 24$. This is all the known examples.

Topological properties of hyperkähler manifolds

The fixed subspace of $\mathrm{Sp}(m)$ on $\Lambda^{p,0}(\mathbb{C}^{2m})^*$ is $\mathbb{C}$ if $p=2j$ for $j=0,\ldots,m$, spanned by $(\omega_2+i\omega_3)^j$, and is 0 otherwise. So the method of Lemma 14.1 gives:

Lemma 14.5

Let (X,J,g) be a compact Kähler $2m$-manifold, with $\mathrm{Hol}(g)=\mathrm{Sp}(m)$. Then

$$H^{2j,0}(X)\cong H^{0,2j}(X)\cong H^{2j,2m}(X)\cong H^{2m,2j}(X)\cong\mathbb{C}$$

for $j=0,\ldots,m$, and otherwise

$$H^{p,0}(X)=H^{0,p}(X)=H^{p,2m}(X)=H^{2m,p}(X)=0.$$

In contrast to Corollary 14.2, a compact hyperkähler manifold (X, J, g) has $H^{2,0}(X) = \mathbb{C}$, so we can't use Corollary 9.10 to deduce (X, J) is projective. For generic $a_1, a_2, a_3 \in \mathbb{R}$ with $a_1^2 + a_2^2 + a_3^2 = 1$, the complex structure $a_1 J_1 + a_2 J_2 + a_3 J_3$ is not projective; using lectures 7 and 9, one can show that the projective complex structures on X are of complex codimension 1 in the family of all hyperkähler complex structures.

As for Corollary 14.3 and Proposition 14.4, we can prove:

Proposition 14.6

Let (X, J, g) be a compact Kähler $2m$-manifold, with $\mathrm{Hol}(g) = \mathrm{Sp}(m)$. Then X is simply-connected.

14.4. Calabi–Yau 2-folds

When $n = 1$, $\mathrm{SU}(1) = \{1\}$, and any Calabi–Yau 1-fold is a torus T^2 with a flat metric g.

Calabi–Yau 2-folds have holonomy $\mathrm{SU}(2) = \mathrm{Sp}(1)$, so they are hyperkähler. This gives them special features. They are well understood, through Kodaira's classification of complex surfaces.

A $K3$ *surface* is a compact, complex surface (X, J) with $h^{1,0} = 0$ and K_X trivial. All Calabi–Yau 2-folds are $K3$ surfaces, and vice versa.

All $K3$ surfaces (X, J) are diffeomorphic, with $\pi_1(X) = \{1\}$, $b^2_+(X) = 3$, and $b^2_-(X) = 19$. The moduli space $\mathcal{M}_{K3}$ of $K3$ surfaces is a 20-dimensional complex space, described by the 'Torelli Theorems'. Some $K3$ surfaces are projective, and some are not. Each $K3$ surface (X, J) has a 20-dimensional family of Calabi–Yau metrics, so the family of Calabi–Yau 2-folds (X, J, g) is 60-dimensional.

The holonomy group $\mathrm{SU}(2) = \mathrm{Sp}(1)$ behaves more like the holonomy groups $\mathrm{Sp}(m)$ for $m > 1$ than like the groups $\mathrm{SU}(n)$ for $n > 2$.

Outlook on Calabi–Yau geometry

The geometry of Calabi–Yau n-folds, especially when $n = 3$, is a huge subject. Much of the impetus comes from String Theory in Theoretical Physics, which uses Calabi-Yau 3-folds as ingredients in their models of the universe. *Mirror Symmetry* is a circle of conjectures coming from String Theory, which relates 'mirror pairs' of Calabi–Yau 3-folds $X, \check{X}$ in a mysterious way. Broadly, the complex geometry of X is equivalent to the symplectic geometry of $\check{X}$, and vice versa.