

Complex manifolds and Kähler Geometry

Lecture 15 of 16: Introduction to moduli spaces

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Plan of talk:

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15. Introduction to moduli spaces

Suppose we want to understand some class \mathcal{C} of geometric objects X up to isomorphism (or some weaker kind of equivalence) — for instance, compact complex manifolds diffeomorphic to some fixed real manifold, holomorphic vector bundles on a fixed complex manifold, etc. A common approach is to try and define a *moduli space* \mathcal{M} of such objects X . As a set $\mathcal{M} = \{[X] : X \in \mathcal{C}\}$ is just the set of isomorphism classes $[X]$ of objects X that we want to classify.

However, usually we want \mathcal{M} to have more geometric structure than this. For example, \mathcal{M} could be a topological space, or a manifold (real or complex), or some kind of singular complex manifold in algebraic geometry – a variety or scheme – or something worse, e.g. an Artin stack. The rule is that the geometric structure on \mathcal{M} must reflect the behaviour of *families* of the objects X under study.

For example, suppose we have a notion of limits in \mathcal{C} , that is, when a sequence $(X_i)_{i=1}^{\infty}$ has $X_i \rightarrow X_{\infty}$ in \mathcal{C} as $i \rightarrow \infty$. Then we would require the topology \mathcal{T} on \mathcal{M} to satisfy $[X_i] \rightarrow [X_{\infty}]$ in \mathcal{M} whenever $X_i \rightarrow X_{\infty}$ in \mathcal{C} , and we could define \mathcal{T} to be the strongest topology on \mathcal{M} such that this is always true.

15.1. Moduli spaces in Differential Geometry

Moduli spaces in differential geometry usually involve a quotient of an infinite-dimensional (often singular) “manifold” by an infinite-dimensional group. Here are two examples from complex geometry.

Example 15.1

Let (X, J) be a compact complex manifold, and $E \rightarrow X$ a complex vector bundle. We wish to study the moduli space \mathcal{M}_E of holomorphic vector bundles $(F, \bar{\partial}_F)$ on X whose underlying complex vector bundle is isomorphic to E .

Choose a fixed $\bar{\partial}$ -operator $\bar{\partial}_0$ on E , with $(0,2)$ -curvature $F_0^{0,2} = \bar{\partial}_0^2$. Given a holomorphic vector bundle $(F, \bar{\partial}_F)$, we can choose an isomorphism $\iota : E \rightarrow F$, and this identifies $\bar{\partial}_F$ with $\bar{\partial}_0 + A$ for some A in $C^\infty(\text{End}(E) \otimes \Lambda^{0,1}X)$.

Example (Continued)

As $(F, \bar{\partial}_F)$ is holomorphic we have

$$F_{\bar{\partial}_0 + A}^{0,2} = F_0^{0,2} + \bar{\partial}_0 A + A \wedge A = 0.$$

Different choices ι, ι' of ι yield A, A' in the same orbit of the infinite-dimensional *gauge group* $\mathcal{G} := C^\infty(\text{Aut}(E))$. It follows that

$$\mathcal{M}_E \cong \{A \in C^\infty(\text{End}(E) \otimes \Lambda^{0,1}X) : F_0^{0,2} + \bar{\partial}_0 A + A \wedge A = 0\} / \mathcal{G}. \quad (15.1)$$

From (15.1) we can define a topology on \mathcal{M}_E , and some kind of singular manifold structure.

Example 15.2

Let X be a fixed compact $2n$ -manifold. We wish to study the moduli space \mathcal{M}_X of complex manifolds (Y, J) such that Y is diffeomorphic to X . For such (Y, J) , we can choose a diffeomorphism $\iota : X \rightarrow Y$, and then $I = \iota^*(J)$ is a complex structure on X , that is, $I \in C^\infty(TX \otimes T^*X)$ with $I^2 = -\text{id}$ and Nijenhuis tensor $N_I \equiv 0$.

Different choices ι, ι' of ι yield I, I' in the same orbit of the infinite-dimensional diffeomorphism group $\text{Diff}(X)$. Hence

$$\mathcal{M}_X \cong \{I \in C^\infty(TX \otimes T^*X) : I^2 = -\text{id}, N_I = 0\} / \text{Diff}(X).$$

For example, if X is an oriented 2-manifold of genus $g > 1$ then \mathcal{M}_X is *Riemann's moduli space* \mathcal{R}_g . It is a manifold of dimension $3g - 3$, with mild singularities (in fact, it is a nonsingular orbifold).

15.2. Moduli spaces in Algebraic Geometry

In algebraic geometry, one takes a different approach. Here, we study a moduli space \mathcal{M} of *algebraic* objects, such as complex projective manifolds, and the goal is to give \mathcal{M} an *algebraic* geometric structure – most often that of a *scheme*. I'm not going to define schemes. Loosely, a \mathbb{C} -scheme is a geometric space locally modelled on the zeroes of finitely many polynomials in \mathbb{C}^n . They form a category $\text{Sch}_{\mathbb{C}}$. Smooth \mathbb{C} -schemes are complex manifolds. It is usually not feasible to write algebro-geometric moduli spaces as quotients by infinite-dimensional gauge groups (though often, in a more complicated way, they are written as quotients by finite-dimensional algebraic groups). Instead, moduli spaces are defined to satisfy a universal property expressed in terms of the category of \mathbb{C} -schemes $\text{Sch}_{\mathbb{C}}$.

Let \mathcal{C} be a category of algebraic objects E we want to form a moduli space for. Then we must define a notion of *family of objects* E_S in \mathcal{C} over a base scheme S , thought of as $E_s \in \mathcal{C}$ for points $s \in S$, varying algebraically with s . For example, if \mathcal{C} is holomorphic vector bundles over a fixed complex projective manifold X , then a family of objects in \mathcal{C} over S is a vector bundle over $X \times S$. Such families form a category \mathcal{C}_S . If $\phi : S \rightarrow T$ is a morphism of schemes, then we should have a pullback functor $\phi^* : \mathcal{C}_T \rightarrow \mathcal{C}_S$.

The nicest kind of moduli space, a *fine moduli scheme* for \mathcal{C} , is a scheme \mathcal{M} such that for any \mathbb{C} -scheme S , there is a 1-1 correspondence between isomorphism classes $[E_S]$ of objects in \mathcal{C}_S and morphisms $\psi_{[E_S]} : S \rightarrow \mathcal{M}$ in $\text{Sch}_{\mathbb{C}}$, such that if $\phi : S \rightarrow T$ is a morphism in $\text{Sch}_{\mathbb{C}}$ and $E_T \in \mathcal{C}_T$ then $\psi_{[\phi^*(E_T)]} = \psi_{[E_T]} \circ \phi$. In this case, $\text{id} : \mathcal{M} \rightarrow \mathcal{M}$ corresponds to a *universal family* $U_{\mathcal{M}}$ in $\mathcal{C}_{\mathcal{M}}$, a family of objects U_m in \mathcal{C} for $m \in \mathcal{M}$, such that every E in \mathcal{C} is isomorphic to U_m for unique $m \in \mathcal{M}$.

Coarse moduli schemes satisfy a weaker universal property.

15.3. General questions in moduli space theory

In any moduli problem, one can ask a number of general questions:

(A) Existence. Does a moduli space exist in some class of spaces, e.g. is there a fine moduli scheme?

(B) Local properties. Fix an object E in \mathcal{C} . What does \mathcal{M} look like in a small neighbourhood of $[E]$, e.g. is it a manifold of some dimension? The study of (B) is called *deformation theory*.

(C) Global properties. Is the moduli space \mathcal{M} compact, Hausdorff, etc.?

(D) Compactification. If \mathcal{M} is not compact, is there a natural compactification $\bar{\mathcal{M}}$? If so, what do points of $\bar{\mathcal{M}} \setminus \mathcal{M}$ parametrize?

(E) Explicit description. Can we describe \mathcal{M} completely, as a topological space/scheme/manifold?

In a few very nice cases – moduli spaces of Riemann surfaces or $K3$ surfaces, for instance – we can answer all these questions.

More usually we can only answer (A)–(C) or (A)–(D).

Often if an object E in \mathcal{C} has a *nontrivial automorphism group*, this makes the moduli space \mathcal{M} singular at $[E]$, or causes other problems. For example, fine moduli schemes usually do not exist when objects have automorphisms.

Stacks (Deligne–Mumford or Artin) are geometric spaces which include an automorphism group at each point. In moduli problems with automorphisms, it may be best to make the moduli space a stack.

Orbifolds are basically smooth Deligne–Mumford stacks.

Murphy's Law

Even if the objects E you are studying are as nice (smooth, nonsingular) as possible, their moduli spaces \mathcal{M} can be very singular. In particular, in some problems (e.g. moduli spaces of smooth surfaces), one can prove that all possible singularities of schemes over \mathbb{Z} occur as singularities of moduli schemes. This is known as *Murphy's Law*. It is one reason why we have to work with schemes rather than manifolds.

There are exceptions, e.g. moduli spaces of Riemann surfaces and Calabi–Yau m -folds are smooth. But unless you have a geometrical reason for your moduli space to be nice, you should expect the worst.

Counting invariants

One very important use of moduli spaces is in defining *invariants*. The basic idea is this: start with some geometric object X (e.g. a compact complex manifold). Form a moduli space \mathcal{M}_α of auxiliary geometric objects on X with topological invariants α (e.g. holomorphic vector bundles on X with Chern character α). Define a number $I(\alpha)$ which ‘counts’ the points in \mathcal{M}_α . If you do this just right, the number $I(\alpha)$ turns out to be unchanged under deformations of X (it is ‘invariant’), and may have other exciting properties as well.

Examples include Donaldson, Seiberg–Witten, Gromov–Witten and Donaldson–Thomas invariants. They are important in 4-manifold theory, Symplectic Geometry, String Theory and Mirror Symmetry.

15.4. Deformations of compact complex manifolds

We now discuss moduli spaces of compact complex manifolds. Our approach will be differential-geometric. As in Example 15.2, if X is a compact $2n$ -manifold, then the moduli space of complex structures on X is

$$\mathcal{M}_X \cong \{J \in C^\infty(TX \otimes T^*X) : J^2 = -\text{id}, N_J = 0\} / \text{Diff}(X). \quad (15.2)$$

Write $[J]$ for $J \text{Diff}(X)$ in \mathcal{M}_X . From (15.2) we can define a topology on \mathcal{M}_X , and a singular smooth structure.

We will study *deformation theory* for compact complex manifolds. That is, if J is a complex structure on X , we wish to describe the topological space \mathcal{M}_X near $[J]$. We will see that modulo automorphisms of (X, J) , \mathcal{M}_X near $[J]$ looks like $\Phi^{-1}(0)$, where $\Phi : U \rightarrow H^2(TX)$ is holomorphic with $\Phi(0) = 0$, for U an open neighbourhood of 0 in $H^1(TX)$, where $H^q(TX)$ are the cohomology groups of the holomorphic vector bundle TX on X , as in §8. A book is K. Kodaira, '*Complex manifolds and deformation of complex structures*'.

Infinitesimal deformations

Suppose $\{J_t : t \in (-\epsilon, \epsilon)\}$ is a family of complex structures on X depending smoothly on $t \in (-\epsilon, \epsilon)$, with $J_0 = J$. Set $K = \left(\frac{d}{dt}J_t\right)|_{t=0}$, so that $K_b^a \in C^\infty(TX \otimes T^*X)$. Then K is an *infinitesimal deformation* of J as a complex structure.

By Taylor's Theorem $J_t = J + tK + O(t^2)$. As J_t is an almost complex structure $J_t^2 = -\text{id}$. Thus

$$\begin{aligned} -\delta_a^c &= (J_a^b + tK_a^b + O(t^2))(J_b^c + tK_b^c + O(t^2)) \\ &= J_a^b J_b^c + t(J_a^b K_b^c + K_a^b J_b^c) + O(t^2), \end{aligned}$$

so $J_a^b K_b^c + K_a^b J_b^c \equiv 0$.

In the notation of §11.2 for tensors on complex manifolds we have $J_a^b = i\delta_\alpha^\beta - i\delta_{\bar{\alpha}}^{\bar{\beta}}$. Thus

$$\begin{aligned} J_a^b K_b^c &= iK_\alpha^\gamma + iK_\alpha^{\bar{\gamma}} - iK_{\bar{\alpha}}^{\bar{\gamma}} - iK_{\bar{\alpha}}^\gamma, \\ K_a^b J_b^c &= iK_\alpha^\gamma - iK_\alpha^{\bar{\gamma}} - iK_{\bar{\alpha}}^{\bar{\gamma}} + iK_{\bar{\alpha}}^\gamma, \end{aligned}$$

so $J_a^b K_b^c + K_a^b J_b^c \equiv 0$ gives $K_\alpha^\beta = K_{\bar{\alpha}}^{\bar{\beta}} = 0$, and $K_a^b = K_\alpha^{\bar{\beta}} + K_{\bar{\alpha}}^\beta$.
 Regard $K_{\bar{\alpha}}^\beta$ as an element κ of $C^\infty(TX \otimes_{\mathbb{C}} \Lambda^{0,1}X)$, where $TX = TX^{1,0}$ is considered as a holomorphic vector bundle.

The Nijenhuis tensor of J_t is $(N_{J_t})_{ab}^c \equiv 0$. One can show that

$$\begin{aligned} (N_{J_t})_{\bar{\alpha}\bar{\beta}}^{\gamma} &= 2it(\nabla_{\bar{\alpha}}K_{\bar{\beta}}^{\gamma} - \nabla_{\bar{\beta}}K_{\bar{\alpha}}^{\gamma}) + O(t^2) \\ &= 2it(\bar{\partial}_{TX}\kappa)_{\bar{\alpha}\bar{\beta}}^{\gamma} + O(t^2), \end{aligned} \tag{15.3}$$

where $\bar{\partial}_{TX} : C^{\infty}(TX \otimes_{\mathbb{C}} \Lambda^{0,1}X) \rightarrow C^{\infty}(TX \otimes_{\mathbb{C}} \Lambda^{0,2}X)$ is as in §8 for the holomorphic vector bundle TX . Hence $\bar{\partial}_{TX}\kappa = 0$. Thus κ defines a cohomology class $[\kappa] \in H^1(TX)$.

Suppose $[\kappa] = 0$ in $H^1(TX)$. Then $\kappa = \bar{\partial}_{TX}v$ for some $v \in C^{\infty}(TX)$. This implies that $K = \mathcal{L}_v J$, where \mathcal{L}_v is the Lie derivative. As $C^{\infty}(TX)$ is the Lie algebra of $\text{Diff}(X)$, J_t lies in the orbit of J to first order in t , so $[J_t] = [J]$ to first order in t .

Conclusion

The *Zariski tangent space* to \mathcal{M}_X at $[J]$ is the finite-dimensional complex vector space $H^1(TX)_J$, the cohomology group of the holomorphic vector bundle TX on (X, J) . Here *Zariski tangent spaces* are defined for schemes and for manifolds (the usual tangent spaces). In this case the Zariski tangent space of \mathcal{M}_X at $[J]$ is first-order deformations of J modulo first-order diffeomorphisms of X .

Second order obstructions

Let (X, J) be a compact complex manifold, and $\eta \in H^1(TX)_J$.
What are the conditions on η for there to exist a family of complex structures $\{J_t : t \in (-\epsilon, \epsilon)\}$ with $J_0 = J$, $(\frac{d}{dt}J_t)|_{t=0} = K$, and $[K_{\bar{\alpha}}^{\beta}] = \eta$?

Write $(\frac{d^2}{dt^2}J_t)|_{t=0} = 2L$, so

$$J_t = J + tK + t^2L + O(t^3),$$

and regard $L_{\bar{\alpha}}^{\beta}$ as an element λ of $C^\infty(TX \otimes_{\mathbb{C}} \Lambda^{0,1}X)$.

As for (15.3), we find that

$$\begin{aligned}
 (N_{J_t})_{\bar{\alpha}\bar{\beta}}^\gamma &= -t^2(K_{\bar{\alpha}}^\delta \nabla_\delta K_{\bar{\beta}}^\gamma - K_{\bar{\beta}}^\delta \nabla_\delta K_{\bar{\alpha}}^\gamma) \\
 &\quad + 2it^2(\nabla_{\bar{\alpha}} L_{\bar{\beta}}^\gamma - \nabla_{\bar{\beta}} L_{\bar{\alpha}}^\gamma) + O(t^3) \\
 &= -t^2[\kappa, \kappa]_{\bar{\alpha}\bar{\beta}}^\gamma + 2it^2(\bar{\partial}_{TX} \lambda)_{\bar{\alpha}\bar{\beta}}^\gamma + O(t^3),
 \end{aligned} \tag{15.4}$$

where

$$\begin{aligned}
 [,] : C^\infty(TX \otimes_{\mathbb{C}} \Lambda^{0,k} X) \times C^\infty(TX \otimes_{\mathbb{C}} \Lambda^{0,l} X) \\
 \longrightarrow C^\infty(TX \otimes_{\mathbb{C}} \Lambda^{0,k+l} X)
 \end{aligned}$$

is defined by

$$\begin{aligned}
 [A, B] = \sum_{\sigma \in \mathcal{S}_{k+l}} \frac{\text{sign}(\sigma)}{(k+l)!} & \left(A_{\bar{\alpha}_{\sigma(1)} \dots \bar{\alpha}_{\sigma(k)}}^\delta \nabla_\delta B_{\bar{\alpha}_{\sigma(k+1)} \dots \bar{\alpha}_{\sigma(k+l)}}^\gamma \right. \\
 & \left. - B_{\bar{\alpha}_{\sigma(1)} \dots \bar{\alpha}_{\sigma(l)}}^\delta \nabla_\delta A_{\bar{\alpha}_{\sigma(l+1)} \dots \bar{\alpha}_{\sigma(k+l)}}^\gamma \right).
 \end{aligned}$$

The case $k = l = 0$ is the usual Lie bracket on vector fields.

Then $\bar{\partial}_{TX}[A, B] = [\bar{\partial}_{TX}A, B] \pm [A, \bar{\partial}_{TX}B]$, so $\bar{\partial}_{TX}[\kappa, \kappa] = 0$ as $\bar{\partial}_{TX}\kappa = 0$, and $[\kappa, \kappa]$ defines a class $[[\kappa, \kappa]]$ in $H^2(TX)_J$. In fact $[\cdot, \cdot]$ extends to $[\cdot, \cdot] : H^k(TX)_J \times H^l(TX)_J \rightarrow H^{k+l}(TX)_J$, and $[[\kappa, \kappa]] = [\eta, \eta]$ as $[\kappa] = \eta$. Now (15.4) implies that $[\kappa, \kappa] = 2i\bar{\partial}_{TX}\lambda$. Hence $[[\kappa, \kappa]] = [\eta, \eta] = 0$ in $H^2(TX)_J$. That is, $[\eta, \eta] = 0$ is a necessary condition for there to exist L such that $J_t = J + tK + t^2L + O(t^3)$ is a family of complex structures on X up to second order.

Conclusion

Deformations of J as a complex structure up to diffeomorphism to first order are given by elements $\eta \in H^1(TX)_J$, that is, $H^1(TX)_J$ is the *Zariski tangent space* of \mathcal{M}_X at $[J]$. There is a symmetric bilinear product $[\cdot, \cdot] : H^1(TX)_J \times H^1(TX)_J \rightarrow H^2(TX)_J$. A necessary condition for $\eta \in H^1(TX)_J$ to extend to a deformation of J to second order is $[\eta, \eta] = 0$ in $H^2(TX)_J$. We call $H^2(TX)_J$ the *obstruction space* of \mathcal{M}_X at $[J]$.

Complex manifolds and Kähler Geometry

Lecture 16 of 16: Deformation theory for compact complex manifolds

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Plan of talk:

- 16 Deformation theory for compact complex manifolds
 - 16.1 The unobstructed case
 - 16.2 The obstructed case
 - 16.3 Moduli spaces of Riemann surfaces
 - 16.4 Moduli of higher-dimensional complex manifolds

16. Deformation theory for compact complex manifolds

Let X be a compact $2n$ -manifold. As in §15.4, the *moduli space of complex structures* on X is

$$\mathcal{M}_X \cong \{J \in C^\infty(TX \otimes T^*X) : \\ J^2 = -\text{id}, N_J = 0\} / \text{Diff}(X).$$

Write $[J]$ for $J \text{Diff}(X)$ in \mathcal{M}_X . We seek to describe \mathcal{M}_X near $[J]$, as a topological space, or \mathbb{C} -scheme, or complex manifold.

In §15 we showed that if $\{J_t : t \in (-\epsilon, \epsilon)\}$ is a smooth family of complex structures on X with $J_0 = J$ and $K = \left(\frac{d}{dt} J_t\right)|_{t=0}$ then $\eta = [K_{\alpha}^{\beta}]$ lies in the cohomology group $H^1(TX)_J$, which parametrizes infinitesimal (first-order) deformations of J as a complex structure up to infinitesimal diffeomorphisms. We call $H^1(TX)_J$ the *Zariski tangent space* to \mathcal{M}_X at $[J]$.

We also explained that there is a symmetric bilinear product $[\cdot, \cdot] : H^1(TX)_J \times H^1(TX)_J \rightarrow H^2(TX)_J$, a generalization of the usual Lie bracket $H^0(TX)_J \times H^0(TX)_J \rightarrow H^0(TX)_J$ on holomorphic vector fields. A necessary condition for $\eta \in H^1(TX)_J$ to extend to a deformation of J up to second order is that $[\eta, \eta] = 0$ in $H^2(TX)_J$. We call $H^2(TX)_J$ the *obstruction space* of \mathcal{M}_X at $[J]$.

16.1. The unobstructed case

Suppose first that $H^2(TX)_J = 0$. Then Kodaira, Nirenberg and Spencer (1958) prove:

Theorem 16.1 (Existence.)

Suppose (X, J) is a compact complex manifold and $H^2(TX)_J = 0$. Then there exists an open neighbourhood U of 0 in $H^1(TX)_J$, and a smooth family $\{J_t : t \in U\}$ of complex structures on X with $J_0 = J$, such that the map $T_0U \rightarrow H^1(TX)_J$ taking $v \mapsto [((\partial_v J_t)|_{t=0})_{\bar{\alpha}}^\beta]$ is the identity map on $H^1(TX)_J$. Furthermore, the J_t depend holomorphically on t , in the sense that there exists a complex manifold \mathcal{X} and a holomorphic submersion $\pi : \mathcal{X} \twoheadrightarrow U$ with $\pi^{-1}(t) \cong (X, J_t)$ for all $t \in U$.

Thus, if $H^2(TX)_J = 0$ then every $\eta \in H^1(TX)_J$ is the first derivative of some actual smooth family $\{J_t : t \in (-\epsilon, \epsilon)\}$ of complex structures on X with $J_0 = J$.

Theorem 16.1 is proved by constructing a power series in t for J_t formally solving the equation $N_{J_t} \equiv 0$, and showing it converges near $t = 0$ in $H^1(TX)_J$.

Next, we want to say that the family of J_t in Theorem 16.1 represents all complex structures close to J ; that is, that the image of the map $U \rightarrow \mathcal{M}_X$ taking $t \mapsto [J_t]$ contains an open neighbourhood of $[J]$ in \mathcal{M}_X .

This property is called *completeness* of the family; also, the family is called *semiuniversal*.

Kodaira and Spencer (1958) prove:

Theorem 16.2 (Completeness.)

*In the situation of Theorem 16.1, suppose $u \in U$, and $\{I_t : t \in (-\epsilon, \epsilon)\}$ is a smooth family of complex structures on X with $I_0 = J_u$. Then there exist $0 < \delta \leq \epsilon$ and a smooth map $\phi : (-\delta, \delta) \rightarrow U$ with $\phi(0) = u$, such that the complex structures I_t and $J_{\phi(t)}$ are identified by a diffeomorphism of X , for all $t \in (-\delta, \delta)$. Also ϕ is unique if $H^0(TX)_J = 0$ (that is, the family is **universal**).*

The complex structures J_t for $t \in U$ may not all be distinct up to isomorphism. The holomorphic automorphism group $\text{Aut}(X, J)$, which has Lie algebra $H^0(TX)_J$, acts on $H^1(TX)_J$. We can take U to be invariant under $\text{Aut}(X, J)$. Then we expect J_s, J_t to be equivalent under diffeomorphisms of X iff s, t are in the same orbit of $\text{Aut}(X, J)$. Hence we expect \mathcal{M}_X near $[J]$ to be identified with $U / \text{Aut}(X, J)$.

16.2. The obstructed case

Theorems 16.1 and 16.2 were generalized to $H^2(TX)_J \neq 0$ by Kuranishi (1965).

Theorem 16.3

Let (X, J) be a compact complex manifold. Then there exists an open neighbourhood U of 0 in $H^1(TX)_J$ and a holomorphic map $\Phi : U \rightarrow H^2(TX)_J$ with $\Phi(t) = [t, t] + O(|t|^3)$ for small $t \in U$, so that $\Phi^{-1}(0)$ is a closed subset of U containing 0. There exists a family of complex structures $\{J_t : t \in \Phi^{-1}(0)\}$ with $J_0 = J$, depending continuously on t .

Actually, $\Phi^{-1}(0)$ has the structure of a complex analytic space, and J_t depends holomorphically on t in the sense of complex analytic spaces. That is, there exists a complex analytic space \mathcal{X} and a holomorphic submersion $\pi : \mathcal{X} \rightarrow \Phi^{-1}(0)$ with $\pi^{-1}(t) \cong (X, J_t)$ for all $t \in \Phi^{-1}(0)$.

Theorem (Continued)

Versions of Theorem 16.1 and 16.2 apply: the natural map $T_0\Phi^{-1}(0) \rightarrow H^1(TX)_J$ is the identity, and the family $\{J_t : t \in \Phi^{-1}(0)\}$ is complete.

Notice how Φ generalizes our second-order obstruction $[\eta, \eta] = 0$ to deforming complex structures.

Again, the J_t for $t \in \Phi^{-1}(0)$ may not all be distinct up to isomorphism. We expect to be able to take U to be invariant under $\text{Aut}(X, J)$ and Φ to be equivariant under $\text{Aut}(X, J)$, so that $\Phi^{-1}(0)$ is invariant, and then we expect \mathcal{M}_X near $[J]$ to be identified with $\Phi^{-1}(0)/\text{Aut}(X, J)$. Really $\Phi^{-1}(0)$ should be a complex analytic space or a scheme, and $\Phi^{-1}(0)/\text{Aut}(X, J)$ should be a stack.

16.3. Moduli spaces of Riemann surfaces

Let (X, J) be a compact Riemann surface of genus g . Then $\dim_{\mathbb{C}} X = 1$, so $H^2(TX)_J = 0$ for dimensional reasons. We can also compute the dimension of $H^1(TX)_J$ using §8 and §9.

When $g = 0$, so $X = \mathbb{C}P^1$, we have $H^0(TX)_J = \mathbb{C}^3$ and $H^1(TX)_J = 0$. The complex structure on $\mathbb{C}P^1$ has no deformations, it is *rigid*.

When $g = 1$, so $X = T^2$, we find that $H^0(TX)_J = H^1(TX)_J = \mathbb{C}$.

Let $g > 1$. By Serre duality

$$H^1(TX)_J \cong H^0(T^*X \otimes K_X)_J^* = H^0(K_X^2)_J^*.$$

K_X is positive as $c_1(K_X) = 2g - 2 > 0$, so $H^1(T^*X \otimes K_X)_J^* = 0$ by the Kodaira Vanishing Theorem in §9.2. Thus the Hirzebruch–Riemann–Roch Theorem in §8.2 gives

$$\begin{aligned} \dim H^1(TX)_J &= \chi(K_X^2) \\ &= \deg K_X^2 + (1 - g) \operatorname{rank} K_X^2 \\ &= 4g - 4 + 1 - g = 3g - 3. \end{aligned}$$

Hence

$$\dim H^1(TX)_J = \begin{cases} 0, & g = 0, \\ 1, & g = 1, \\ 3g - 3, & g > 1. \end{cases} \quad (16.1)$$

Let X be a fixed compact, oriented 2-manifold of genus g . There are two sensible ways to form a moduli space \mathcal{M}_X of complex structures on X : up to oriented diffeomorphisms $\text{Diff}^+(X)$, or up to diffeomorphisms isotopic to the identity $\text{Diff}_0(X)$. Here $\text{Diff}_0(X)$ is a normal subgroup in $\text{Diff}^+(X)$, with discrete quotient

$$\Gamma_g^+ = \text{Diff}^+(X) / \text{Diff}_0(X),$$

the *mapping class group*. This gives two different moduli spaces

$$\begin{aligned} \mathcal{R}_g &\cong \{J \text{ oriented complex structure on } X\} / \text{Diff}^+(X), \\ \mathcal{T}_g &\cong \{J \text{ oriented complex structure on } X\} / \text{Diff}_0(X), \end{aligned}$$

with $\mathcal{R}_g \cong \mathcal{T}_g / \Gamma_g^+$.

Here \mathcal{T}_g is *Teichmüller space*. It is a complex manifold of dimension (16.1), as you expect from Theorems 16.1-16.2. Elements of \mathcal{T}_g are called 'marked Riemann surfaces', as an element of the moduli space \mathcal{T}_g is an isomorphism class of genus g Riemann surfaces (Y, J) 'marked' with an isotopy class of diffeomorphisms $X \rightarrow Y$ for fixed X . \mathcal{R}_g is *Riemann's moduli space*. It is a complex orbifold, with quotient singularities at $[J]$ representing (X, J) with extra finite symmetry groups.

16.4. Moduli of higher-dimensional complex manifolds

Moduli of del Pezzo surfaces

A *del Pezzo surface* is a Fano 2-manifold, that is, a compact complex 2-manifold (X, J) with K_X negative. Such X have Kähler metrics with positive Ricci curvature. They are well understood. If X is a surface then contracting vectors TX with 2-forms $\Lambda^2 T^*X = K_X$ gives an isomorphism of holomorphic vector bundles $TX \otimes K_X \cong T^*X$, so that $TX \cong T^*X \otimes K_X^{-1}$. Hence

$$H^2(TX)_J \cong H^2(T^*X \otimes K_X^{-1})_J.$$

If X is del Pezzo then K_X^{-1} is positive, so $H^2(T^*X \otimes K_X^{-1})_J = 0$ by the Kodaira Vanishing Theorem in §9.2, and $H^2(TX)_J = 0$. Thus, deformations of del Pezzo surfaces are unobstructed, and Theorems 16.1-16.2 apply.

Moduli of Calabi–Yau manifolds

Now suppose (X, J) is a compact complex manifold, admitting Kähler metrics, with $\dim_{\mathbb{C}} X = m$, and with trivial canonical bundle K_X . Then X has Calabi–Yau metrics g by the Calabi Conjecture. Choose a nonzero holomorphic section Ω of K_X . Then $v \mapsto v \cdot \Omega$ defines an isomorphism of holomorphic vector bundles $TX \rightarrow \Lambda^{m-1} T^*X$. So

$$H^q(TX) \cong H^q(\Lambda^{m-1} T^*X) \cong H^{m-1,q}(X).$$

Thus the Zariski tangent space for deformations of J is $H^{m-1,1}(X)$, and obstruction space $H^{m-1,2}(X)$.

The obstruction space $H^{m-1,2}(X)$ may be nonzero – always when $m = 3$. But, the (Bogomolov–)Tian–Todorov Theorem says that in Theorem 16.3, the Kuranishi map Φ is identically zero, so that the moduli space of deformations of J is locally a complex manifold isomorphic to $H^{m-1,1}(X)$.

Theorem 16.4 (Bogomolov–Tian–Todorov)

Let (X, J) be a compact Kähler m -fold with $K_X \cong \mathcal{O}_X$. Then the universal family of deformations of J is smooth of dimension $h^{m-1,1}(X)$.

Sketch proof of the Bogomolov–Tian–Todorov Theorem

As in §15.4, the second order obstruction to deforming J in direction $\eta \in H^1(TX)_J$ is $[\eta, \eta] = 0$ in $H^2(TX)_J$, where $[\cdot, \cdot] : H^1(TX)_J \times H^1(TX)_J \rightarrow H^2(TX)_J$ is a symmetric bilinear product. We will show that for X Calabi–Yau, $[\cdot, \cdot] \equiv 0$, so that the second order obstructions to deforming J vanish, and $\Phi(t) = O(t^3)$ in Theorem 16.3. The full proof shows that n^{th} derivatives of Φ vanish for $n = 2, 3, \dots$, so that $\Phi \equiv 0$ as Φ is holomorphic. Let $\kappa, \lambda \in C^\infty(TX \otimes_{\mathbb{C}} \Lambda^{0,1}X)$. Then we defined $[\kappa, \lambda]$ in §15.4. Contracting the TX factor with Ω gives $\kappa \cdot \Omega \in C^\infty(\Lambda^{m-1,1}X)$, and similarly for λ , $[\kappa, \lambda]$. A computation by Tian shows that

$$([\kappa, \lambda]) \cdot \Omega = \frac{\partial(\kappa \cdot \Omega)}{\Omega} \wedge (\lambda \cdot \Omega) + (\kappa \cdot \Omega) \wedge \frac{\partial(\lambda \cdot \Omega)}{\Omega}. \quad (16.2)$$

Here $\kappa \cdot \Omega$ is an $(m-1, 1)$ -form, so $\partial(\kappa \cdot \Omega)$ is an $(m, 1)$ -form, and $(\partial(\kappa \cdot \Omega))/\Omega$ is a $(0, 1)$ -form.

Sketch proof of the Bogomolov–Tian–Todorov Theorem

Let $\eta, \zeta \in H^1(TX)_J \cong H^{m-1,1}(X)$. As any class in $H^{m-1,1}(X)$ can be represented by an $(m-1, 1)$ -form α with $\partial\alpha = \bar{\partial}\alpha = 0$, we can choose representatives κ, λ for η, ζ with $\bar{\partial}_{TX}\kappa = \bar{\partial}_{TX}\lambda = 0$ and $\partial(\kappa \cdot \Omega) = \partial(\lambda \cdot \Omega) = 0$. So (16.2) shows that $([\kappa, \lambda]) \cdot \Omega = \partial(\kappa \cdot (\lambda \cdot \Omega))$ is ∂ -exact, and thus $[\kappa, \lambda]$ is $\bar{\partial}_{TX}$ -exact. Hence $[\eta, \zeta] = [[\kappa, \lambda]] = 0$ in $H^2(TX)_J$. So $[\cdot, \cdot] : H^1(TX)_J \times H^1(TX)_J \rightarrow H^2(TX)_J$ is zero.

An explanation for the B–T–T theorem is that singularities of moduli spaces are not actually caused by obstructions per se, but by the Zariski tangent spaces jumping in dimension from point to point. In the Calabi–Yau case the Zariski tangent space is $H^{m-1,1}(X)$, which is a chunk of de Rham cohomology, and its dimension is fixed topologically (e.g. when $m = 2$ it is $b^2(X) - 2$, and when $m = 3$ it is $\frac{1}{2}b^3(X) - 1$). So the tangent spaces cannot jump in dimension, and the moduli spaces are smooth.

The period map

Let X be a Calabi–Yau m -fold. Define the moduli space of complex structures on X by

$$\mathcal{M}_X \cong \{J \text{ oriented C–Y complex structure on } X\} / \text{Diff}_0(X).$$

The B–T–T Theorem essentially says \mathcal{M}_X is a complex manifold of dimension $h^{m-1,1}(X)$, though it may not be Hausdorff.

Define the *period map*

$$\Pi : \mathcal{M}_X \rightarrow \mathbb{P}(H^m(X; \mathbb{C}))$$

by $\Pi : [J] \mapsto H^{m,0}(X)_J$. One can show that Π is a *holomorphic immersion*, that is, locally it identifies \mathcal{M}_X with a complex submanifold of $\mathbb{P}(H^m(X; \mathbb{C}))$. Our isomorphism

$T_{[J]}\mathcal{M}_X \cong H^{m-1,1}(X)_J$ depended on a choice of $[\Omega]$ in $H^{m,0}(X)_J$; without this choice $T_{[J]}\mathcal{M}_X \cong H^{m-1,1}(X)_J \otimes H^{m,0}(X)_J^*$, which is a vector subspace of $T_{H^{m,0}(X)_J}\mathbb{P}(H^m(X; \mathbb{C}))$.

In good cases one can describe the image of Π explicitly, and so understand \mathcal{M}_X . For example, when $m = 2$ (the moduli space of K3 surfaces), classes $\langle \theta \rangle \in \Pi(\mathcal{M}_X)$ must satisfy $\theta \cup \theta = 0$ and $\theta \cup \bar{\theta} > 0$ in $H^4(X; \mathbb{C})$, and $\Pi(\mathcal{M}_X)$ is the complement of certain hyperplanes in the set of such $\langle \theta \rangle$.

When $m = 3$, the intersection form \cup makes $H^3(X; \mathbb{C})$ into a complex symplectic vector space, and $\mathbb{P}(H^3(X; \mathbb{C}))$ into a complex contact manifold. The image $\Pi(\mathcal{M}_X)$ is a complex Legendrian submanifold (the projectivization of a complex Lagrangian cone in $H^3(X; \mathbb{C})$).

Mirror Symmetry gives a conjectural power series expansion for $\Pi(\mathcal{M}_X)$ near a singular point in the moduli space in terms of the Gromov–Witten invariants of a ‘mirror’ Calabi–Yau 3-fold \check{X} .