Derived differential geometry

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see website
people.maths.ox.ac.uk/∼joyce/dmanifolds.html,
and papers arXiv:1001.0023, arXiv:1104.4951, arXiv:1206.4207, arXiv:1208.4948, arXiv:1409.6908, arXiv:1509.05672, and arXiv:1510.07444.
These slides available at
people.maths.ox.ac.uk/∼joyce/talks.html.

Plan of talk:

1. Introduction

Derived Differential Geometry (DDG) is the study of *derived smooth manifolds* and *derived smooth orbifolds*, where 'derived' is in the sense of the Derived Algebraic Geometry (DAG) of Jacob Lurie and Toën–Vezzosi. Derived manifolds include ordinary smooth manifolds, but also many singular objects.

Derived manifolds and orbifolds form higher categories – 2-categories **dMan**, **dOrb** or **mKur**, **Kur** in my set-up, and ∞-categories in the set-ups of Spivak–Borisov–Noel.

Many interesting moduli spaces over $\mathbb{R}$ or $\mathbb{C}$ in both algebraic and differntial geometry are naturally derived manifolds or derived orbifolds, including those used to define Donaldson, Donaldson–Thomas, Gromov–Witten and Seiberg–Witten invariants, Floer theories, and Fukaya categories.

A compact, oriented derived manifold or orbifold $\mathbf{X}$ has a *virtual class* in homology (or a *virtual chain* if $\partial\mathbf{X} \neq \emptyset$), which can be used to define these enumerative invariants, Floer theories, ...

Introduction
D-manifolds
Differential geometry of d-manifolds
D-manifold and d-orbifold structures on moduli spaces

Different definitions of derived manifolds and orbifolds

There are several versions of 'derived manifolds' and 'derived orbifolds' in the literature, in order of increasing simplicity:

- Spivak's ∞-category $\mathbf{DerMan_{Spi}}$ of derived manifolds (2008).
- Borisov–Noël's ∞-category $\mathbf{DerMan_{BN}}$ (2011,2012).
- My d-manifolds and d-orbifolds (2010–2016), which form strict 2-categories **dMan**, **dOrb**.
- My μ-Kuranishi spaces, m-Kuranishi spaces and Kuranishi spaces (2014), which form a category **mKur** and weak 2-categories **mKur**, **Kur**.

  Here μ-, m-Kuranishi spaces are types of derived manifold, and Kuranishi spaces a type of derived orbifold.

In fact the Kuranishi space approach is motivated by earlier work by Fukaya, Oh, Ohta and Ono in symplectic geometry (1999,2009–) whose 'Kuranishi spaces' are really a prototype kind of derived orbifold, from before the invention of DAG.

Relation between these definitions

- Borisov–Noel (2011) prove an equivalence of ∞-categories $\mathbf{DerMan_{Spi}} \simeq \mathbf{DerMan_{BN}}$.
- Borisov (2012) gives a 2-functor $\pi_2(\mathbf{DerMan_{BN}}) \to \mathbf{dMan}$ which is nearly an equivalence of 2-categories (e.g. it is a 1-1 correspondence on equivalence classes of objects), where $\pi_2(\mathbf{DerMan_{BN}})$ is the 2-category truncation of $\mathbf{DerMan_{BN}}$.
- I prove (2016) equivalences of 2-categories $\mathbf{dMan} \simeq \mathbf{mKur}$, $\mathbf{dOrb} \simeq \mathbf{Kur}$ and of categories $\mathrm{Ho}(\mathbf{dMan}) \simeq \mathrm{Ho}(\mathbf{mKur})$ $\simeq \boldsymbol{\mu}\mathbf{Kur}$, where $\mathrm{Ho}(\cdots)$ is the homotopy category.

Thus all these notions of derived manifold are more-or-less equivalent. Kuranishi spaces are easiest. There is a philosophical difference between $\mathbf{DerMan_{Spi}}, \mathbf{DerMan_{BN}}$ (locally modelled on $X \times_Z Y$ for smooth maps of manifolds $g : X \to Z$, $h : Y \to Z$) and $\mathbf{dMan}, \boldsymbol{\mu}\mathbf{Kur}, \mathbf{mKur}$ (locally modelled on $s^{-1}(0)$ for E a vector bundle over a manifold V with $s : V \to E$ a smooth section).

Two ways to define ordinary manifolds

Definition 1.1

A *manifold* of dimension n is a Hausdorff, second countable topological space X with a sheaf $\mathcal{O}_X$ of $\mathbb{R}$-algebras (or C^∞-rings) locally isomorphic to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$, where $\mathcal{O}_{\mathbb{R}^n}$ is the sheaf of smooth functions $f : \mathbb{R}^n \to \mathbb{R}$.

Definition 1.2

A *manifold* of dimension n is a Hausdorff, second countable topological space X equipped with an atlas of charts $\{(V_i, \psi_i) : i \in I\}$, where $V_i \subseteq \mathbb{R}^n$ is open, and $\psi_i : V_i \to X$ is a homeomorphism with an open subset $\mathrm{Im}\,\psi_i$ of X for all $i \in I$, and $\psi_j^{-1} \circ \psi_i : \psi_i^{-1}(\mathrm{Im}\,\psi_j) \to \psi_j^{-1}(\mathrm{Im}\,\psi_i)$ is a diffeomorphism of open subsets of $\mathbb{R}^n$ for all $i, j \in I$.

If you define derived manifolds by generalizing Definition 1.1, you get something like d-manifolds; if you generalize Definition 1.2, you get something like (m-)Kuranishi spaces.

2. D-manifolds

2.1. C^∞-rings

Let X be a manifold, and write $C^\infty(X)$ for the smooth functions $c : X \to \mathbb{R}$. Then $C^\infty(X)$ is an $\mathbb{R}$-*algebra*: we can add smooth functions $(c,d) \mapsto c+d$, and multiply them $(c,d) \mapsto cd$, and multiply by $\lambda \in \mathbb{R}$.

But there are many more operations on $C^\infty(X)$ than this, e.g. if $c : X \to \mathbb{R}$ is smooth then $\exp(c) : X \to \mathbb{R}$ is smooth, giving $\exp : C^\infty(X) \to C^\infty(X)$, which is algebraically independent of addition and multiplication.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth. Define $\Phi_f : C^\infty(X)^n \to C^\infty(X)$ by $\Phi_f(c_1,\ldots,c_n)(x) = f\bigl(c_1(x),\ldots,c_n(x)\bigr)$ for all $x \in X$. Then addition comes from $f : \mathbb{R}^2 \to \mathbb{R}$, $f : (x,y) \mapsto x+y$, multiplication from $(x,y) \mapsto xy$, etc. This huge collection of algebraic operations Φ_f make $C^\infty(X)$ into an algebraic object called a C^∞-*ring*.

Definition

A C^∞-*ring* is a set $\mathfrak{C}$ together with n-fold operations $\Phi_f : \mathfrak{C}^n \to \mathfrak{C}$ for all smooth maps $f : \mathbb{R}^n \to \mathbb{R}$, $n \geqslant 0$, satisfying:

Let $m,n \geqslant 0$, and $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1,\ldots,m$ and $g : \mathbb{R}^m \to \mathbb{R}$ be smooth functions. Define $h : \mathbb{R}^n \to \mathbb{R}$ by

$$h(x_1,\ldots,x_n) = g(f_1(x_1,\ldots,x_n),\ldots,f_m(x_1\ldots,x_n)),$$

for $(x_1,\ldots,x_n) \in \mathbb{R}^n$. Then for all $c_1,\ldots,c_n$ in $\mathfrak{C}$ we have

$$\Phi_h(c_1,\ldots,c_n) = \Phi_g(\Phi_{f_1}(c_1,\ldots,c_n),\ldots,\Phi_{f_m}(c_1,\ldots,c_n)).$$

Also defining $\pi_j : (x_1,\ldots,x_n) \mapsto x_j$ for $j = 1,\ldots,n$ we have $\Phi_{\pi_j} : (c_1,\ldots,c_n) \mapsto c_j$.

A *morphism* of C^∞-rings is $\phi : \mathfrak{C} \to \mathfrak{D}$ with $\Phi_f \circ \phi^n = \phi \circ \Phi_f : \mathfrak{C}^n \to \mathfrak{D}$ for all smooth $f : \mathbb{R}^n \to \mathbb{R}$. Write $\mathbf{C^\infty Rings}$ for the category of C^∞-rings.

Any C^∞-ring $\mathfrak{C}$ is automatically an $\mathbb{R}$-algebra. A *module* over a C^∞-ring $\mathfrak{C}$ is a module over $\mathfrak{C}$ as an $\mathbb{R}$-algebra.

Examples of C^∞-rings

Then $C^\infty(X)$ is a C^∞-ring for any manifold X, and from $C^\infty(X)$ we can recover X up to canonical isomorphism.
If $f : X \to Y$ is smooth then $f^* : C^\infty(Y) \to C^\infty(X)$ is a morphism of C^∞-rings; conversely, if $\phi : C^\infty(Y) \to C^\infty(X)$ is a morphism of C^∞-rings then $\phi = f^*$ for some unique smooth $f : X \to Y$. This gives a *full and faithful functor* $F : \mathbf{Man} \to \mathbf{C^\infty Rings^{op}}$ by $F : X \mapsto C^\infty(X)$, $F : f \mapsto f^*$.
Thus, we can think of manifolds as examples of C^∞-rings, and C^∞-rings as generalizations of manifolds. But there are many more C^∞-rings than manifolds. For example, $C^0(X)$ is a C^∞-ring for any topological space X.

2.2. C^∞-schemes

We can now develop the whole machinery of scheme theory in algebraic geometry, replacing rings by C^∞-rings throughout.
A *C^∞-ringed space* $\underline{X} = (X, \mathcal{O}_X)$ is a topological space X with a sheaf of C^∞-rings $\mathcal{O}_X$. It is *local* if the stalks $\mathcal{O}_{X,x}$ for $x \in X$ are local $\mathbb{R}$-algebras with residue field $\mathbb{R}$. Write $\mathbf{LC^\infty RS}$ for the category of local C^∞-ringed spaces.
The *global sections functor* $\Gamma : \mathbf{LC^\infty RS} \to \mathbf{C^\infty Rings^{op}}$ maps $\Gamma : (X, \mathcal{O}_X) \mapsto \mathcal{O}_X(X)$. It has a right adjoint, the *spectrum functor* $\mathrm{Spec} : \mathbf{C^\infty Rings^{op}} \to \mathbf{LC^\infty RS}$. That is, for each C^∞-ring $\mathfrak{C}$ we construct a local C^∞-ringed space $\mathrm{Spec}\,\mathfrak{C}$. Points $x \in \mathrm{Spec}\,\mathfrak{C}$ are $\mathbb{R}$-algebra morphisms $x : \mathfrak{C} \to \mathbb{R}$. We don't use prime ideals.
On the subcategory of *complete* C^∞-rings, Spec is full and faithful.

A local C^∞-ringed space $\underline{X}$ is called an *affine C^∞-scheme* if $\underline{X} \cong \operatorname{Spec} \mathfrak{C}$ for some C^∞-ring $\mathfrak{C}$. It is a *C^∞-scheme* if X can be covered by open $U \subseteq X$ with $(U, \mathcal{O}_X|_U)$ an affine C^∞-scheme. Write $\mathbf{C^\infty Sch}$ for the full subcategory of C^∞-schemes in $\mathbf{LC^\infty RS}$.
If X is a manifold, define a C^∞-scheme $\underline{X} = (X, \mathcal{O}_X)$ by $\mathcal{O}_X(U) = C^\infty(U)$ for all open $U \subseteq X$. Then $\underline{X} \cong \operatorname{Spec} C^\infty(X)$. This defines a full and faithful embedding $\mathbf{Man} \hookrightarrow \mathbf{C^\infty Sch}$. So we can regard manifolds as examples of C^∞-schemes.
All fibre products exist in $\mathbf{C^\infty Sch}$. In manifolds $\mathbf{Man}$, fibre products $X \times_{g,Z,h} Y$ need exist only if $g : X \to Z$ and $h : Y \to Z$ are transverse. When g, h are not transverse, the fibre product $X \times_{g,Z,h} Y$ exists in $\mathbf{C^\infty Sch}$, but is not a manifold.
We also define *quasicoherent sheaves* on a C^∞-scheme $\underline{X}$, and write $\operatorname{qcoh}(\underline{X})$ for the abelian category of quasicoherent sheaves.
A C^∞-scheme $\underline{X}$ has a well-behaved *cotangent sheaf* $T^*\underline{X}$.

2.3. Differential graded C^∞-rings

We can define derived $\mathbb{C}$-schemes by replacing $\mathbb{C}$-algebras A by *dg $\mathbb{C}$-algebras* $A^\bullet$ in the definition of $\mathbb{C}$-scheme — commutative differential graded $\mathbb{C}$-algebras in degrees $\leqslant 0$, of the form $\cdots \to A^{-2} \xrightarrow{\mathrm{d}} A^{-1} \xrightarrow{\mathrm{d}} A^0$, where A^0 is an ordinary $\mathbb{C}$-algebra. The corresponding 'classical' $\mathbb{C}$-algebra is $H^0(A^\bullet) = A^0/\mathrm{d}[A^{-1}]$.
There is a parallel notion of *dg C^∞-ring* $\mathfrak{C}^\bullet$, of the form $\cdots \to \mathfrak{C}^{-2} \xrightarrow{\mathrm{d}} \mathfrak{C}^{-1} \xrightarrow{\mathrm{d}} \mathfrak{C}^0$, where $\mathfrak{C}^0$ is an ordinary C^∞-ring, and $\mathfrak{C}^{-1}, \mathfrak{C}^{-2}, \ldots$ are modules over $\mathfrak{C}^0$. The corresponding 'classical' C^∞-ring is $H^0(\mathfrak{C}^\bullet) = \mathfrak{C}^0/\mathrm{d}[\mathfrak{C}^{-1}]$.
One could use dg C^∞-rings to define 'derived C^∞-schemes'; an alternative is to use *simplicial C^∞-rings*, see Spivak arXiv:0810.5175, Borisov–Noel arXiv:1112.0033, and Borisov arXiv:1212.1153.

Square zero dg C^∞-rings

My d-spaces are a 2-category truncation of derived C^∞-schemes. To define them, I use a special class of dg C^∞-rings called *square zero dg C^∞-rings*, which form a 2-category $\mathbf{SZC^\infty Rings}$.

A dg C^∞-ring $\mathfrak{C}^\bullet$ is *square zero* if $\mathfrak{C}^i = 0$ for $i < -1$ and $\mathfrak{C}^{-1} \cdot \mathrm{d}[\mathfrak{C}^{-1}] = 0$. Then $\mathfrak{C}$ is $\mathfrak{C}^{-1} \xrightarrow{\mathrm{d}} \mathfrak{C}^0$, and $\mathrm{d}[\mathfrak{C}^{-1}]$ is a square zero ideal in the (ordinary) C^∞-ring $\mathfrak{C}^0$, and $\mathfrak{C}^{-1}$ is a module over the 'classical' C^∞-ring $H^0(\mathfrak{C}^\bullet) = \mathfrak{C}^0/\mathrm{d}[\mathfrak{C}^{-1}]$.

A 1-*morphism* $\alpha^\bullet : \mathfrak{C}^\bullet \to \mathfrak{D}^\bullet$ in $\mathbf{SZC^\infty Rings}$ is maps $\alpha^0 : \mathfrak{C}^0 \to \mathfrak{D}^0$, $\alpha^{-1} : \mathfrak{C}^{-1} \to \mathfrak{D}^{-1}$ preserving all the structure.

Then $H^0(\alpha^\bullet) : H^0(\mathfrak{C}) \to H^0(\mathfrak{D})$ is a morphism of C^∞-rings.

For 1-morphisms $\alpha^\bullet, \beta^\bullet : \mathfrak{C}^\bullet \to \mathfrak{D}^\bullet$ a 2-*morphism* $\eta : \alpha^\bullet \Rightarrow \beta^\bullet$ is a C^∞-derivation $\eta : \mathfrak{C}^0 \to \mathfrak{D}^{-1}$ with $\beta^0 = \alpha^0 + \mathrm{d} \circ \eta$, $\beta^{-1} = \alpha^{-1} + \eta \circ \mathrm{d}$.

There is an embedding of (2-)categories $\mathbf{C^\infty Rings} \subset \mathbf{SZC^\infty Rings}$ as the (2-)subcategory of $\mathfrak{C}^\bullet$ with $\mathfrak{C}^{-1} = 0$.

Examples of square zero dg C^∞-rings

Let V be a manifold, $E \to V$ a vector bundle, and $s : V \to E$ a smooth section. Then we call (V, E, s) a *Kuranishi neighbourhood* (compare Kuranishi spaces); for d-orbifolds, we take V an orbifold.

Associate a square zero dg C^∞-ring $\mathfrak{C}^{-1} \xrightarrow{\mathrm{d}} \mathfrak{C}^0$ to (V, E, s) by

$$\mathfrak{C}^0 = C^\infty(V)/I_s^2, \qquad \mathfrak{C}^{-1} = C^\infty(E^*)/I_s \cdot C^\infty(E^*),$$
$$\mathrm{d}(\epsilon + I_s \cdot C^\infty(E^*)) = \epsilon(s) + I_s^2,$$

where $I_s = C^\infty(E^*) \cdot s \subset C^\infty(V)$ is the ideal generated by s.

The d-manifold $\mathbf{X}$ associated to (V, E, s) is $\mathrm{Spec}\, \mathfrak{C}^\bullet$. It only knows about functions on V up to $O(s^2)$, and sections of E up to $O(s)$.

2.4. D-spaces and d-manifolds

A *d-space* $\mathbf{X}$ is a topological space X with a sheaf of square zero dg-C^∞-rings $\mathcal{O}^\bullet_{\mathbf{X}} = \mathcal{O}^{-1}_X \xrightarrow{\mathrm{d}} \mathcal{O}^0_{\mathbf{X}}$, such that $\underline{X} = (X, H^0(\mathcal{O}^\bullet_{\mathbf{X}}))$ and $(X, \mathcal{O}^0_{\mathbf{X}})$ are C^∞-schemes, and $\mathcal{O}^{-1}_X$ is quasicoherent over $\underline{X}$. We call $\underline{X}$ the *underlying classical C^∞-scheme*.

D-spaces form a strict 2-category **dSpa**, with 1-morphisms and 2-morphisms defined using sheaves of 1-morphisms and 2-morphisms in $\mathbf{SZC^\infty Rings}$ in the obvious way.

All (2-category) fibre products exist in **dSpa**.

C^∞-schemes include into d-spaces as those $\mathbf{X}$ with $\mathcal{O}^{-1}_X = 0$. Thus we have inclusions of (2-)categories $\mathbf{Man} \subset \mathbf{C^\infty Sch} \subset \mathbf{dSpa}$, so manifolds are examples of d-spaces.

A d-space $\mathbf{X}$ has a *cotangent complex* $\mathbb{L}^\bullet_{\mathbf{X}}$, a 2-term complex $\mathbb{L}^{-1}_{\mathbf{X}} \xrightarrow{\mathrm{d}_{\mathbf{X}}} \mathbb{L}^0_{\mathbf{X}}$ of quasicoherent sheaves on $\underline{X}$. Such complexes form a 2-category $\mathrm{qcoh}^{[-1,0]}(\underline{X})$.

D-manifolds and generalizations

A *d-manifold* $\mathbf{X}$ of *virtual dimension* $n \in \mathbb{Z}$ is a d-space $\mathbf{X}$ whose topological space X is Hausdorff and second countable, and such that $\mathbf{X}$ is covered by open d-subspaces $\mathbf{Y} \subset \mathbf{X}$ with equivalences $\mathbf{Y} \simeq U \times_{g,W,h} V$, where U, V, W are manifolds with $\dim U + \dim V - \dim W = n$, and $g : U \to W$, $h : V \to W$ are smooth maps, and $U \times_{g,W,h} V$ is the fibre product in the 2-category **dSpa**. (The 2-category structure is essential here.)

Write **dMan** for the full 2-subcategory of d-manifolds in **dSpa**.

Alternatively, we can write the local models as $\mathbf{Y} \simeq V \times_{0,E,s} V$, where V is a manifold, $E \to V$ a vector bundle, $s : V \to E$ a smooth section, and $n = \dim V - \mathrm{rank}\, E$. We call such $V \times_{0,E,s} V$ *affine d-manifolds*.

I also define 2-categories $\mathbf{dMan^b}, \mathbf{dMan^c}$ of *d-manifolds with boundary* and *corners*, and orbifold versions $\mathbf{dOrb}, \mathbf{dOrb^b}, \mathbf{dOrb^c}$, *d-orbifolds*, using Deligne–Mumford C^∞-stacks.

3. Differential geometry of d-manifolds
Tangent and obstruction spaces of d-manifolds

If $\mathbf{X}$ is a d-manifold, its cotangent complex $\mathbb{L}^\bullet_{\mathbf{X}}$ is *perfect*, that is, $\mathbb{L}^\bullet_{\mathbf{X}}$ is equivalent locally on $\underline{X}$ in the 2-category $\mathrm{qcoh}^{[-1,0]}(\underline{X})$ of 2-term complexes of quasicoherent sheaves on $\underline{X}$ to a complex of vector bundles $\mathcal{E}^{-1} \to \mathcal{E}^0$, and $\mathrm{rank}\,\mathcal{E}^0 - \mathrm{rank}\,\mathcal{E}^{-1} = \mathrm{vdim}\,\mathbf{X}$.
For $x \in \mathbf{X}$, define the *tangent space* $T_x\mathbf{X} = H^0(\mathbb{L}_{\mathbf{X}}|_x)^*$ and the *obstruction space* $O_x\mathbf{X} = H^{-1}(\mathbb{L}_{\mathbf{X}}|_x)^*$, with $\dim T_x\mathbf{X} - \dim O_x\mathbf{X} = \mathrm{vdim}\,\mathbf{X}$. A 1-morphism of d-manifolds $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ induces functorial linear maps $T_x\mathbf{f} : T_x\mathbf{X} \to T_{f(x)}\mathbf{Y}$ and $O_x\mathbf{f} : O_x\mathbf{X} \to O_{f(x)}\mathbf{Y}$.

Theorem

A 1-morphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ in **dMan** *is étale (a local equivalence) if and only if $T_x\mathbf{f} : T_x\mathbf{X} \to T_{f(x)}\mathbf{Y}$ and $O_x\mathbf{f} : O_x\mathbf{X} \to O_{f(x)}\mathbf{Y}$ are isomorphisms for all $x \in \mathbf{X}$.*

3.1. D-transversality and fibre products

Let $g : X \to Z$, $h : Y \to Z$ be smooth maps of manifolds. Then g, h are *transverse* if for all $x \in X$, $y \in Y$ with $g(x) = h(y) = z$ in Z, the map $T_xg \oplus T_yh : T_xX \oplus T_yY \to T_zZ$ is surjective.
Similarly, we call 1-morphisms $\mathbf{g} : \mathbf{X} \to \mathbf{Z}$, $\mathbf{h} : \mathbf{Y} \to \mathbf{Z}$ in **dMan** *d-transverse* if for all $x \in \mathbf{X}$, $y \in \mathbf{Y}$ with $\mathbf{g}(x) = \mathbf{h}(y) = z$ in $\mathbf{Z}$, the map $O_x\mathbf{g} \oplus O_y\mathbf{h} : O_x\mathbf{X} \oplus O_y\mathbf{Y} \to O_z\mathbf{Z}$ is surjective.

Theorem

Let $\mathbf{g} : \mathbf{X} \to \mathbf{Z}$ and $\mathbf{h} : \mathbf{Y} \to \mathbf{Z}$ be d-transverse 1-morphisms in **dMan**. *Then a fibre product $\mathbf{W} = \mathbf{X} \times_{\mathbf{g},\mathbf{Z},\mathbf{h}} \mathbf{Y}$ exists in the 2-category* **dMan**, *with $\mathrm{vdim}\,\mathbf{W} = \mathrm{vdim}\,\mathbf{X} + \mathrm{vdim}\,\mathbf{Y} - \mathrm{vdim}\,\mathbf{Z}$.*

If $\mathbf{Z}$ is a manifold, $O_z\mathbf{Z} = 0$ and d-transversality is trivial, giving:

Corollary

All fibre products of the form $\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y}$ with $\mathbf{X}, \mathbf{Y}$ d-manifolds and $\mathbf{Z}$ a manifold exist in **dMan**.

3.2. Gluing by equivalences

A 1-morphism $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$ in $\mathbf{dMan}$ is an *equivalence* if there exist $\mathbf{g}:\mathbf{Y}\to\mathbf{X}$ and 2-morphisms $\eta:\mathbf{g}\circ\mathbf{f}\Rightarrow\mathbf{id_X}$ and $\zeta:\mathbf{f}\circ\mathbf{g}\Rightarrow\mathbf{id_Y}$.

Theorem

Let $\mathbf{X},\mathbf{Y}$ be d-manifolds, $\emptyset\neq\mathbf{U}\subseteq\mathbf{X}$, $\emptyset\neq\mathbf{V}\subseteq\mathbf{Y}$ open d-submanifolds, and $\mathbf{f}:\mathbf{U}\to\mathbf{V}$ an equivalence. Suppose the topological space $Z=X\cup_{U=V}Y$ made by gluing X,Y using f is Hausdorff. Then there exists a d-manifold $\mathbf{Z}$, unique up to equivalence, open $\hat{\mathbf{X}},\hat{\mathbf{Y}}\subseteq\mathbf{Z}$ with $\mathbf{Z}=\hat{\mathbf{X}}\cup\hat{\mathbf{Y}}$, equivalences $\mathbf{g}:\mathbf{X}\to\hat{\mathbf{X}}$ and $\mathbf{h}:\mathbf{Y}\to\hat{\mathbf{Y}}$, and a 2-morphism $\eta:\mathbf{g}|_{\mathbf{U}}\Rightarrow\mathbf{h}\circ\mathbf{f}$.

The theorem generalizes to gluing families of d-manifolds $\mathbf{X}_i:i\in I$ by equivalences on double overlaps $\mathbf{X}_i\cap\mathbf{X}_j$, with (weak) conditions on triple overlaps $\mathbf{X}_i\cap\mathbf{X}_j\cap\mathbf{X}_k$.

3.3. D-manifold bordism, and virtual classes

Let Y be a manifold. Define the *bordism group* $B_k(Y)$ to have elements $\sim$-equivalence classes $[X,f]$ of pairs (X,f), where X is a compact oriented k-manifold and $f:X\to Y$ is smooth, and $(X,f)\sim(X',f')$ if there exists a $(k+1)$-manifold with boundary W and a smooth $e:W\to Y$ with $\partial W\cong X\amalg -X'$ and $e|_{\partial W}\cong f\amalg f'$. It is an abelian group, with $[X,f]+[X',f']=[X\amalg X,f\amalg f']$.

Similarly, define the *derived bordism group* $dB_k(Y)$ with elements $\approx$-equivalence classes $[\mathbf{X},\mathbf{f}]$ of pairs (X,f), where $\mathbf{X}$ is a compact oriented d-manifold with $\mathrm{vdim}\,\mathbf{X}=k$ and $\mathbf{f}:\mathbf{X}\to\mathbf{Y}=F_{\mathbf{Man}}^{\mathbf{dMan}}(Y)$ is a 1-morphism in $\mathbf{dMan}$, and $(\mathbf{X},\mathbf{f})\approx(\mathbf{X}',\mathbf{f}')$ if there exists a d-manifold with boundary $\mathbf{W}$ with $\mathrm{vdim}\,\mathbf{W}=k+1$ and a 1-morphism $\mathbf{e}:\mathbf{W}\to\mathbf{Y}$ in $\mathbf{dMan^b}$ with $\partial\mathbf{W}\simeq\mathbf{X}\amalg-\mathbf{X}'$ and $\mathbf{e}|_{\partial\mathbf{W}}\cong\mathbf{f}\amalg\mathbf{f}'$. It is an abelian group, with $[\mathbf{X},\mathbf{f}]+[\mathbf{X}',\mathbf{f}']=[\mathbf{X}\amalg\mathbf{X},\mathbf{f}\amalg\mathbf{f}']$.

There is a morphism $\Pi_{\rm bo}^{\rm dbo}: B_k(Y)\to dB_k(Y)$ mapping $[X,f]\mapsto [F_{\bf Man}^{\bf dMan}(X),F_{\bf Man}^{\bf dMan}(f)]$.

Theorem (first proved by Spivak for his derived manifolds)

$\Pi_{\rm bo}^{\rm dbo}: B_k(Y)\to dB_k(Y)$ *is an isomorphism for all* $k\in\mathbb{Z}$.

This holds because every d-manifold can be perturbed to a manifold. Composing $(\Pi_{\rm bo}^{\rm dbo})^{-1}$ with the projection $B_k(Y)\to H_k(Y,\mathbb{Z})$ gives a morphism $\Pi_{\rm dbo}^{\rm hom}: dB_k(Y)\to H_k(Y,\mathbb{Z})$. We can interpret this as a *virtual class map* for compact oriented d-manifolds. Virtual classes (in homology over $\mathbb{Q}$) also exist for compact oriented d-orbifolds. In arXiv:1509.0672 I define new (co)homology theories of manifolds, called *M-(co)homology*. I will give direct constructions of virtual classes (or virtual chains) for compact d-manifolds and d-orbifolds (with corners) in M-(co)homology.

4. D-manifold and d-orbifold structures on moduli spaces

Theorem 4.1

Let $\mathcal{V}$ *be a Banach manifold,* $\mathcal{E}\to\mathcal{V}$ *a Banach vector bundle, and* $s:\mathcal{V}\to\mathcal{E}$ *a smooth Fredholm section, with constant Fredholm index* $n\in\mathbb{Z}$. *Then there is a canonical d-manifold* $\boldsymbol{X}$ *with topological space* $X=s^{-1}(0)$ *and* $\mathrm{vdim}\,\boldsymbol{X}=n$.

Nonlinear elliptic equations, when written as maps between suitable Hölder or Sobolev spaces, are the zeroes of Fredholm sections of a Banach vector bundle over a Banach manifold. Thus we have:

Corollary 4.2

Let $\mathcal{M}$ *be a moduli space of solutions of a nonlinear elliptic equation on a compact manifold, with fixed topological invariants. Then* $\mathcal{M}$ *extends to a d-manifold* $\boldsymbol{\mathcal{M}}$.

The virtual dimension of $\boldsymbol{\mathcal{M}}$ at $x\in\boldsymbol{\mathcal{M}}$ is the index of the linearization of the elliptic p.d.e. at x, given by the A–S Index Theorem.

Truncation functors from other structures

Theorem 4.3

Suppose X is a Hausdorff, second countable topological space equipped with any of the following geometric structures, each of constant virtual dimension $n \in \mathbb{Z}$:

(a) *A $\mathbb{C}$-scheme or Deligne–Mumford $\mathbb{C}$-stack with perfect obstruction theory in the sense of Behrend and Fantechi (where X is the underlying complex analytic space).*
(b) *A quasi-smooth derived $\mathbb{C}$-scheme or D–M $\mathbb{C}$-stack.*
(c) *An M-polyfold or polyfold Fredholm structure in the sense of Hofer, Wysocki and Zehnder.*
(d) *A Kuranishi structure in the sense of Fukaya–Oh–Ohta–Ono.*
(e) *A Kuranishi atlas in the sense of McDuff and Wehrheim.*

Then X may be given the structure of a d-manifold or d-orbifold, natural up to equivalence in $\mathbf{dMan}, \mathbf{dOrb}$, *with* $\mathrm{vdim}\,\mathbf{X} = n$. *We can also allow corners in* (c)–(e), *with* $\mathbf{X} \in \mathbf{dMan^c}, \mathbf{dOrb^c}$.

Combining Theorem 4.3 with results from the literature shows that many interesting moduli spaces over $\mathbb{R}$ or $\mathbb{C}$, in both differential and algebraic geometry, have the structure of d-manifolds or d-orbifolds, natural up to equivalence. This includes almost every moduli space used in any enumerative invariant problem over $\mathbb{R}$ or $\mathbb{C}$.

I hope in future to develop a new approach to defining d-orbifold structures on moduli spaces, based on Grothendieck's method of representable functors in algebraic geometry. Given a moduli problem, the idea is to define a weak 'moduli 2-functor'

$$F : (\mathbf{dMan^{aff}})^{\mathbf{op}} \longrightarrow \mathbf{Groupoids}$$

such that $F(\mathbf{S})$ is the groupoid of families of objects in the moduli problem over a base affine d-manifold $\mathbf{S}$, and then prove by verifying some representability criteria that F is 'representable', that is, F is 2-naturally equivalent to $\mathrm{Hom}(-, \boldsymbol{\mathcal{M}})$ for a d-orbifold $\boldsymbol{\mathcal{M}}$, which is then unique up to equivalence in $\mathbf{dOrb}$.