Donaldson–Thomas theory: introduction and open problems

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Joyce-Song arXiv:0810.5645,
Joyce arXiv:0910.0105,
Kontsevich-Soibelman arXiv:0811.2435,
arXiv:0910.4315,
and arXiv:1006.2706

These slides available at

`www.maths.ox.ac.uk/∼joyce/talks.html`

1. Calabi–Yau manifolds

A *Calabi–Yau m-fold* is a compact $2m$-dimensional manifold X equipped with four geometric structures:

- a Riemannian metric g;
- a complex structure J;
- a symplectic form (Kähler form) ω; and
- a complex volume form Ω.

These satisfy pointwise compatibility conditions: $\omega(u,v) = g(Ju,v)$, $|\Omega|_g \equiv 2^{m/2}$, Ω is of type $(m,0)$ w.r.t. J, and p.d.e.s: J is integrable, and $\mathrm{d}\omega \equiv \mathrm{d}\Omega \equiv 0$. Usually we also require $H^1(X;\mathbb{R}) = 0$.

This is a rich geometric structure, and very interesting from several points of view.

Complex algebraic geometry: (X, J) is a projective complex manifold. That is, we can embed X as a complex submanifold of $\mathbb{CP}^N$ for some $N \gg 0$, and then X is the zero set of finitely many homogeneous polynomials on $\mathbb{C}^{N+1}$. Also Ω is a holomorphic section of the canonical bundle K_X, so K_X is trivial, and $c_1(X)=0$.

Analysis: For fixed (X, J), Yau's solution of the Calabi Conjecture by solving a nonlinear elliptic p.d.e. shows that there exists a family of Kähler metrics g on X making X Calabi–Yau.

Combining complex algebraic geometry and analysis proves the existence of huge numbers of examples of Calabi–Yau m-folds.

Riemannian geometry: (X, g) is a Ricci-flat Riemannian manifold with holonomy group $\mathrm{Hol}(g) \subseteq \mathrm{SU}(m)$.

Symplectic geometry: (X, ω) is a symplectic manifold with $c_1(X) = 0$.

Calibrated geometry: there is a distinguished class of minimal submanifolds in (X, g) called special Lagrangian m-folds.

String Theory: a branch of theoretical physics aiming to combine Quantum Theory and General Relativity. String Theorists believe that space-time is not 4 dimensional, but 10-dimensional, and is locally modelled on $\mathbb{R}^{3,1} \times X$, where $\mathbb{R}^{3,1}$ is Minkowski space, our observed universe, and X is a Calabi–Yau 3-fold with radius of order 10^{-33}cm, the Planck length.

String Theorists believe that each Calabi–Yau 3-fold X has a quantization, a *Super Conformal Field Theory* (SCFT), not yet rigorously defined. Invariants of X such as the Dolbeault groups $H^{p,q}(X)$ and the Gromov–Witten invariants of X translate to properties of the SCFT. Using physical reasoning they made amazing predictions about Calabi–Yau 3-folds, an area known as *Mirror Symmetry*, conjectures which are slowly turning into theorems.

Part of the picture is that Calabi–Yau 3-folds should occur in pairs $X, \hat{X}$, such that $H^{p,q}(X) \cong H^{3-p,q}(\hat{X})$, and the complex geometry of X is somehow equivalent to the symplectic geometry of $\hat{X}$, and vice versa. This is very strange. It is an exciting area in which to work.

Today we are doing algebraic geometry, though the other points of view will be lurking in the background. So from now on, a *Calabi–Yau 3-fold* X will mean a smooth projective 3-fold X over a field $\mathbb{K}$, with trivial canonical bundle K_X, and with $H^1(\mathcal{O}_X) = 0$. The field $\mathbb{K}$ will almost always be $\mathbb{C}$, but we are interested in extending this, say to $\mathbb{K}$ algebraically closed of characteristic zero. We take X to be equipped with an ample line bundle $\mathcal{L}$ (roughly equivalent to the Kähler form ω: take $c_1(\mathcal{L}) = [\omega] \in H^2(X;\mathbb{Z})$), with which we define *stability* of coherent sheaves.

2. Coherent sheaves on X

Let X be a Calabi–Yau 3-fold. A *holomorphic vector bundle* $\pi : E \to X$ of *rank* r is a complex manifold E with a holomorphic map $\pi : E \to X$ whose fibres are complex vector spaces $\mathbb{C}^r$. A *morphism* $\phi : E \to F$ of holomorphic vector bundles $\pi : E \to X$, $\pi' : F \to X$ is a holomorphic map $\phi : E \to F$ with $\pi' \circ \phi \equiv \pi$, that is linear on the vector space fibres. Then $\mathrm{Hom}(E, F)$ is a finite-dimensional vector space. Holomorphic vector bundles form an exact category $\mathrm{Vect}(X)$.

A holomorphic vector bundle E has topological invariants, the *Chern character* $\mathrm{ch}_*(E)$ in $H^{\mathrm{even}}(X, \mathbb{Q})$, with $\mathrm{ch}_0(E) = r$, the rank of E. Holomorphic vector bundles are very natural objects to study.

However, vector bundles have two disadvantages:

• If $\phi : E \to F$ is a morphism of vector bundles, then the kernel $\operatorname{Ker}\phi$ and cokernel $\operatorname{Coker}\phi$ need not be vector bundles, as their rank may not be constant on X.

• Moduli spaces (coarse moduli schemes) of (semistable) holomorphic vector bundles need not be compact (proper). But compactness of moduli spaces is essential to define counting invariants unchanged under deformations.

To get round these, we enlarge from Vect(X) to the category coh(X) of *coherent sheaves* on X. Intuitively, you can think of a coherent sheaf $\mathcal{E}$ as being like a holomorphic vector bundle on a (possibly singular) submanifold S in X.

Then coh(X) is an *abelian category*. That is, it has a good notion of *exact sequence* $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$, and every morphism $\phi : \mathcal{E} \to \mathcal{F}$ has a kernel and a cokernel. It is the smallest abelian category containing Vect(X).

Let X be a projective $\mathbb{K}$-scheme, and $\mathcal{E}, \mathcal{F} \in \mathrm{coh}(X)$. Then one can define the Ext *groups* $\mathrm{Ext}^i(\mathcal{E}, \mathcal{F})$ for $i = 0, 1, \ldots$. If X is smooth of dimension m with canonical bundle K_X, then $\mathrm{Ext}^i(\mathcal{E}, \mathcal{F})$ are finite-dimensional vector spaces over $\mathbb{K}$ for $i = 0, \ldots, m$, and are zero for $i > m$, and satisfy *Serre duality*

$$\mathrm{Ext}^i(\mathcal{F}, \mathcal{E}) \cong \mathrm{Ext}^{m-i}(\mathcal{E}, \mathcal{F} \otimes K_X)^*. \quad (1)$$

The groups $\mathrm{Ext}^0(\mathcal{E}, \mathcal{F}), \mathrm{Ext}^1(\mathcal{E}, \mathcal{F})$ have easy interpretations: $\mathrm{Ext}^0(\mathcal{E}, \mathcal{F}) = \mathrm{Hom}(\mathcal{E}, \mathcal{F})$, and elements of $\mathrm{Ext}^1(\mathcal{E}, \mathcal{F})$ classify exact sequences $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{E} \to 0$ in $\mathrm{coh}(X)$, that is, extensions of $\mathcal{E}$ by $\mathcal{F}$.

The *Euler form* of $\mathcal{E},\mathcal{F}$ is

$$\bar{\chi}(\mathcal{E},\mathcal{F})=\sum_{i=0}^{m}(-1)^i\dim_{\mathbb{K}}\mathrm{Ext}^i(\mathcal{E},\mathcal{F}).$$

By the Hirzebruch–Riemann–Roch Theorem, it is given by

$$\bar{\chi}(\mathcal{E},\mathcal{F})=\deg\bigl(\mathrm{ch}(\mathcal{E})^\vee\cdot\mathrm{ch}(\mathcal{F})\cdot\mathrm{td}(TX)\bigr)_m,$$

where $\mathrm{ch}(\mathcal{E})$ is the *Chern character* of $\mathcal{E}$, a topological invariant. The *Grothendieck group* $K_0(\mathrm{coh}(X))$ is the group generated by isomorphism classes $[\mathcal{E}]$ of $\mathcal{E}\in\mathrm{coh}(X)$ with the relation $[\mathcal{F}]=[\mathcal{E}]+[\mathcal{G}]$ for each exact sequence $0\to\mathcal{E}\to\mathcal{F}\to\mathcal{G}\to 0$. The *numerical Grothendieck group* $K(\mathrm{coh}(X))$ is the quotient of $K_0(\mathrm{coh}(X))$ by the subgroup of $[\mathcal{E}]$ with $\bar{\chi}(\mathcal{E},\mathcal{F})=0$ for all $\mathcal{F}\in\mathrm{coh}(X)$. Then $\bar{\chi}$ descends to $K(\mathrm{coh}(X))$, i.e. to $\bar{\chi}([E],[F])$. The Chern character embeds $\mathrm{ch}:K(\mathrm{coh}(X))\hookrightarrow H^{\mathrm{even}}(X;\mathbb{Q})$.

Now suppose X is a Calabi–Yau 3-fold, $K_X \cong \mathcal{O}_X$. Then Serre duality becomes $\mathrm{Ext}^i(\mathcal{F},\mathcal{E}) \cong \mathrm{Ext}^{3-i}(\mathcal{E},\mathcal{F})^*$. Hence:

• The Euler form $\bar\chi$ is antisymmetric.

• The full Ext groups are determined solely by $\mathrm{Ext}^0 = \mathrm{Hom}$ and Ext^1.

• For $\mathcal{E},\mathcal{F} \in \mathrm{coh}(X)$, we have

$$\begin{aligned} &\dim \mathrm{Hom}(\mathcal{E},\mathcal{F}) - \dim \mathrm{Ext}^1(\mathcal{E},\mathcal{F}) \\ &- \dim \mathrm{Hom}(\mathcal{F},\mathcal{E}) + \dim \mathrm{Ext}^1(\mathcal{F},\mathcal{E}) = \bar\chi([\mathcal{E}],[\mathcal{F}]). \end{aligned} \tag{2}$$

This means that in some ways, coherent sheaves on a Calabi–Yau 3-fold behave like a 1-dimensional category (like coherent sheaves on a curve), not a 3-dimensional category, as we have only $\mathrm{Ext}^0, \mathrm{Ext}^1$ to worry about.

Aside: Starting with the abelian category coh(X), one forms the (*bounded*) *derived category* $D(X) = D^b(\text{coh}(X))$, a triangulated category, which contains coh(X) as a subcategory. The Ext groups satisfy $\text{Ext}^i(\mathcal{E},\mathcal{F}) = \text{Hom}_{D(X)}(\mathcal{E},\mathcal{F}[i])$, where $[i]$ is shift by i. Derived categories appear in Kontsevich's Homological Mirror Symmetry Conjecture. It is an important problem to extend Donaldson–Thomas theory from coh(X) to $D^b(\text{coh}(X))$.

3. Donaldson–Thomas invariants

Let X be a Calabi–Yau 3-fold, and $\mathcal{L}$ an ample line bundle on X. This induces a notion of *Gieseker stability* on coh(X). Write τ for the stability condition coming from $\mathcal{L}$. It depends on $\mathcal{L}$, so a different ample line bundle $\tilde{\mathcal{L}}$ induces a different stability condition $\tilde\tau$. Given $\alpha \in K(X)$, we can form the moduli spaces $\mathcal{M}^\alpha_{\rm st}(\tau), \mathcal{M}^\alpha_{\rm ss}(\tau)$ of τ-(semi)stable sheaves E in coh(X) with $[E] = \alpha$ in $K(X)$. We can regard these as *coarse moduli schemes*, or as *Artin stacks*. *Donaldson–Thomas invariants* $DT^\alpha(\tau)$ are $\mathbb{Z}$-valued invariants 'counting' τ-(semi)stable sheaves in class $\alpha \in K(X)$.

Milestones in D–T theory

- Thomas (1998): defined $DT^\alpha(\tau) \in \mathbb{Z}$, proved deformation-invariant.
- MNOP (2003): conjectured relation between $DT^\alpha(\tau)$ for α of rank 1 and Gromov–Witten invariants $GW_g(\alpha)$ of X.
- Behrend (2005): wrote $DT^\alpha(\tau)$ as a weighted Euler characteristic $\int_{\mathcal{M}_{\mathrm{st}}^\alpha(\tau)} \nu \, \mathrm{d}\chi$, where ν is the 'Behrend function'.
- Joyce–Song (2008): defined *generalized D–T invariants* $\bar{DT}^\alpha(\tau) \in \mathbb{Q}$, proved wall-crossing formula under change of τ and deformation-invariance.
- Kontsevich–Soibelman (2008): defined *motivic D–T invariants*, with wall-crossing (depends on conjectures).
- Kontsevich–Soibelman (2010): defined *cohomological Hall algebras*, categorified D-T theory (depends on conjectures).

3.1 Thomas math.AG/9806111

Form the coarse moduli schemes $\mathcal{M}_{\rm st}^\alpha(\tau)$, $\mathcal{M}_{\rm ss}^\alpha(\tau)$ of τ-(semi)stable $E\in\mathrm{coh}(X)$ with $[E]=\alpha$ in $K(X)$. Then $\mathcal{M}_{\rm ss}^\alpha(\tau)$ is a proper projective $\mathbb{K}$-scheme, and $\mathcal{M}_{\rm st}^\alpha(\tau)$ is an open subscheme. Thomas showed that $\mathcal{M}_{\rm st}^\alpha(\tau)$ has a (*symmetric*) *obstruction theory*. When $\mathcal{M}_{\rm st}^\alpha(\tau)=\mathcal{M}_{\rm ss}^\alpha(\tau)$, i.e., when there are no strictly semistable sheaves in class α, $\mathcal{M}_{\rm st}^\alpha(\tau)$ is also proper. Using the obstruction theory, one defines a *virtual class* $[\mathcal{M}_{\rm st}^\alpha(\tau)]^{\rm vir}$ in $A_0(\mathcal{M}_{\rm st}^\alpha(\tau))$, and sets

$$DT^\alpha(\tau)=\int_{[\mathcal{M}_{\rm st}^\alpha(\tau)]^{\rm vir}}1\in\mathbb{Z}.$$

Thomas proved $DT^\alpha(\tau)$ is unchanged by continuous deformations of X, that is, it is independent of the complex structure J of X up to deformation. This is a strong statement, as deforming X can change $\mathrm{coh}(X)$ and $\mathcal{M}_{\rm st}^\alpha(\tau)$ radically.

3.2 MNOP math.AG/0312059

Let $\mathcal{E}$ be a torsion-free coherent sheaf of rank 1 on X. The reflexive hull $(\mathcal{E}^\vee)^\vee$ is a line bundle, and $\mathcal{E} \hookrightarrow (\mathcal{E}^\vee)^\vee$ is an isomorphism except in codimension $\geqslant 2$. Line bundles on X are classified by $H^2(X;\mathbb{Z})$. So we may as well take $(\mathcal{E}^\vee)^\vee = \mathcal{O}_X$. Then $\mathcal{E}$ is a subsheaf of $\mathcal{O}_X$, the *ideal sheaf* $\mathcal{I}_S$ of a subscheme S of X with $\dim S \leqslant 1$.
In rank 1, stable=semistable=torsion-free. So when $\operatorname{rank}\alpha = 1$, $DT^\alpha(\tau)$ is defined and 'counts' ideal sheaves $\mathcal{I}_S$ for $\dim S \leqslant 1$ – roughly, counts curves in X with a given homology class and genus. The *MNOP Conjecture* relates $DT^\alpha(\tau)$ for α of rank 1 with the Gromov–Witten invariants $GW_g(\beta)$ of X. It is still unproven.

Aside: higher rank D–T invariants

One evil thing MNOP did is make most of the world believe that all D–T invariants do is count ideal sheaves. But in fact D–T invariants count vector bundles and coherent sheaves of any rank.

Very little is known about the 'meaning' of higher rank D–T invariants.

Questions: What information is contained in the system of invariants $\bar{DT}^\alpha(\tau)$ for all α, especially when rank $\alpha > 1$?

Do they depend on more than just $\chi(X)$ and the Gromov–Witten invariants $GW_g(\alpha)$?

Is there some rank d such that all $\bar{DT}^\alpha(\tau)$ for rank $\alpha > d$ may be written in terms of $\bar{DT}^\beta(\tau)$ for rank $\beta \leqslant d$? (True (?) for Donaldson invariants with $d = 2$.)

3.3 Behrend math.AG/0507523

Kai Behrend showed that $DT^\alpha(\tau)$ is a *weighted Euler characteristic*

$$DT^\alpha(\tau)=\int_{\mathcal{M}_{\rm st}^\alpha(\tau)}\nu\,{\rm d}\chi, \qquad (3)$$

where ν is the 'Behrend function', a $\mathbb{Z}$-valued constructible function on $\mathcal{M}_{\rm st}^\alpha(\tau)$ depending only on the scheme structure of $\mathcal{M}_{\rm st}^\alpha(\tau)$. We think of ν as a *multiplicity function*. If $\mathcal{M}_{\rm st}^\alpha(\tau)$ is a k-fold point $\mathrm{Spec}\,\mathbb{C}[z]/(z^k)$ then $\nu\equiv k$. If $\mathcal{M}_{\rm st}^\alpha(\tau)$ is smooth of dimension d then $\nu\equiv(-1)^d$. Suppose U is a complex manifold, $f:U\rightarrow\mathbb{C}$ is holomorphic, and $\mathfrak{M}$ is a $\mathbb{C}$-scheme locally isomorphic to $\mathrm{Crit}(f)$ as a complex analytic space. Then

$$\nu_{\mathfrak{M}}(x)=(-1)^{\dim U}\bigl(1-\chi(MF_f(x))\bigr), \quad (4)$$

with $MF_f(x)$ the *Milnor fibre* of f at x.

3.4 Joyce–Song arXiv/0810.5645

Joyce–Song defined *generalized D–T invariants* $\bar{DT}^\alpha(\tau)$ in $\mathbb{Q}$, defined for all $\alpha \in K(X)$, such that

• $\bar{DT}^\alpha(\tau)$ is unchanged by deformations of the underlying Calabi–Yau 3-fold.

• If $\mathcal{M}_{\rm st}^\alpha(\tau) = \mathcal{M}_{\rm ss}^\alpha(\tau)$ then $\bar{DT}^\alpha(\tau) = DT^\alpha(\tau)$.

• The $\bar{DT}^\alpha(\tau)$ transform according to a known transformation law under change of stability condition, of the form

$$\bar{DT}^\alpha(\tilde\tau) = \sum_{\substack{\text{iso. classes}\\ \text{of } \Gamma, I, \kappa}} \pm\, U(\Gamma, I, \kappa; \tau, \tilde\tau)\cdot \prod_{i\in I} \bar{DT}^{\kappa(i)}(\tau)\cdot \prod_{\substack{\text{edges}\\ i - j \text{ in } \Gamma}} \bar\chi(\kappa(i), \kappa(j)). \qquad (5)$$

Here Γ is a connected, simply-connected undirected graph with vertices I, $\kappa : I \to K(X)$ has $\sum_{i\in I}\kappa(i) = \alpha$, and $U(\Gamma, I, \kappa; \tau, \tilde\tau)$ in $\mathbb{Q}$ are explicit combinatorial coefficients.

The $\bar{DT}^\alpha(\tau)$ lie in $\mathbb{Q}$ rather than $\mathbb{Z}$ because strictly semistable sheaves E must be 'counted' with rational weights.

Suppose E is stable and rigid in class α. Then $kE = E\oplus\cdots\oplus E$ is strictly semistable in class $k\alpha$, for $k \geqslant 2$. Calculations show that E contributes 1 to $\bar{DT}^\alpha(\tau)$, and kE contributes $1/k^2$ to $\bar{DT}^{k\alpha}(\tau)$.

Define new invariants $\hat{DT}^\alpha(\tau) \in \mathbb{Q}$ by

$$\bar{DT}^\alpha(\tau) = \sum_{k\geqslant 1:k \text{ divides } \alpha} \frac{1}{k^2}\hat{DT}^{\alpha/k}(\tau). \quad (6)$$

Then the kE for $k \geqslant 1$ above contribute 1 to $\hat{DT}^\alpha(\tau)$ and 0 to $\hat{DT}^{k\alpha}(\tau)$ for $k > 1$.

Conjecture. *Suppose τ is generic, in the sense that $\tau(\alpha) = \tau(\beta)$ implies $\bar{\chi}(\alpha,\beta) = 0$. Then $\hat{DT}^\alpha(\tau) \in \mathbb{Z}$ for all $\alpha \in K(X)$.*

The $\hat{DT}^\alpha(\tau)$ may be interpreted as 'numbers of BPS states' in String Theory.

Aside: counting special Lagrangians.

Let X be a Calabi–Yau 3-fold with 'mirror' Calabi–Yau $\hat{X}$. Then mirror symmetry is supposed to identify $D^b(\mathrm{coh}(X))$ with $D^b\mathrm{Fuk}(\hat{X})$, where $\mathrm{Fuk}(\hat{X})$ is the Fukaya category of Lagrangians in $\hat{X}$. (Semi)stable coherent sheaves in $D^b(\mathrm{coh}(X))$ are expected to be identified with special Lagrangians in $D^b\mathrm{Fuk}(\hat{X})$. So we expect $\widehat{DT}^\alpha(\tau)$ to be identified with an invariant 'counting' special Lagrangians in $\hat{X}$.

I actually got into this subject from the mirror side: my work on singularities of special Lagrangians led me to conjecture the existence of an interesting invariant ‘counting’ special Lagrangians (see hep-th/9907013), presumably mirror to D–T invariants, though I didn’t know that then. The shape of the wall-crossing formula (5) as a sum over trees Γ is motivated from the special Lagrangian side: given a special Lagrangian $\mathbb{Q}$-homology sphere $\hat{L}$ in $\hat{X}$, as you deform the complex structure $\hat{J}$, you expect $\hat{L}$ to break up into a tree of special Lagrangian $\mathbb{Q}$-homology spheres.

Outstanding problems from Joyce–Song

- Extend field from $\mathbb{C}$ to $\mathbb{K}$ algebraically closed of characteristic 0 – now done, with Vittoria Bussi. Includes a strictly algebraic proof of the 'Behrend function identities', proved in Joyce–Song using gauge theory.
- Prove $\hat{DT}^\alpha(\tau) \in \mathbb{Z}$ for generic τ.
- Extend from coh(X) to $D^b($coh$(X))$. Combine methods of Bussi and Huybrechts–Thomas to prove 'Behrend function identities' for $D^b($coh$(X))$?
- Prove that can write $\mathcal{M}^\alpha_{\mathrm{st}}(\tau)$ globally in the form Crit(f) for $f : U \to \mathbb{C}$ a holomorphic function on a complex manifold U. Done near a point in $\mathcal{M}^\alpha_{\mathrm{st}}(\tau)$ in J–S.

3.5. Kontsevich–Soibelman arXiv/0811.2435

Behrend showed that D–T invariants are weighted Euler characteristics $DT^\alpha(\tau) = \chi\bigl(\mathcal{M}_{\rm st}^\alpha(\tau),\nu\bigr)$. Similarly, Joyce–Song use Euler characteristics to do the actual 'counting' to define the invariants $\bar{DT}^\alpha(\tau), \hat{DT}^\alpha(\tau)$.

The *Euler characteristic* χ is defined for finite type schemes X over $\mathbb{C}$, or general fields $\mathbb{K}$ (usually characteristic 0). It satisfies $\chi(X\times Y)=\chi(X)\chi(Y)$, and if $Z\subseteq Y$ is closed then $\chi(Y)=\chi(Y\setminus Z)+\chi(Z)$.

Define the Grothendieck group $K(\mathrm{Sch}_{\mathbb{C}})$ to be the commutative ring generated by isomorphism classes $[X]$ of finite type $\mathbb{C}$-schemes X, with relations $[Y]=[Y\setminus Z]+[Z]$ if $Z\subseteq Y$ is closed, and with multiplication $[X]\cdot[Y]=[X\times Y]$.

Then the Euler characteristic induces a ring homomorphism $\chi : K(\mathsf{Sch}_{\mathbb{C}}) \to \mathbb{Z}$. It is an example of a *motivic invariant* of $\mathbb{C}$-schemes. A general motivic invariant is a map $\Upsilon : \{\mathbb{C}-\text{schemes}\} \to R$ for a commutative ring R, which factors through a ring morphism $K(\mathsf{Sch}_{\mathbb{C}}) \to R$. Examples are (virtual) Poincaré polynomials, (virtual) Hodge polynomials, and the universal motivic invariant with $R = K(\mathsf{Sch}_{\mathbb{C}})$ and $\Upsilon(X) = [X]$.

Kontsevich and Soibelman outlined a very general version of D–T theory, with lots of exciting new ideas in it. One aspect was that they wanted to define ‘motivic’ D–T invariants in which Euler characteristics are replaced by another motivic invariant of $\mathbb{C}$-schemes Υ, so that roughly we have

$$DT^\alpha_{\text{mot}}(\tau) = \Upsilon\big(\mathcal{M}^\alpha_{\text{st}}(\tau), \nu_{\text{mot}}\big), \quad (7)$$

where ν_{mot} is a ‘motivic Behrend function’ – very roughly a constructible function $\nu_{\text{mot}} : \mathcal{M}^\alpha_{\text{st}}(\tau) \to R$, but actually more complicated. So, if $\Upsilon(X)$ is the virtual Poincaré polynomial $P_X(t)$, with $P_X(-1) = \chi(X)$, then $DT^\alpha_{\text{mot}}(\tau)$ would be a polynomial in t, with $DT^\alpha_{\text{mot}}(\tau)(-1) = DT^\alpha(\tau)$. They are refinements of ordinary D–T invariants, containing more information.

Working over motivic invariants has advantages and disadvantages. An advantage: suppose Υ is a motivic invariant of $\mathbb{C}$-schemes with values in a ring R, and $\Upsilon(\mathrm{GL}(n,\mathbb{C}))$ is invertible in R for all $n = 1, 2, \ldots$. This holds for instance if Υ is virtual Poincaré polynomials and $R = \mathbb{Q}(t)$ is rational functions in t.

Now let $\mathfrak{R}$ be a finite type Artin $\mathbb{C}$-stack (with affine geometric stabilizers). We can write $\mathfrak{R} = \coprod_{i=1}^{k} \mathfrak{R}_i$ for $\mathfrak{R}_1, \ldots, \mathfrak{R}_k$ locally closed substacks with $\mathfrak{R}_i \cong [X_i / \mathrm{GL}(n_i, \mathbb{C})]$ for finite type $\mathbb{C}$-schemes X_i. Define

$$\Upsilon(\mathfrak{R}) = \sum_{i=1}^{k} \frac{\Upsilon(X_i)}{\Upsilon(\mathrm{GL}(n,\mathbb{C}))}.$$

This is a well-behaved extension of Υ to Artin stacks. It does not work for Euler characteristics, as you divide by zero.

This is very helpful: to 'count' strictly semistables correctly, one should work not with the coarse moduli scheme $\mathcal{M}_{\mathrm{ss}}^{\alpha}(\tau)$, but with the moduli stack $\mathfrak{M}_{\mathrm{ss}}^{\alpha}(\tau)$. The Euler characteristic $\chi(\mathfrak{M}_{\mathrm{ss}}^{\alpha}(\tau))$ is undefined, and getting round this causes a lot of work in Joyce–Song.

A disadvantage: suppose $\phi : X \to Y$ is an étale locally trivial fibration of $\mathbb{C}$-schemes, with fibre F. Euler characteristics satisfy $\chi(X) = \chi(Y)\chi(F)$, but other motivic invariants Υ do not have $\Upsilon(X) = \Upsilon(Y)\Upsilon(F)$. Because of this, with Euler characteristics, one works with constructible functions $\mathrm{CF}(Y)$, but with motivic invariants one needs 'stack functions' $K_0(\mathrm{Sch}_Y)\otimes_{\mathbb{Z}}R$.

Other features of Kontsevich–Soibelman:

• motivic D–T invariants $DT^{\alpha}_{\text{mot}}(\tau)$ will not be deformation-invariant in general.

• rather than working in the abelian category coh(X) with Gieseker stability, as in Joyce–Song, they worked in the derived category $D^b(\text{coh}(X))$ with Bridgeland stability conditions (no examples of Bridgeland stability conditions on $D^b(\text{coh}(X))$ for X compact C–Y 3-fold currently known).

• K–S have their own wall-crossing formula, essentially equivalent to J–S (5), but more popular with general public.

• most results depend on conjectures, not yet proved.

Outstanding problems from Kontsevich–Soibelman 2008:

There are many, but one area I'm interested in is to define 'motivic Behrend functions' rigorously, and prove they satisfy suitable 'motivic Behrend function identities' needed to make the K–S integration map an algebra morphism (more later). Two contexts for this:

(a) quiver with (polynomial) superpotential – toy model, but still difficult?

(b) compact C–Y 3-fold, using (algebraic) almost closed 1-form methods rather than superpotentials.

3.6. Kontsevich–Soibelman arXiv/1006.2706

One direction in which to generalize D–T theory is to make it motivic, as in K–S 2008. A second direction is to *categorify* it. The basic idea is this. Consider for simplicity a D–T moduli space $\mathcal{M}_{\mathrm{st}}^{\alpha}(\tau)$ with $\mathcal{M}_{\mathrm{st}}^{\alpha}(\tau)=\mathcal{M}_{\mathrm{ss}}^{\alpha}(\tau)$. Then $DT^{\alpha}(\tau)=\chi\bigl(\mathcal{M}_{\mathrm{st}}^{\alpha}(\tau),\nu\bigr)$ is a kind of generalized Euler characteristic of $\mathcal{M}_{\mathrm{st}}^{\alpha}(\tau)$.

Now $\chi(X)=\sum_{i=0}^{\dim X}(-1)^i\dim_{\mathbb{C}}H^i(X;\mathbb{C})$. So it seems reasonable that there might exist some natural 'generalized cohomology' $H^*_{\mathrm{gen}}\bigl(\mathcal{M}_{\mathrm{st}}^{\alpha}(\tau),\mathbb{C}\bigr)$ of $\mathcal{M}_{\mathrm{st}}^{\alpha}(\tau)$, such that

$$DT^{\alpha}(\tau)=\sum_{i\geqslant 0}(-1)^i\dim_{\mathbb{C}}H^i_{\mathrm{gen}}\bigl(\mathcal{M}_{\mathrm{st}}^{\alpha}(\tau),\mathbb{C}\bigr).$$

This process is called 'categorification', as we are moving one place rightwards in the sequence: integers, vector spaces, categories, 2-categories, ...
Here are some reasons to believe that such a 'generalized cohomology' $H^*_{\text{gen}}\bigl(\mathcal{M}^\alpha_{\text{st}}(\tau),\mathbb{C}\bigr)$ might exist, and be important:

- in String Theory, there is a notion of 'algebras of BPS states' – these are a chunk of the String Theory, not yet mathematically defined, but at least a graded vector space and perhaps an associative algebra. Their graded dimensions are 'numbers of BPS states'. The integer invariants $DT^\alpha(\tau)$, $\hat{DT}{}^\alpha(\tau)$ are interpreted as 'numbers of BPS states'.

• exotic cohomology theories – 'Floer cohomology' – occur in several areas of geometry connected to moduli spaces and to physics. One such is the instanton Floer homology $HF^*(Y)$ of compact 3-manifolds Y (must be a $\mathbb{Q}$-homology sphere?). This satisfies $\sum_i (-1)^i \dim HF^i(Y) = \mathrm{Cass}(Y)$, where $\mathrm{Cass}(Y)$ is the *Casson invariant* of Y. But D–T invariants are based on Casson invariants, and were originally called 'holomorphic Casson invariants'.

• A heuristic argument of Richard Thomas says we can regard $\mathcal{M}_{\rm st}^{\alpha}(\tau)$ (at least for moduli spaces of vector bundles E) as the critical locus of the *holomorphic Chern–Simons functional* CS, a holomorphic function on an infinite-dimensional complex manifold of connections on E.
Using this, Joyce–Song proved that $\mathcal{M}_{\rm st}^{\alpha}(\tau)$ is isomorphic near a point $[E]$ as a complex analytic space to Crit(f) for $f : U \to \mathbb{C}$ a holomorphic function on a finite-dimensional complex manifold.

If you have a holomorphic function $f : U \to \mathbb{C}$ for U a complex manifold, there (at least) are two ways (morally equivalent?) in which you can make a 'generalized cohomology theory' $H^*_{\text{gen}}(U, f)$:

(a) there is a *perverse sheaf of vanishing cycles* Q on U, supported on $\text{Crit}(f)$. The hypercohomology $\mathbb{H}^*(Q)$ of Q is the generalized cohomology we want. Think of perverse sheaves as a 'categorification' of constructible functions, and Q as a categorification of the Behrend function $\nu_{\text{Crit}(f)}$.

(b) Think of $\operatorname{Re} f : U \to \mathbb{R}$ as a smooth function, and let $\tilde{f} : U \to \mathbb{R}$ be a small perturbation of $\operatorname{Re} f$ which is a Morse function. Then compute the Morse homology $H^*_{\tilde{f}}(U;\mathbb{C})$ of U with respect to $\tilde{f}$. Note that we expect U to be noncompact, so $H^*_{\tilde{f}}(U;\mathbb{C})$ is not just $H^*(U;\mathbb{C})$, but depends on $\tilde{f}$. It also may not be defined in general, as one needs compactness properties for gradient flow lines of $\tilde{f}$. But these should hold if $\tilde{f}$ perturbs $\operatorname{Re} f$, f holomorphic.

Floer homology theories are motivated in exactly this way, as computing Morse homology of infinite-dimensional manifolds of connections/submanifolds etc., w.r.t. gradient flow of an interesting functional.

Aside: Donaldson and Segal, *Gauge theory and higher dimensions II*, arXiv:0902.3239, 2009, roughly speaking contains a proposal to categorify D–T invariants counting holomorphic vector bundles on a C–Y 3-fold X by a similar method to instanton Floer homology of 3-manifolds, using G_2 geometry on $X \times \mathbb{R}$.
Kai Behrend (unpublished) also has a programme aimed at categorifying D–T invariants.

There is a lot in K–S 2010. One strand of it aims to define some kind of generalized cohomology $H^*_{\rm gen}\bigl(\mathcal{M}^\alpha_{\rm st}(\tau),\mathbb{C}\bigr)$, or better $H^*_{\rm gen}\bigl(\mathfrak{M},\mathbb{C}\bigr)$ for $\mathfrak{M}$ the moduli stack of coherent sheaves on the C–Y 3-fold X, with an associative (or twisted associative) multiplication, and such that

$$DT^\alpha(\tau)=\sum_{i\geqslant 0}(-1)^i\dim_{\mathbb{C}}H^i_{\rm gen}\bigl(\mathcal{M}^\alpha_{\rm st}(\tau),\mathbb{C}\bigr).$$

It is worked out for quivers without and with potential, rather than C–Y 3-folds. We think of $H^*_{\rm gen}\bigl(\mathfrak{M},\mathbb{C}\bigr)$ as a kind of *Ringel–Hall algebra*, in a sense I'll explain later. When $\mathfrak{M}$ is a smooth stack, $H^*_{\rm gen}\bigl(\mathfrak{M},\mathbb{C}\bigr)$ is the usual *stack cohomology* $H^*_{\rm sta}\bigl(\mathfrak{M},\mathbb{C}\bigr)$, given by $H^*_{\rm sta}\bigl([X/G],\mathbb{C}\bigr)=H^*_G(X;\mathbb{C})$ for a quotient stack $[X/G]$, where $H^*_G(X;\mathbb{C})$ is the G-equivariant cohomology of X.

4. Ringel–Hall algebras

A fundamental part of Joyce–Song and Kontsevich–Soibelman 2008 and 2010 is the use of *Ringel–Hall algebras*. This is a general method for constructing an associative algebra $H(\mathcal{A})$ from an abelian category $\mathcal{A}$.

We first explain the idea over finite fields. Suppose $\mathcal{A}$ is an abelian category over the finite field $\mathbb{F}_q$, and $\mathrm{Hom}(\mathcal{E},\mathcal{F}), \mathrm{Ext}^1(\mathcal{E},\mathcal{F})$ are finite-dimensional over $\mathbb{F}_q$ for all $\mathcal{E},\mathcal{F}\in\mathcal{A}$ and $i \geqslant 0$ (and so finite). Define $H(\mathcal{A})$ to be the $\mathbb{Q}$-vector space spanned by isomorphism classes $[\mathcal{E}]$ of objects $\mathcal{E}$ in $\mathcal{A}$. Define a $\mathbb{Q}$-bilinear product $*$ on $H(\mathcal{A})$ by

$$[\mathcal{E}]*[\mathcal{G}] = \sum_{\substack{\text{iso. classes of}\\ \text{exact sequences}\\ 0\rightarrow\mathcal{E}\xrightarrow{\alpha}\mathcal{F}\xrightarrow{\beta}\mathcal{G}\rightarrow 0}} \frac{1}{\left|\begin{smallmatrix}\{\gamma:\mathcal{F}\rightarrow\mathcal{F}:\\ \gamma\circ\alpha=\alpha,\\ \beta\circ\gamma=\beta\}\end{smallmatrix}\right|}\cdot[\mathcal{F}].$$

This is a finite sum as $\mathrm{Ext}^1(\mathcal{G},\mathcal{E})$ is finite, and the fraction is well-defined as $\mathrm{Hom}(\mathcal{F},\mathcal{F})$ is finite.
The important point is that $*$ is *associative*. To see this, note that

$$([\mathcal{E}_1]*[\mathcal{E}_2])*[\mathcal{E}_3]=[\mathcal{E}_1]*([\mathcal{E}_2]*[\mathcal{E}_3])$$
$$=\sum_{[\mathcal{F}]}\frac{n_{\mathcal{F}}\prod_{i=1}^3|\mathrm{Aut}(\mathcal{E}_i)|}{|\mathrm{Aut}(\mathcal{F})|}\cdot[\mathcal{F}],$$

where $n_{\mathcal{F}}$ is the number of filtrations $0=\mathcal{F}_0\subset\mathcal{F}_1\subset\mathcal{F}_2\subset\mathcal{F}_3=\mathcal{F}$ with $\mathcal{F}_i/\mathcal{F}_{i-1}\cong\mathcal{E}_i$ for $i=1,2,3$. So $H(\mathcal{A})$ is an associative algebra, with identity [0].
For $\mathcal{A}$ in which $\mathrm{Hom}(\mathcal{E},\mathcal{F}),\mathrm{Ext}^1(\mathcal{E},\mathcal{F})$ are not finite for all $\mathcal{E},\mathcal{F}$, we must adapt this construction. We need $H(\mathcal{A})$ to be spanned not just by isomorphism classes $[\mathcal{E}]$ of $\mathcal{E}\in\mathcal{A}$, but by 'families of objects in $\mathcal{A}$', and the 'counting' of exact sequences should be done using some motivic invariant.

Here is a version of Ringel–Hall algebras which works in great generality. Let $\mathcal{A}$ be an abelian category in which one can 'do algebraic geometry', i.e. we have a moduli stack $\mathfrak{M}_{\mathcal{A}}$ of objects in $\mathcal{A}$, an Artin $\mathbb{K}$-stack locally of finite type. E.g. $\mathcal{A} = \mathrm{coh}(X)$ for X a projective $\mathbb{K}$-scheme, $\mathcal{A} = \text{mod-}\mathbb{K}Q/I$, (Q, I) quiver with relations.
We then define the vector space of 'stack functions' $\mathrm{SF}(\mathcal{M}_{\mathcal{A}})$ on $\mathfrak{M}_{\mathcal{A}}$. This is generated by equivalence classes $[\mathfrak{R}, \phi]$ of morphisms $\phi : \mathfrak{R} \to \mathfrak{M}_{\mathcal{A}}$ for $\mathfrak{R}$ a finite type Artin $\mathbb{K}$-stack, with motivic relation

$$[\mathfrak{R}, \phi] = [\mathfrak{R} \setminus \mathfrak{S}, \phi|_{\mathfrak{R}\setminus\mathfrak{S}}] + [\mathfrak{S}, \phi|_{\mathfrak{S}}]$$

when $\mathfrak{S}$ is a closed substack of $\mathfrak{R}$.

Think of stack functions $\mathrm{SF}(\mathcal{M}_{\mathcal{A}})$ as a generalization of constructible functions $\mathrm{CF}(\mathcal{M}_{\mathcal{A}})$ on $\mathcal{M}_{\mathcal{A}}$. Using the short exact sequences $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ in $\mathcal{A}$, we define an associative, noncommutative multiplication $*$ on $\mathrm{SF}(\mathcal{M}_{\mathcal{A}})$, making $\mathrm{SF}(\mathcal{M}_{\mathcal{A}})$ into a 'Ringel–Hall algebra'.
Joyce also defines a Lie subalgebra $\mathrm{SF}^{\mathrm{ind}}(\mathcal{M}_{\mathcal{A}})$ of 'indecomposable' stack functions in $\mathrm{SF}(\mathcal{M}_{\mathcal{A}})$, which is closed under the Lie bracket $[f,g]=f*g-g*f$, but not under the Ringel–Hall product $*$. Roughly, $\mathrm{SF}^{\mathrm{ind}}(\mathcal{M}_{\mathcal{A}})$ is generated by $[\mathfrak{R},\phi]$ in which $\mathfrak{R}$ has stabilizer groups of rank 1.
Suppose τ is a stability condition on $\mathcal{A}$ such that the open substacks $\mathfrak{M}^{\alpha}_{\mathrm{ss}}(\tau)$ of τ-semistable objects in $\mathcal{A}$ in class $\alpha \in K(\mathcal{A})$ are of finite type. Then we have elements $\delta^{\alpha}_{\mathrm{ss}}(\tau)=\bigl[\mathfrak{M}^{\alpha}_{\mathrm{ss}}(\tau),\mathrm{inc}\bigr]$ in $\mathrm{SF}(\mathcal{M}_{\mathcal{A}})$, thought of as the characteristic function of $\mathfrak{M}^{\alpha}_{\mathrm{ss}}(\tau)$.

The existence of unique Harder–Narasimhan filtrations for the stability condition τ means that we have an identity in $\mathrm{SF}(\mathcal{M}_{\mathcal{A}})$:

$$1_{\mathfrak{M}_{\mathcal{A}}^{\alpha}} = \sum_{\substack{\alpha_1,\ldots,\alpha_n \in K(\mathcal{A})\setminus 0: \\ \alpha_1+\cdot+\alpha_n=\alpha, \\ \tau(\alpha_1)>\cdots>\tau(\alpha_n)}} \delta_{\mathrm{ss}}^{\alpha_1}(\tau) * \cdots \delta_{\mathrm{ss}}^{\alpha_n}(\tau). \quad (8)$$

If $\tau, \tilde{\tau}$ are different stability conditions, by inverting this we can write $\delta_{\mathrm{ss}}^{\alpha}(\tilde{\tau})$ as a sum of products of $\delta_{\mathrm{ss}}^{\beta}(\tau)$ in $\mathrm{SF}(\mathcal{M}_{\mathcal{A}})$. This gives a *wall-crossing formula* in $\mathrm{SF}(\mathcal{M}_{\mathcal{A}})$.
Joyce also defines $\epsilon^{\alpha}(\tau) \in \mathrm{SF}^{\mathrm{ind}}(\mathfrak{M}_{\mathcal{A}})$, essentially by taking log of the generating function of the $\delta_{\mathrm{ss}}^{\alpha}(\tau)$, and gives a wall-crossing formula for them using $[\,,\,]$.
To find out more than you want to know about this, read all of D. Joyce, *'Configurations in abelian categories. I,II,...,∞'*.

Note that $\mathrm{SF}(\mathcal{M}_{\mathcal{A}}), \mathrm{SF}^{\mathrm{ind}}(\mathcal{M}_{\mathcal{A}})$ are huge algebras, can't describe them explicitly.

Now for the interesting bit: suppose $\mathcal{A}$ is a 3-Calabi–Yau category, which means Serre duality $\mathrm{Ext}^i(\mathcal{F},\mathcal{E}) \cong \mathrm{Ext}^{3-i}(\mathcal{E},\mathcal{F})^*$ holds in $\mathcal{A}$, and maybe extra assumptions as well.

Then Joyce–Song define a Lie algebra morphism $\Phi : \mathrm{SF}^{\mathrm{ind}}(\mathcal{M}_{\mathcal{A}}) \to \mathcal{L}(\mathcal{A})$, where $\mathcal{L}(\mathcal{A})$ is the small, explicit Lie algebra over $\mathbb{Q}$ generated by symbols λ^α for $\alpha \in K(\mathcal{A})$, with $[\lambda^\alpha, \lambda^\beta] = (-1)^{\bar\chi(\alpha,\beta)}\bar\chi(\alpha,\beta)\lambda^{\alpha+\beta}$.

Similarly, given a motivic invariant Υ with values in a ring R in which $\Upsilon(\mathrm{GL}(n,\mathbb{K}))$ is invertible for $k \geqslant 1$, K–S 2008 'define' an algebra morphism $\Psi : \mathrm{SF}(\mathcal{M}_{\mathcal{A}}) \to Q(\mathcal{A},R)$, where $Q(\mathcal{A},R)$ is the R-algebra with basis of symbols ρ^α for $\alpha \in K(\mathcal{A})$, and multiplication $\rho^\alpha \star \rho^\beta = \mathbb{L}^{\frac{1}{2}\bar\chi(\alpha,\beta)}\rho^{\alpha+\beta}$.

We use these morphisms Φ, Ψ to define D–T invariants. J–S define $\bar{DT}^\alpha(\tau)$ by

$$\Phi(\epsilon^\alpha(\tau)) = -\bar{DT}^\alpha(\tau)\lambda^\alpha.$$

Kontsevich and Soibelman define

$$\Psi(\delta_{\rm ss}^\alpha(\tau)) = -DT_{\rm mot}^\alpha(\tau)\rho^\alpha.$$

Then the wall-crossing formulae for $\epsilon^\alpha(\tau)$, $\delta_{\rm ss}^\alpha(\tau)$ and Φ, Ψ (Lie) algebra morphisms gives wall-crossing formulae for $\bar{DT}^\alpha(\tau)\lambda^\alpha$, $DT_{\rm mot}^\alpha(\tau)\rho^\alpha$ in the (Lie) algebras $\mathcal{L}(\mathcal{A})$, $Q(\mathcal{A}, R)$, and hence wall-crossing formulae for $\bar{DT}^\alpha(\tau), DT_{\rm mot}^\alpha(\tau)$.

My morphism $\Phi : \mathrm{SF}^{\mathrm{ind}}(\mathcal{M}_{\mathcal{A}}) \to \mathcal{L}(\mathcal{A})$ essentially involves taking Euler characteristics weighted by the Behrend function ν of $\mathcal{M}_{\mathcal{A}}$. The fact that Φ is a Lie algebra morphism follows from two 'Behrend function identities'

$$\nu(E_1 \oplus E_2) = (-1)^{\bar\chi([E_1],[E_2])}\nu(E_1)\nu(E_2), \quad (9)$$

$$\begin{aligned}&\int_{\substack{[\lambda]\in\mathbb{P}(\mathrm{Ext}^1(E_2,E_1)):\\ \lambda\,\Leftrightarrow\, 0\to E_1\to F\to E_2\to 0}}\nu(F)\,\mathrm{d}\chi\\ &-\int_{\substack{[\lambda']\in\mathbb{P}(\mathrm{Ext}^1(E_1,E_2)):\\ \lambda'\,\Leftrightarrow\, 0\to E_2\to F'\to E_1\to 0}}\nu(F')\,\mathrm{d}\chi\\ &=\Big(\dim\mathrm{Ext}^1(E_2,E_1)-\dim\mathrm{Ext}^1(E_1,E_2)\Big)\\ &\qquad\nu_{\mathfrak{M}}(E_1\oplus E_2). \qquad (10)\end{aligned}$$

Similarly, the K–S morphism Ψ takes a motivic invariant Υ weighted by the (not fully defined) 'motivic Behrend function' ν_{mot}, and is an algebra morphism because of (unproved) 'motivic Behrend function identities'.

Aside: The proof of the Behrend function identities (9)–(10) in Joyce–Song involved first showing (using gauge theory) that we can write the moduli stack $\mathfrak{M}$ locally as $[\mathrm{Crit}(f)/G]$ for G a complex group, U a complex manifold acted on by G, and $f : U \to \mathbb{C}$ a G-equivariant holomorphic function. Then we used the formula

$$\nu_{\mathfrak{M}}(uG) = (-1)^{\dim U - \dim G}\bigl(1 - \chi(MF_f(u))\bigr)$$

for $u \in \mathrm{Crit}(f) \subseteq U$, where $MF_f(u)$ is the Milnor fibre of f at u. The gauge theory part is non-algebraic, and works only over the field $\mathbb{K} = \mathbb{C}$.

Now, Vittoria Bussi and I have found a new, algebraic proof of (9)–(10), using 'almost closed 1-forms', which works over $\mathbb{K}$ algebraically closed of characteristic zero. This suggests that almost closed 1-form methods may be useful in motivic D–T theory as in K–S 2008, as a substitute for their formal power series methods.

Question: Let ω be an algebraic almost closed 1-form on a smooth $\mathbb{K}$-scheme U. That is, ω is an algebraic 1-form on U, and $\mathrm{d}w \in I_\omega \cdot \Lambda^2 T^*U$, where I_ω is the ideal of functions vanishing on $\omega^{-1}(0)$.

Can you define a 'motivic Behrend function' ν_{mot} of $\omega^{-1}(0)$ with the expected properties? Can you prove 'motivic Behrend function identities' for it?

Another question: We could also ask whether almost closed 1-forms are enough to do categorified D–T theory, e.g. can you define a ‘perverse sheaf of vanishing cycles’ Q supported on $\omega^{-1}(0)$ for ω an almost closed 1-form, which agrees with the usual definition when $\omega = \mathrm{d}f$. My guess would be that you cannot, but I’d like to hear what people who know about perverse sheaves think.

You can think of stack functions $\mathrm{SF}(\mathfrak{M})$ as being a generalized cohomology theory of $\mathfrak{M}$, in that they are a functor $\{\text{stacks}\} \rightarrow \{\text{algebras}\}$ with the same kind of push-forwards, pullbacks etc. as a cohomology theory. So, we can imagine repeating the Ringel–Hall algebra construction, but replacing $\mathrm{SF}(\mathfrak{M})$ by some other 'generalized cohomology theory' $H^*_{\mathrm{gen}}(\mathfrak{M})$. This is the basic idea of Kontsevich and Soibelman 2010.

Questions: Suppose we apply some version of the K–S 2010 'cohomological Hall algebra' construction to simple categories $\mathcal{A}$ coming from algebra, e.g. representations of quivers with potential. Do we obtain algebras of interest in (higher) representation theory?

What can representation theory tell us about D–T theory? E.g. should generating functions of D–T invariants be characters of some interesting algebra of 'cohomological Hall algebra' type, and if so, can we use them to make predictions on properties of generating functions of D–T invariants, e.g. modularity?

5. Geometric structures on moduli spaces

Kinds of space used in complex algebraic geometry, in decreasing order of 'niceness':

- complex manifolds (very nice)
- varieties (nice)
- schemes (not bad): Thomas' $DT^\alpha(\tau)$.
- algebraic spaces (getting worse)
- Deligne–Mumford stacks (not nice)
- Artin stacks (horrible): our $\bar{D}T^\alpha(\tau)$.
- higher/derived stacks (deeply horrible)
- derived Artin (k,l)-stacks (yuck . . .)

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- d-manifolds and d-orbifolds (gorgeous)

– see `http://people.maths.ox.ac.uk/~joyce/dmanifolds.html`

An important issue in extending D–T theory is: what geometric structure do you put on moduli spaces $\mathcal{M}$ of coherent sheaves on a Calabi–Yau m-fold X?
The deformation theory of coherent sheaves E concerns by the Ext groups $\mathrm{Ext}^i(E,E)$. So one way to talk about different geometric structures on moduli spaces $\mathcal{M}$ is to ask what information they store about $\mathrm{Ext}^*(E,E)$ at each point $[E]$.
The *coarse moduli scheme* $\mathcal{M}_{\mathrm{st}}^\alpha(\tau)$ of *stable* sheaves E has $T_{[E]}\mathcal{M}_{\mathrm{st}}^\alpha(\tau)\cong\mathrm{Ext}^1(E,E)$. Although schemes do not remember Ext^0, for E stable $\mathrm{Ext}^0(E,E)=\mathbb{C}$ is standard. In the C–Y 3 case we then have $\mathrm{Ext}^2(E,E)\cong\mathrm{Ext}^1(E,E)^*$ and $\mathrm{Ext}^3(E,E)\cong\mathrm{Ext}^0(E,E)^*$. This recovers the whole of $\mathrm{Ext}^*(E,E)$ from $T_{[E]}\mathcal{M}_{\mathrm{st}}^\alpha(\tau)$. This is why for Thomas' $DT^\alpha(\tau)$ when $\mathcal{M}_{\mathrm{ss}}^\alpha(\tau)=\mathcal{M}_{\mathrm{st}}^\alpha(\tau)$, schemes are enough.

In fact Thomas works with more structure on $\mathcal{M} = \mathcal{M}^\alpha_{\mathrm{st}}(\tau)$: a (*symmetric*) *obstruction theory*. This is an object $E^\bullet$ in the derived category $D^b(\mathrm{coh}(\mathcal{M}))$ of coherent sheaves on $\mathcal{M}$, and a morphism $\phi : E^\bullet \to \mathbb{L}_{\mathcal{M}}$ in $D^b(\mathrm{coh}(\mathcal{M}))$, where $\mathbb{L}_{\mathcal{M}}$ is the cotangent complex of $\mathcal{M}$.

This is really a shadow of some deeper 'derived' geometry: morally speaking the singular scheme $\mathcal{M}$ is the classical part (1-category truncation) of a 'derived scheme' $\boldsymbol{\mathcal{M}}$, where $\boldsymbol{\mathcal{M}}$ is 'smooth' in a derived sense – a 'derived complex manifold'. There should be an inclusion $i : \mathcal{M} \hookrightarrow \boldsymbol{\mathcal{M}}$. Then $\phi : E^\bullet \to \mathbb{L}_{\mathcal{M}}$ is essentially $\mathrm{d}i : i^*(T^*\boldsymbol{\mathcal{M}}) \to T^*\mathcal{M}$ in a derived sense.

Behrend (2005) showed that when the obstruction theory is symmetric, the virtual class $DT^\alpha(\tau) = \chi\big(\mathcal{M}^\alpha_{\rm st}(\tau), \nu\big)$ depends only on the scheme structure $\mathcal{M}^\alpha_{\rm st}(\tau)$, not on the choice of symmetric obstruction theory. This corresponds to the fact that $T_{[E]}\mathcal{M}^\alpha_{\rm st}(\tau)$ determines $\mathrm{Ext}^*(E,E)$ in the C–Y 3 case.

For strictly semistables E, the coarse moduli scheme $\mathcal{M}^\alpha_{\rm ss}(\tau)$ is not a good model. In this case we can have $\mathrm{Ext}^0(E,E) \not\cong \mathbb{C}$, and $T_{[E]}\mathcal{M}^\alpha_{\rm ss}(\tau) \not\cong \mathrm{Ext}^1(E,E)$. So $\mathcal{M}^\alpha_{\rm ss}(\tau)$ tells you almost nothing about $\mathrm{Ext}^*(E,E)$. This is why Thomas' definition fails when $\mathcal{M}^\alpha_{\rm ss}(\tau) \neq \mathcal{M}^\alpha_{\rm st}(\tau)$.

Joyce–Song and Kontsevich–Soibelman (2008) worked principally with *Artin stacks*. If $\mathfrak{M}$ is the moduli stack of coherent sheaves E then each point $[E]$ in $\mathfrak{M}$ has a stabilizer group $\mathrm{Iso}_{\mathfrak{M}}(E) \cong \mathrm{Aut}(E)$, whose Lie algebra is $\mathrm{Ext}^0(E,E)$, and $T_{[E]}\mathfrak{M} \cong \mathrm{Ext}^1(E,E)$. So the stack $\mathfrak{M}$ knows about $\mathrm{Ext}^0(E,E), \mathrm{Ext}^1(E,E)$. In the C–Y 3 case $\mathrm{Ext}^2(E,E) \cong \mathrm{Ext}^1(E,E)^*$ and $\mathrm{Ext}^3(E,E) \cong \mathrm{Ext}^0(E,E)^*$, so $\mathfrak{M}$ determines $\mathrm{Ext}^*(E,E)$. This is one reason why J–S and K–S work.

Questions:

- What kind of geometric structure on moduli spaces of coherent sheaves on a C–Y 3-fold X is 'best' for doing motivic and categorified D–T theory? Are Artin stacks enough, or do we need some kind of derived stack?

In K–S 2008 we already need some extra structure on the moduli stack $\mathfrak{M}$ to make motivic D–T invariants, 'orientation data'.

J–S show that an atlas A for the moduli stack $\mathfrak{M}$ can locally be written in the form Crit(f) for U a complex manifold and $f : U \to \mathbb{C}$ holomorphic. That is, Crit(f) is the zeroes of a non-algebraic closed 1-form. Behrend shows that a scheme with a symmetric obstruction theory is locally the zeroes $\omega^{-1}(0)$ of an algebraic almost closed 1-form ω on a smooth scheme U. Conversely, given an algebraic almost closed 1-form ω on U, then $\omega^{-1}(0)$ has a symmetric obstruction theory. Maulik, Pandharipande and Thomas give examples of an almost closed 1-form ω such that $\omega^{-1}(0)$ is not locally isomorphic to Crit(f) for holomorphic f.

Hence, an atlas A for $\mathfrak{M}$ has a property – being locally of the form Crit(f) for holomorphic f – that is *stronger* than having a symmetric obstruction theory, but currently has no algebraic description.

Questions:

- Is there an *algebraic* condition on $\mathfrak{M}$, stronger than having a symmetric obstruction theory, which is equivalent to atlases A for $\mathfrak{M}$ being locally of the form Crit(f)? If so, can one give an algebraic proof that this condition holds?
- What structure on $\mathfrak{M}$ best reflects the fact that $\mathfrak{M}$ is morally of the form Crit(f) for f a holomorphic function on an infinite-dimensional manifold?

Questions:

• Is there an interesting generalization of D-T theory to Calabi-Yau m-folds for $m>3$? I do not think there will be *deformation-invariant* D–T style invariants when $m>3$. But it seems possible that there may be well-behaved 'motivic Behrend functions' and a K–S style 'integration map' which yield motivic D–T invariants when $m>3$ with nice wall-crossing formulae. We should probably be thinking in terms of derived stacks, since Artin moduli stacks would lose information about $\mathrm{Ext}^i(E,E)$ for $2\leqslant i\leqslant m-2$.