Donaldson–Thomas theory of Calabi–Yau 3-folds, and generalizations

Lecture 3 of 3: D-critical loci, perverse sheaves, and motives

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Based on papers arXiv:1211.3259, arXiv:1304.4508, arXiv:1305.6302, arXiv:1305.6428, arXiv:1312.0090, arXiv:1506.04024 by Ben-Bassat, Brav, Bussi, Dupont, Joyce, Meinhardt and Szendrői

Plan of talk:

6. D-critical loci and d-critical stacks

We will explain how to define 'd-critical loci' and 'd-critical stacks', classical truncations of -1-shifted symplectic derived schemes and stacks.

Theorem (Joyce arXiv:1304.4508)

Let X be a classical $\mathbb{K}$-scheme. Then there exists a canonical sheaf $\mathcal{S}_X$ of $\mathbb{K}$-vector spaces on X, such that if $R \subseteq X$ is Zariski open and $i : R \hookrightarrow U$ is a closed embedding of R into a smooth $\mathbb{K}$-scheme U, and $I_{R,U} \subseteq \mathcal{O}_U$ is the ideal vanishing on $i(R)$, then

$$\mathcal{S}_X|_R \cong \mathrm{Ker}\left(\frac{\mathcal{O}_U}{I_{R,U}^2} \xrightarrow{\ \mathrm{d}\ } \frac{T^*U}{I_{R,U} \cdot T^*U}\right).$$

Also $\mathcal{S}_X$ splits naturally as $\mathcal{S}_X = \mathcal{S}_X^0 \oplus \mathbb{K}_X$, where $\mathbb{K}_X$ is the sheaf of locally constant functions $X \to \mathbb{K}$.

The meaning of the sheaves $\mathcal{S}_X, \mathcal{S}_X^0$

If $X = \mathrm{Crit}(f : U \to \mathbb{A}^1)$ then taking $R = X$, $i =$inclusion, we see that $f + I_{X,U}^2$ is a section of $\mathcal{S}_X$. Also $f|_{X^{\mathrm{red}}} : X^{\mathrm{red}} \to \mathbb{K}$ is locally constant, and if $f|_{X^{\mathrm{red}}} = 0$ then $f + I_{X,U}^2$ is a section of $\mathcal{S}_X^0$. Note that $f + I_{X,U} = f|_X$ in $\mathcal{O}_X = \mathcal{O}_U / I_{X,U}$. The theorem means that $f + I_{X,U}^2$ makes sense *intrinsically on* X, without reference to the embedding of X into U.

That is, if $X = \mathrm{Crit}(f : U \to \mathbb{A}^1)$ then we can remember f up to second order in the ideal $I_{X,U}$ as a piece of data on X, not on U.

Suppose $X = \mathrm{Crit}(f : U \to \mathbb{A}^1) = \mathrm{Crit}(g : V \to \mathbb{A}^1)$ is written as a critical locus in two different ways. Then $f + I_{X,U}^2$, $g + I_{X,V}^2$ are sections of $\mathcal{S}_X$, so we can ask whether $f + I_{X,U}^2 = g + I_{X,V}^2$. This gives a way to compare isomorphic critical loci in different smooth classical schemes.

6.1. The definition of d-critical loci

Definition (Joyce arXiv:1304.4508)

An (*algebraic*) *d-critical locus* (X, s) is a classical $\mathbb{K}$-scheme X and a global section $s \in H^0(\mathcal{S}_X^0)$ such that X may be covered by Zariski open $R \subseteq X$ with an isomorphism $i : R \to \mathrm{Crit}(f : U \to \mathbb{A}^1)$ identifying $s|_R$ with $f + I_{R,U}^2$, for f a regular function on a smooth $\mathbb{K}$-scheme U.

That is, a d-critical locus (X, s) is a $\mathbb{K}$-scheme X which may Zariski locally be written as a critical locus $\mathrm{Crit}(f : U \to \mathbb{A}^1)$, and the section s remembers f up to second order in the ideal $I_{X,U}$. We also define *complex analytic d-critical loci*, with X a complex analytic space locally modelled on $\mathrm{Crit}(f : U \to \mathbb{C})$ for U a complex manifold and f holomorphic.

Orientations on d-critical loci

Theorem (Joyce arXiv:1304.4508)

Let (X, s) be an algebraic d-critical locus and X^{red} the reduced $\mathbb{K}$-subscheme of X. Then there is a natural line bundle $K_{X,s}$ on X^{red} called the **canonical bundle**, *such that if (X, s) is locally modelled on $\mathrm{Crit}(f : U \to \mathbb{A}^1)$ then $K_{X,s}$ is locally modelled on $K_U^{\otimes^2}|_{\mathrm{Crit}(f)^{\mathrm{red}}}$, for K_U the usual canonical bundle of U.*

Definition

Let (X, s) be a d-critical locus. An *orientation* on (X, s) is a choice of square root line bundle $K_{X,s}^{1/2}$ for $K_{X,s}$ on X^{red}.

This is related to *orientation data* in Kontsevich–Soibelman 2008.

A truncation functor from -1-symplectic derived schemes

Theorem 6.1 (Brav, Bussi and Joyce arXiv:1305.6302)

Let $(\boldsymbol{X},\omega)$ be a -1-shifted symplectic derived $\mathbb{K}$-scheme. Then the classical $\mathbb{K}$-scheme $X=t_0(\boldsymbol{X})$ extends naturally to an algebraic d-critical locus (X,s). The canonical bundle of (X,s) satisfies $K_{X,s}\cong\det\mathbb{L}_{\boldsymbol{X}}|_{X^{\rm red}}$.

That is, we define a *truncation functor* from -1-shifted symplectic derived $\mathbb{K}$-schemes to algebraic d-critical loci. Examples show this functor is not full. Think of d-critical loci as *classical truncations* of -1-shifted symplectic derived $\mathbb{K}$-schemes.
An alternative semi-classical truncation, used in D–T theory, is *schemes with symmetric obstruction theory*. D-critical loci appear to be more useful, for both categorified and motivic D–T theory.

Corollaries 5.3–5.4 in lecture 2 imply:

Corollary 6.2

Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a classical moduli $\mathbb{K}$-scheme of coherent sheaves, or complexes of coherent sheaves, on Y. Then $\mathcal{M}$ extends naturally to a d-critical locus $(\mathcal{M},s)$. The canonical bundle satisfies $K_{\mathcal{M},s}\cong\det(\mathcal{E}^\bullet)|_{\mathcal{M}^{\rm red}}$, where $\phi:\mathcal{E}^\bullet\rightarrow\mathbb{L}_{\mathcal{M}}$ is the (symmetric) obstruction theory on $\mathcal{M}$ defined by Thomas or Huybrechts and Thomas.

6.2. D-critical stacks

To generalize the d-critical loci in §6.1 to Artin stacks, we need a good notion of sheaves on Artin stacks. This is already well understood. Roughly, a sheaf $\mathcal{S}$ on an Artin stack X assigns a sheaf $\mathcal{S}(U,\varphi)$ on U (in the usual sense for schemes) for each smooth morphism $\varphi: U \to X$ with U a scheme, and a morphism $\mathcal{S}(\alpha,\eta): \alpha^*(\mathcal{S}(V,\psi)) \to \mathcal{S}(U,\varphi)$ (often an isomorphism) for each 2-commutative diagram

$$\begin{array}{ccccc} & & V & & \\ & \overset{\alpha}{\nearrow} & \eta\Uparrow & \overset{\psi}{\searrow} & \\ U & & \xrightarrow{\quad\varphi\quad} & & X \end{array} \qquad (1)$$

with U, V schemes and φ, ψ smooth, such that $\mathcal{S}(\alpha,\eta)$ have the obvious associativity properties. So, we pass from stacks X to schemes U by working with smooth atlases $\varphi: U \to X$.

The definition of d-critical stacks

Generalizing d-critical loci to stacks is now straightforward. As above, on each scheme U we have a canonical sheaf $\mathcal{S}_U^0$. If $\alpha: U \to V$ is a morphism of schemes we have pullback morphisms $\alpha^\star: \alpha^{-1}(\mathcal{S}_V^0) \to \mathcal{S}_U^0$ with associativity properties.
So, for any classical Artin stack X, we define a sheaf $\mathcal{S}_X^0$ on X by $\mathcal{S}_X(U,\varphi) = \mathcal{S}_U^0$ for all smooth $\varphi: U \to X$ with U a scheme, and $\mathcal{S}(\alpha,\eta) = \alpha^\star$ for all diagrams (1).
A global section $s \in H^0(\mathcal{S}_X^0)$ assigns $s(U,\varphi) \in H^0(\mathcal{S}_U^0)$ for all smooth $\varphi: U \to X$ with $\alpha^\star[\alpha^{-1}(s(V,\psi))] = s(U,\varphi)$ for all diagrams (1). We call (X,s) a *d-critical stack* if $\big(U, s(U,\varphi)\big)$ is a d-critical locus for all smooth $\varphi: U \to X$.
That is, if X is a d-critical stack then any smooth atlas $\varphi: U \to X$ for X is a d-critical locus.

A truncation functor from −1-symplectic derived stacks

As for the scheme case in §6.1, we prove:

Theorem 6.3 (Ben-Bassat, Brav, Bussi, Joyce arXiv:1312.0090)

Let $(\boldsymbol{X},\omega)$ be a -1-shifted symplectic derived Artin stack. Then the classical Artin stack $X=t_0(\boldsymbol{X})$ extends naturally to a d-critical stack (X,s), with canonical bundle $K_{X,s}\cong\det\mathbb{L}_{\boldsymbol{X}}|_{X^{\rm red}}$.

Corollary

Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a classical moduli stack of coherent sheaves F on Y, or complexes $F^\bullet$ in $D^b\mathrm{coh}(Y)$ with $\mathrm{Ext}^{<0}(F^\bullet,F^\bullet)=0$. Then $\mathcal{M}$ extends naturally to a d-critical locus $(\mathcal{M},s)$ with canonical bundle $K_{\mathcal{M},s}\cong\det(\mathcal{E}^\bullet)|_{\mathcal{M}^{\rm red}}$, where $\phi:\mathcal{E}^\bullet\to\mathbb{L}_{\mathcal{M}}$ is the natural obstruction theory on $\mathcal{M}$.

Canonical bundles and orientations

For schemes, a d-critical locus (U,s) has a canonical bundle $K_{U,s}\to U^{\rm red}$, and an orientation on (U,s) is a square root $K_{U,s}^{1/2}$. Similarly, a d-critical stack (X,s) has a *canonical bundle* $K_{X,s}\to X^{\rm red}$. For any smooth $\varphi:U\to X$ with U a scheme we have $K_{X,s}(U^{\rm red},\varphi^{\rm red})=K_{U,s(U,\varphi)}\otimes(\det\mathbb{L}_{U/X})^{\otimes^{-2}}$. An *orientation* on (X,s) is a choice of square root $K_{X,s}^{1/2}$ for $K_{X,s}$. Note that as $(\det\mathbb{L}_{U/X})^{\otimes^{-2}}$ has a natural square root, an orientation for (X,s) gives an orientation for $\big(U,s(U,\varphi)\big)$ for any smooth atlas $\varphi:U\to X$.

7. Categorification using perverse sheaves

7.1. Perverse sheaves

It's not easy to explain what perverse sheaves are. We can think of a perverse sheaf as a *system of coefficients for cohomology*. Let X be a complex manifold. The cohomology group $H^k(X;\mathbb{Q})$ is the sheaf cohomology group $H^k(X,\mathbb{Q}_X)$, where $\mathbb{Q}_X$ is the constant sheaf with fibre $\mathbb{Q}$. Working in complexes of sheaves of $\mathbb{Q}$-modules on X, consider the shifted sheaf $\mathbb{Q}_X[\dim_{\mathbb{C}} X]$. This is an example of a perverse sheaf. The shift means that Poincaré duality for X has the nice form $\mathbb{H}^i_{\rm cs}(\mathbb{Q}_X[\dim_{\mathbb{C}} X])\cong \mathbb{H}^{-i}(\mathbb{Q}_X[\dim_{\mathbb{C}} X])^*$.
If instead X is a singular complex variety, rather than considering $H^*(X;\mathbb{Q})$, it can be helpful (e.g. in 'intersection cohomology', and to preserve nice properties like Poincaré duality) to consider cohomology $\mathbb{H}^*(X,\mathcal{P}^\bullet)$ with coefficients in a complex $\mathcal{P}^\bullet$ on X (a 'perverse sheaf') which treats the singularities of X in a special way.

Let U be a complex manifold, and $f:U\to\mathbb{C}$ a holomorphic function. Then one can define a perverse sheaf $\mathcal{PV}^\bullet_{U,f}$ on $\mathrm{Crit}\, f$ called the *perverse sheaf of vanishing cycles*, with nice properties. The *vanishing cohomology* $\mathbb{H}^\bullet(\mathcal{PV}^\bullet_{U,f})$ measures how $H^*(f^{-1}(c);\mathbb{Q})$ changes as c passes through critical values of f.
Kai Behrend observed that the pointwise Euler characteristic $\chi_{\mathcal{PV}^\bullet_{U,f}}:\mathrm{Crit}\, f\to\mathbb{Z}$ is the Behrend function of $\mathrm{Crit}\, f$, as used in classical Donaldson–Thomas theory.

Theorem 7.1 (Brav-Bussi-Dupont-Joyce-Szendrői arXiv:1211.3259)

Let (X,s) be an algebraic d-critical locus over $\mathbb{K}$, with an orientation $K^{1/2}_{X,s}$. Then we can construct a canonical perverse sheaf $P^\bullet_{X,s}$ on X, such that if (X,s) is locally modelled on $\mathrm{Crit}(f:U\to\mathbb{A}^1)$, then $P^\bullet_{X,s}$ is locally modelled on the perverse sheaf of vanishing cycles $\mathcal{PV}^\bullet_{U,f}$ of (U,f).
Similarly, we can construct a natural $\mathscr{D}$-module $D^\bullet_{X,s}$ on X, and when $\mathbb{K}=\mathbb{C}$ a natural mixed Hodge module $M^\bullet_{X,s}$ on X.

Sketch of the proof of Theorem 7.1

Roughly, we prove the theorem by taking a Zariski open cover $\{R_i : i \in I\}$ of X with $R_i \cong \mathrm{Crit}(f_i : U_i \to \mathbb{A}^1)$, and showing that $\mathcal{PV}^\bullet_{U_i,f_i}$ and $\mathcal{PV}^\bullet_{U_j,f_j}$ are canonically isomorphic on $R_i \cap R_j$, so we can glue the $\mathcal{PV}^\bullet_{U_i,f_i}$ to get a global perverse sheaf $P^\bullet_{X,s}$ on X.
In fact things are more complicated: the (local) isomorphisms $\mathcal{PV}^\bullet_{U_i,f_i} \cong \mathcal{PV}^\bullet_{U_j,f_j}$ are only canonical *up to sign*. To make them canonical, we use the orientation $K^{1/2}_{X,s}$ to define natural principal $\mathbb{Z}_2$-bundles Q_i on R_i, such that $\mathcal{PV}^\bullet_{U_i,f_i} \otimes_{\mathbb{Z}_2} Q_i \cong \mathcal{PV}^\bullet_{U_j,f_j} \otimes_{\mathbb{Z}_2} Q_j$ is canonical, and then we glue the $\mathcal{PV}^\bullet_{U_i,f_i} \otimes_{\mathbb{Z}_2} Q_i$ to get $P^\bullet_{X,s}$.

Theorem 7.1 and Corollary 6.2 imply:

Corollary

Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a classical moduli $\mathbb{K}$-scheme of coherent sheaves, or complexes of coherent sheaves, on Y, with (symmetric) obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$. Suppose we are given a square root $\det(\mathcal{E}^\bullet)^{1/2}$ for $\det(\mathcal{E}^\bullet)$ (i.e. **orientation data**, *K–S). Then we have a natural perverse sheaf $P^\bullet_{\mathcal{M},s}$ on $\mathcal{M}$.*

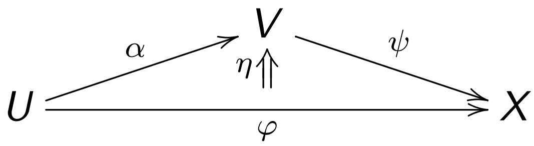
(Compare Kiem and Li arXiv:1212.6444).
The *hypercohomology* $\mathbb{H}^*(P^\bullet_{\mathcal{M},s})$ is a finite-dimensional graded vector space. The pointwise Euler characteristic $\chi(P^\bullet_{\mathcal{M},s})$ is the *Behrend function* $\nu_{\mathcal{M}}$ of $\mathcal{M}$. Thus

$$\textstyle\sum_{i\in\mathbb{Z}}(-1)^i \dim \mathbb{H}^i(P^\bullet_{\mathcal{M},s}) = \chi(\mathcal{M},\nu_{\mathcal{M}}).$$

Now by Behrend 2005, the Donaldson–Thomas invariant of $\mathcal{M}$ is $DT(\mathcal{M}) = \chi(\mathcal{M},\nu_{\mathcal{M}})$. So, $\mathbb{H}^*(P^\bullet_{\mathcal{M},s})$ is a graded vector space with dimension $DT(\mathcal{M})$, that is, a *categorification* of $DT(\mathcal{M})$.

Extension to Artin stacks

Let (X, s) be a d-critical stack, with an orientation $K_{X,s}^{1/2}$. Then for any smooth $\varphi : U \to X$ with U a scheme, $\big(U, s(U, \varphi)\big)$ is an oriented d-critical locus, so as above, BBDJS constructs a perverse sheaf $P^\bullet_{U,\varphi}$ on U. Given a diagram



with U, V schemes and φ, ψ smooth, we can construct a natural isomorphism $P^\bullet_{\alpha,\eta} : \alpha^*(P^\bullet_{V,\psi})[\dim\varphi - \dim\psi] \to P^\bullet_{U,\varphi}$.
All this data $P^\bullet_{U,\varphi}, P^\bullet_{\alpha,\eta}$ is equivalent to a perverse sheaf on X.

Thus we prove:

Theorem (Ben-Bassat, Brav, Bussi, Joyce arXiv:1312.0090)

Let (X, s) be a d-critical stack, with an orientation $K_{X,s}^{1/2}$. Then we can construct a canonical perverse sheaf $P^\bullet_{X,s}$ on X.

Corollary

Suppose Y is a Calabi–Yau 3-fold and $\mathcal{M}$ a classical moduli stack of coherent sheaves F on Y, or of complexes $F^\bullet$ in $D^b\operatorname{coh}(Y)$ with $\operatorname{Ext}^{<0}(F^\bullet, F^\bullet) = 0$, with (symmetric) obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$. Suppose we are given a square root $\det(\mathcal{E}^\bullet)^{1/2}$ for $\det(\mathcal{E}^\bullet)$. Then we construct a natural perverse sheaf $P^\bullet_{\mathcal{M},s}$ on $\mathcal{M}$.

The *hypercohomology* $\mathbb{H}^*(P^\bullet_{\mathcal{M},s})$ is a categorification of the Donaldson–Thomas theory of Y.

7.2. Algebraic structures on perverse sheaves

Let $(\boldsymbol{X}, \omega_{\boldsymbol{X}})$ be a -1-shifted symplectic derived scheme, and $\boldsymbol{i} : \boldsymbol{L} \to \boldsymbol{X}$ a Lagrangian, in the sense of PTVV.
Choose an orientation $K^{1/2}_{X,s}$ for $(\boldsymbol{X}, \omega_{\boldsymbol{X}})$. There is then a notion of relative orientation for $\boldsymbol{i} : \boldsymbol{L} \to \boldsymbol{X}$, choose one of these.
We get a perverse sheaf $P^\bullet_{\boldsymbol{X},\omega_{\boldsymbol{X}}}$ on $\boldsymbol{X}$, by Theorem 7.1.

Conjecture 7.2 (Joyce; see Amorim and Ben-Bassat arXiv:1601.01536)

There is a natural morphism in $D^b_c(\boldsymbol{L})$

$$\mu_{\boldsymbol{L}} : \mathbb{Q}_{\boldsymbol{L}}[\operatorname{vdim} \boldsymbol{L}] \longrightarrow \boldsymbol{i}^!(P^\bullet_{\boldsymbol{X},\omega_{\boldsymbol{X}}}), \qquad (2)$$

with given local models in 'Darboux form' presentations for $\boldsymbol{X}, \boldsymbol{L}$*. These* $\mu_{\boldsymbol{L}}$ *should satisfy a package of properties under products, composition of Lagrangian correspondences, Verdier duality, etc.*

This conjecture has important consequences.

We already know local models for $\boldsymbol{i} : \boldsymbol{L} \to \boldsymbol{X}$ and $\mu_{\boldsymbol{L}}$ in (2) (Joyce–Safronov arXiv:1506.04024). What makes the conjecture difficult is that local models are not enough: $\mu_{\boldsymbol{L}}$ is a morphism of complexes, not of (perverse) sheaves, and such morphisms do not glue like sheaves. For instance, one could imagine $\mu_{\boldsymbol{L}}$ to be globally nonzero, but zero on the sets of an open cover of $\boldsymbol{L}$.
So to construct $\mu_{\boldsymbol{L}}$, we have to do a gluing problem in an ∞-category, probably using hypercovers. I have a sketch of one way to do this (over $\mathbb{C}$). It is not easy.
Maybe gluing local models naïvely is not the best approach for this problem, need some more advanced Lurie-esque technology? Any help would be appreciated.

An application: Cohomological Hall Algebras

Let Y be a Calabi–Yau 3-fold, and $\boldsymbol{\mathcal{M}}$ the derived moduli stack of coherent sheaves (or suitable complexes) on Y, with its -1-shifted symplectic structure ω. Then Theorem 6.3 makes the classical stack $\mathcal{M}$ into a d-critical stack $(\mathcal{M}, s)$. Suppose we have 'orientation data' for Y, i.e. an orientation $K^{1/2}_{\mathcal{M},s}$, with compatibility condition on exact sequences. Then we have a perverse sheaf $P^\bullet_{\mathcal{M},s}$, with hypercohomology $\mathbb{H}^*(P^\bullet_{\mathcal{M},s})$.
We would like to define an associative multiplication on $\mathbb{H}^*(P^\bullet_{\mathcal{M},s})$, making it into a *Cohomological Hall Algebra*, in the style of Kontsevich and Soibelman (arXiv:1006.2706).

Let $\boldsymbol{\mathcal{E}xact}$ be the derived stack of short exact sequences $0 \to F_1 \to F_2 \to F_3 \to 0$ in $\mathrm{coh}(Y)$ (or distinguished triangles in $D^b\,\mathrm{coh}(Y)$), with projections $\pi_1, \pi_2, \pi_3 : \boldsymbol{\mathcal{E}xact} \to \boldsymbol{\mathcal{M}}$.

Claim (Oren Ben-Bassat, work in progress?)

$\pi_1 \times \pi_2 \times \pi_3 : \boldsymbol{\mathcal{E}xact} \to (\boldsymbol{\mathcal{M}}, \omega) \times (\boldsymbol{\mathcal{M}}, -\omega) \times (\boldsymbol{\mathcal{M}}, \omega)$ is Lagrangian in -1-shifted symplectic.

Then apply the stack version of Conjecture 7.2 and manipulate using Verdier duality of perverse sheaves to get the COHA multiplication.

8. Motives of d-critical loci and d-critical stacks

By similar (but easier) arguments to those used to construct the perverse sheaves $P^\bullet_{X,s}$ in §7.1, we prove:

Theorem 8.1 (Bussi, Joyce and Meinhardt arXiv:1305.6428)

Let (X,s) be a finite type algebraic d-critical locus over $\mathbb{K}$, with an orientation $K^{1/2}_{X,s}$. Then we can construct a natural motive $MF_{X,s}$ in a certain ring of $\hat\mu$-equivariant motives $\bar{\mathcal{M}}^{\hat\mu}_X$ on X, such that if (X,s) is locally modelled on $\mathrm{Crit}(f:U\to\mathbb{A}^1)$, then $MF_{X,s}$ is locally modelled on $\mathbb{L}^{-\dim U/2}([X]-MF^{\mathrm{mot}}_{U,f})$, where $MF^{\mathrm{mot}}_{U,f}$ is the **motivic Milnor fibre** *or* **motivic nearby cycle** *of f.*

Here a *motivic invariant* I of finite type $\mathbb{K}$-varieties X in a ring R is $I(X)\in R$ satisfying $I(X)=I(X\setminus Y)+I(Y)$ for $Y\subset X$ closed and $I(X\times Y)=I(X)I(Y)$ for all X,Y. The obvious example is Euler characteristics, but there are others, e.g. virtual Poincaré polynomials. The ring $\bar{\mathcal{M}}^{\hat\mu}_X$ is R for 'universal motivic invariants'.

Relation to motivic D–T invariants

Theorem 8.1 and Corollary 6.1 imply:

Corollary

Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a finite type classical moduli $\mathbb{K}$-scheme of (complexes of) coherent sheaves on Y, with (symmetric) obstruction theory $\phi:\mathcal{E}^\bullet\to\mathbb{L}_{\mathcal{M}}$. Suppose we are given a square root $\det(\mathcal{E}^\bullet)^{1/2}$ for $\det(\mathcal{E}^\bullet)$ (i.e. **orientation data**, *K–S). Then we have a natural motive $MF^\bullet_{\mathcal{M},s}$ on $\mathcal{M}$.*

This motive $MF^\bullet_{\mathcal{M},s}$ is essentially the motivic Donaldson–Thomas invariant of $\mathcal{M}$ defined (partially conjecturally) by Kontsevich and Soibelman, arXiv:0811.2435. K–S work with motivic Milnor fibres of formal power series at each point of $\mathcal{M}$. Our results show the formal power series can be taken to be a regular function, and clarify how the motivic Milnor fibres vary in families over $\mathcal{M}$.

Extension to Artin stacks

We can also generalize Theorem 8.1 to d-critical stacks:

Theorem (Ben-Bassat, Brav, Bussi, Joyce arXiv:1312.0090)

Let (X, s) be an oriented d-critical stack, of finite type and locally a global quotient. Then we can construct a natural motive $MF_{X,s}$ in a certain ring of $\hat{\mu}$-equivariant motives $\bar{\mathcal{M}}_X^{\mathrm{st},\hat{\mu}}$ on X, such that if $\varphi : U \to X$ is smooth and U is a scheme then

$$\varphi^*(MF_{X,s}) = \mathbb{L}^{\dim\varphi/2} \odot MF_{U,s(U,\varphi)},$$

where $MF_{U,s(U,\varphi)}$ for the scheme case is as in BJM above.

For CY3 moduli stacks, these $MF_{X,s}$ are basically Kontsevich–Soibelman's motivic Donaldson–Thomas invariants.