D-critical loci, and categorification of Donaldson–Thomas theory using perverse sheaves

Lecture 2 of 3

Dominic Joyce, Oxford University

January 2014

Based on: arXiv:1304.4508, arXiv:1211.3259, and arXiv:1312.0090.

Joint work with Oren Ben-Bassat, Chris Brav, Vittoria Bussi, Delphine Dupont, and Balázs Szendrői. Funded by the EPSRC.

Plan of talk:

4. D-critical loci

Theorem (Joyce arXiv:1304.4508)

Let X be a classical $\mathbb{K}$-scheme. Then there exists a canonical sheaf $\mathcal{S}_X$ of $\mathbb{K}$-vector spaces on X, such that if $R \subseteq X$ is Zariski open and $i : R \hookrightarrow U$ is a closed embedding of R into a smooth $\mathbb{K}$-scheme U, and $I_{R,U} \subseteq \mathcal{O}_U$ is the ideal vanishing on $i(R)$, then

$$\mathcal{S}_X|_R \cong \mathrm{Ker}\left(\frac{\mathcal{O}_U}{I_{R,U}^2} \xrightarrow{\mathrm{d}} \frac{T^*U}{I_{R,U} \cdot T^*U}\right).$$

Also $\mathcal{S}_X$ splits naturally as $\mathcal{S}_X = \mathcal{S}_X^0 \oplus \mathbb{K}_X$, where $\mathbb{K}_X$ is the sheaf of locally constant functions $X \to \mathbb{K}$.

The meaning of the sheaves $\mathcal{S}_X, \mathcal{S}_X^0$

If $X = \mathrm{Crit}(f : U \to \mathbb{A}^1)$ then taking $R = X$, $i =$inclusion, we see that $f + I_{X,U}^2$ is a section of $\mathcal{S}_X$. Also $f|_{X^{\mathrm{red}}} : X^{\mathrm{red}} \to \mathbb{K}$ is locally constant, and if $f|_{X^{\mathrm{red}}} = 0$ then $f + I_{X,U}^2$ is a section of $\mathcal{S}_X^0$. Note that $f + I_{X,U} = f|_X$ in $\mathcal{O}_X = \mathcal{O}_U/I_{X,U}$. The theorem means that $f + I_{X,U}^2$ makes sense *intrinsically on* X, without reference to the embedding of X into U.

That is, if $X = \mathrm{Crit}(f : U \to \mathbb{A}^1)$ then we can remember f up to second order in the ideal $I_{X,U}$ as a piece of data on X, not on U.

Suppose $X = \mathrm{Crit}(f : U \to \mathbb{A}^1) = \mathrm{Crit}(g : V \to \mathbb{A}^1)$ is written as a critical locus in two different ways. Then $f + I_{X,U}^2$, $g + I_{X,V}^2$ are sections of $\mathcal{S}_X$, so we can ask whether $f + I_{X,U}^2 = g + I_{X,V}^2$. This gives a way to compare isomorphic critical loci in different smooth classical schemes.

The definition of d-critical loci

Definition (Joyce arXiv:1304.4508)

An (*algebraic*) *d-critical locus* (X, s) is a classical $\mathbb{K}$-scheme X and a global section $s \in H^0(\mathcal{S}_X^0)$ such that X may be covered by Zariski open $R \subseteq X$ with an isomorphism $i : R \to \mathrm{Crit}(f : U \to \mathbb{A}^1)$ identifying $s|_R$ with $f + I_{R,U}^2$, for f a regular function on a smooth $\mathbb{K}$-scheme U.

That is, a d-critical locus (X, s) is a $\mathbb{K}$-scheme X which may Zariski locally be written as a critical locus $\mathrm{Crit}(f : U \to \mathbb{A}^1)$, and the section s remembers f up to second order in the ideal $I_{X,U}$.
We also define *complex analytic d-critical loci*, with X a complex analytic space locally modelled on $\mathrm{Crit}(f : U \to \mathbb{C})$ for U a complex manifold and f holomorphic.

Orientations on d-critical loci

Theorem (Joyce arXiv:1304.4508)

Let (X, s) be an algebraic d-critical locus and X^{red} the reduced $\mathbb{K}$-subscheme of X. Then there is a natural line bundle $K_{X,s}$ on X^{red} called the **canonical bundle**, *such that if (X, s) is locally modelled on $\mathrm{Crit}(f : U \to \mathbb{A}^1)$ then $K_{X,s}$ is locally modelled on $K_U^{\otimes^2}|_{\mathrm{Crit}(f)^{\mathrm{red}}}$, for K_U the usual canonical bundle of U.*

Definition

Let (X, s) be a d-critical locus. An *orientation* on (X, s) is a choice of square root line bundle $K_{X,s}^{1/2}$ for $K_{X,s}$ on X^{red}.

This is related to *orientation data* in Kontsevich–Soibelman 2008.

A truncation functor from −1-symplectic derived schemes

Theorem (Brav, Bussi and Joyce arXiv:1305.6302)

Let $(\mathbf{X}, \omega)$ be a -1-shifted symplectic derived $\mathbb{K}$-scheme. Then the classical $\mathbb{K}$-scheme $X = t_0(\mathbf{X})$ extends naturally to an algebraic d-critical locus (X, s). The canonical bundle of (X, s) satisfies $K_{X,s} \cong \det \mathbb{L}_{\mathbf{X}}|_{X^{\rm red}}$.

That is, we define a *truncation functor* from -1-shifted symplectic derived $\mathbb{K}$-schemes to algebraic d-critical loci. Examples show this functor is not full. Think of d-critical loci as *classical truncations* of -1-shifted symplectic derived $\mathbb{K}$-schemes.
An alternative semi-classical truncation, used in D–T theory, is *schemes with symmetric obstruction theory*. D-critical loci appear to be better, for both categorified and motivic D–T theory.

The corollaries in lecture 1, §2 imply:

Corollary

Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a classical moduli $\mathbb{K}$-scheme of coherent sheaves, or complexes of coherent sheaves, on Y. Then $\mathcal{M}$ extends naturally to a d-critical locus $(\mathcal{M}, s)$. The canonical bundle satisfies $K_{\mathcal{M},s} \cong \det(\mathcal{E}^\bullet)|_{\mathcal{M}^{\rm red}}$, where $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$ is the (symmetric) obstruction theory on $\mathcal{M}$ defined by Thomas or Huybrechts and Thomas.

Corollary

Let (S, ω) be a classical smooth symplectic $\mathbb{K}$-scheme, and $L, M \subseteq S$ be smooth algebraic Lagrangians. Then $X = L \cap M$ extends naturally to a d-critical locus (X, s). The canonical bundle satisfies $K_{X,s} \cong K_L|_{X^{\rm red}} \otimes K_M|_{X^{\rm red}}$. Hence, choices of square roots $K_L^{1/2}, K_M^{1/2}$ give an orientation for (X, s).

5. D-critical stacks

To generalize the d-critical loci in §4 to Artin stacks, we need a good notion of sheaves on Artin stacks. This is already well understood. Roughly, a sheaf $\mathcal{S}$ on an Artin stack X assigns a sheaf $\mathcal{S}(U,\varphi)$ on U (in the usual sense for schemes) for each smooth morphism $\varphi : U \to X$ with U a scheme, and a morphism $\mathcal{S}(\alpha,\eta) : \alpha^*(\mathcal{S}(V,\psi)) \to \mathcal{S}(U,\varphi)$ (often an isomorphism) for each 2-commutative diagram

$$\begin{array}{ccccc} & & V & & \\ & \overset{\alpha}{\nearrow} & \eta\Uparrow & \overset{\psi}{\searrow} & \\ U & & \xrightarrow[\varphi]{\hspace{4em}} & & X \end{array} \qquad (1)$$

with U, V schemes and φ, ψ smooth, such that $\mathcal{S}(\alpha,\eta)$ have the obvious associativity properties. So, we pass from stacks X to schemes U by working with smooth atlases $\varphi : U \to X$.

The definition of d-critical stacks

Generalizing d-critical loci to stacks is now straightforward. As in §6, on each scheme U we have a canonical sheaf $\mathcal{S}_U^0$. If $\alpha : U \to V$ is a morphism of schemes we have pullback morphisms $\alpha^\star : \alpha^{-1}(\mathcal{S}_V^0) \to \mathcal{S}_U^0$ with associativity properties.

So, for any classical Artin stack X, we define a sheaf $\mathcal{S}_X^0$ on X by $\mathcal{S}_X(U,\varphi) = \mathcal{S}_U^0$ for all smooth $\varphi : U \to X$ with U a scheme, and $\mathcal{S}(\alpha,\eta) = \alpha^\star$ for all diagrams (2).

A global section $s \in H^0(\mathcal{S}_X^0)$ assigns $s(U,\varphi) \in H^0(\mathcal{S}_U^0)$ for all smooth $\varphi : U \to X$ with $\alpha^\star[\alpha^{-1}(s(V,\psi))] = s(U,\varphi)$ for all diagrams (2). We call (X,s) a *d-critical stack* if $\bigl(U, s(U,\varphi)\bigr)$ is a d-critical locus for all smooth $\varphi : U \to X$.

That is, if X is a d-critical stack then any smooth atlas $\varphi : U \to X$ for X is a d-critical locus.

A truncation functor from -1-symplectic derived stacks

As for the scheme case in §4, we prove:

Theorem (Ben-Bassat, Brav, Bussi, Joyce)

Let $(\mathbf{X},\omega)$ be a -1-shifted symplectic derived Artin stack. Then the classical Artin stack $X=t_0(\mathbf{X})$ extends naturally to a d-critical stack (X,s), with canonical bundle $K_{X,s}\cong\det\mathbb{L}_{\mathbf{X}}|_{X^{\rm red}}$.

Corollary

Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a classical moduli stack of coherent sheaves F on Y, or complexes $F^\bullet$ in $D^b\mathrm{coh}(Y)$ with $\mathrm{Ext}^{<0}(F^\bullet,F^\bullet)=0$. Then $\mathcal{M}$ extends naturally to a d-critical locus $(\mathcal{M},s)$ with canonical bundle $K_{\mathcal{M},s}\cong\det(\mathcal{E}^\bullet)|_{\mathcal{M}^{\rm red}}$, where $\phi:\mathcal{E}^\bullet\to\mathbb{L}_{\mathcal{M}}$ is the natural obstruction theory on $\mathcal{M}$.

Canonical bundles and orientations

For schemes, a d-critical locus (U,s) has a canonical bundle $K_{U,s}\to U^{\rm red}$, and an orientation on (U,s) is a square root $K_{U,s}^{1/2}$. Similarly, a d-critical stack (X,s) has a *canonical bundle* $K_{X,s}\to X^{\rm red}$. For any smooth $\varphi:U\to X$ with U a scheme we have $K_{X,s}(U^{\rm red},\varphi^{\rm red})=K_{U,s(U,\varphi)}\otimes(\det\mathbb{L}_{U/X})^{\otimes^{-2}}$. An *orientation* on (X,s) is a choice of square root $K_{X,s}^{1/2}$ for $K_{X,s}$. Note that as $(\det\mathbb{L}_{U/X})^{\otimes^{-2}}$ has a natural square root, an orientation for (X,s) gives an orientation for $\bigl(U,s(U,\varphi)\bigr)$ for any smooth atlas $\varphi:U\to X$.

6. Categorification using perverse sheaves

Theorem (Brav, Bussi, Dupont, Joyce, Szendrői arXiv:1211.3259)

Let (X,s) be an algebraic d-critical locus over $\mathbb{K}$, with an orientation $K_{X,s}^{1/2}$. Then we can construct a canonical perverse sheaf $P_{X,s}^\bullet$ on X, such that if (X,s) is locally modelled on $\mathrm{Crit}(f:U\to\mathbb{A}^1)$, then $P_{X,s}^\bullet$ is locally modelled on the perverse sheaf of vanishing cycles $\mathcal{PV}_{U,f}^\bullet$ of (U,f).
Similarly, we can construct a natural $\mathscr{D}$-module $D_{X,s}^\bullet$ on X, and when $\mathbb{K}=\mathbb{C}$ a natural mixed Hodge module $M_{X,s}^\bullet$ on X.

Sketch of the proof of the theorem

Roughly, we prove the theorem by taking a Zariski open cover $\{R_i : i\in I\}$ of X with $R_i\cong\mathrm{Crit}(f_i:U_i\to\mathbb{A}^1)$, and showing that $\mathcal{PV}_{U_i,f_i}^\bullet$ and $\mathcal{PV}_{U_j,f_j}^\bullet$ are canonically isomorphic on $R_i\cap R_j$, so we can glue the $\mathcal{PV}_{U_i,f_i}^\bullet$ to get a global perverse sheaf $P_{X,s}^\bullet$ on X.
In fact things are more complicated: the (local) isomorphisms $\mathcal{PV}_{U_i,f_i}^\bullet\cong\mathcal{PV}_{U_j,f_j}^\bullet$ are only canonical *up to sign*. To make them canonical, we use the orientation $K_{X,s}^{1/2}$ to define natural principal $\mathbb{Z}_2$-bundles Q_i on R_i, such that $\mathcal{PV}_{U_i,f_i}^\bullet\otimes_{\mathbb{Z}_2}Q_i\cong\mathcal{PV}_{U_j,f_j}^\bullet\otimes_{\mathbb{Z}_2}Q_j$ is canonical, and then we glue the $\mathcal{PV}_{U_i,f_i}^\bullet\otimes_{\mathbb{Z}_2}Q_i$ to get $P_{X,s}^\bullet$.

The first corollary in lecture 1, §2 implies:

Corollary

Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a classical moduli $\mathbb{K}$-scheme of coherent sheaves, or complexes of coherent sheaves, on Y, with (symmetric) obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$. Suppose we are given a square root $\det(\mathcal{E}^\bullet)^{1/2}$ for $\det(\mathcal{E}^\bullet)$ (i.e. ***orientation data****, K–S). Then we have a natural perverse sheaf $P^\bullet_{\mathcal{M},s}$ on $\mathcal{M}$.*

(Compare Kiem and Li arXiv:1212.6444).
The *hypercohomology* $\mathbb{H}^*(P^\bullet_{\mathcal{M},s})$ is a finite-dimensional graded vector space. The pointwise Euler characteristic $\chi(P^\bullet_{\mathcal{M},s})$ is the *Behrend function* $\nu_{\mathcal{M}}$ of $\mathcal{M}$. Thus

$$\sum_{i\in\mathbb{Z}}(-1)^i \dim \mathbb{H}^i(P^\bullet_{\mathcal{M},s}) = \chi(\mathcal{M},\nu_{\mathcal{M}}).$$

Now by Behrend 2005, the Donaldson–Thomas invariant of $\mathcal{M}$ is $DT(\mathcal{M}) = \chi(\mathcal{M},\nu_{\mathcal{M}})$. So, $\mathbb{H}^*(P^\bullet_{\mathcal{M},s})$ is a graded vector space with dimension $DT(\mathcal{M})$, that is, a *categorification* of $DT(\mathcal{M})$.

Categorifying Lagrangian intersections

The second corollary in lecture 1, §2 implies:

Corollary

Let (S,ω) be a classical smooth symplectic $\mathbb{K}$-scheme of dimension $2n$, and $L, M \subseteq S$ be smooth algebraic Lagrangians, with square roots $K_L^{1/2}, K_M^{1/2}$ of their canonical bundles. Then we have a natural perverse sheaf $P^\bullet_{L,M}$ on $X = L \cap M$.

This is related to Behrend and Fantechi 2009. We think of the hypercohomology $\mathbb{H}^*(P^\bullet_{L,M})$ as being morally related to the *Lagrangian Floer cohomology* $HF^*(L,M)$ by

$$\mathbb{H}^i(P^\bullet_{L,M}) \approx HF^{i+n}(L,M).$$

We are working on defining 'Fukaya categories' for algebraic/complex symplectic manifolds using these ideas (§8).

Extension to Artin stacks

Let (X, s) be a d-critical stack, with an orientation $K_{X,s}^{1/2}$. Then for any smooth $\varphi : U \to X$ with U a scheme, $\big(U, s(U,\varphi)\big)$ is an oriented d-critical locus, so as above, BBDJS constructs a perverse sheaf $P^\bullet_{U,\varphi}$ on U. Given a diagram

$$\begin{array}{ccccc} & & V & & \\ & \overset{\alpha}{\nearrow} & \eta\Uparrow & \overset{\psi}{\searrow} & \\ U & & \xrightarrow{\quad\varphi\quad} & & X \end{array}$$

with U, V schemes and φ, ψ smooth, we can construct a natural isomorphism $P^\bullet_{\alpha,\eta} : \alpha^*(P^\bullet_{V,\psi})[\dim\varphi - \dim\psi] \to P^\bullet_{U,\varphi}$.
All this data $P^\bullet_{U,\varphi}, P^\bullet_{\alpha,\eta}$ is equivalent to a perverse sheaf on X.

Thus we prove:

Theorem (Ben-Bassat, Brav, Bussi, Joyce)

Let (X, s) be a d-critical stack, with an orientation $K_{X,s}^{1/2}$. Then we can construct a canonical perverse sheaf $P^\bullet_{X,s}$ on X.

Corollary

Suppose Y is a Calabi–Yau 3-fold and $\mathcal{M}$ a classical moduli stack of coherent sheaves F on Y, or of complexes $F^\bullet$ in $D^b\mathrm{coh}(Y)$ with $\mathrm{Ext}^{<0}(F^\bullet, F^\bullet) = 0$, with (symmetric) obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$. Suppose we are given a square root $\det(\mathcal{E}^\bullet)^{1/2}$ for $\det(\mathcal{E}^\bullet)$. Then we construct a natural perverse sheaf $P^\bullet_{\mathcal{M},s}$ on $\mathcal{M}$.

The *hypercohomology* $\mathbb{H}^*(P^\bullet_{\mathcal{M},s})$ is a categorification of the Donaldson–Thomas theory of Y.

7. Algebraic structures on perverse sheaves

Let $(\mathbf{X}, \omega_{\mathbf{X}})$ be a -1-shifted symplectic derived scheme, and $\mathbf{i}: \mathbf{L} \to \mathbf{X}$ a Lagrangian, in the sense of PTVV.
Choose an orientation $K_{X,s}^{1/2}$ for $(\mathbf{X}, \omega_{\mathbf{X}})$. There is then a notion of relative orientation for $\mathbf{i}: \mathbf{L} \to \mathbf{X}$, choose one of these.
We get a perverse sheaf $P^\bullet_{\mathbf{X},\omega_{\mathbf{X}}}$ on $\mathbf{X}$, by BBDJS in §6.

Conjecture

There is a natural morphism in $D^b_c(\mathbf{L})$

$$\mu_{\mathbf{L}} : \mathbb{Q}_{\mathbf{L}}[\mathrm{vdim}\,\mathbf{L}] \longrightarrow \mathbf{i}^!(P^\bullet_{\mathbf{X},\omega_{\mathbf{X}}}), \tag{2}$$

with given local models in 'Darboux form' presentations for $\mathbf{X}, \mathbf{L}$.

This Conjecture has important consequences (§8, §11).

I already know local models for $\mathbf{i}: \mathbf{L} \to \mathbf{X}$ and $\mu_{\mathbf{L}}$ in (2). What makes the Conjecture difficult is that local models are not enough: $\mu_{\mathbf{L}}$ is a morphism of complexes, not of (perverse) sheaves, and such morphisms do not glue like sheaves. For instance, one could imagine $\mu_{\mathbf{L}}$ to be globally nonzero, but zero on the sets of an open cover of $\mathbf{L}$.
So to construct $\mu_{\mathbf{L}}$, we have to do a gluing problem in an ∞-category, probably using hypercovers. I have a sketch of one way to do this (over $\mathbb{C}$). It is not easy.
Maybe gluing local models naïvely is not the best approach for this problem, need some more advanced Lurie-esque technology? Any help would be appreciated.

8. 'Fukaya categories' of complex symplectic manifolds

Let (S, ω) be a complex symplectic manifold, with $\dim_{\mathbb{C}} S = 2n$, and $L, M \subset S$ be complex Lagrangians (not supposed compact or closed). The intersection $L \cap M$, as a complex analytic space, has a d-critical structure s (Vittoria Bussi, work in progress). Given square roots of canonical bundles $K_L^{1/2}, K_M^{1/2}$, we get an orientation on $(L \cap M, s)$, and so a perverse sheaf $P^\bullet_{L,M}$ on $L \cap M$.
I claim that we should think of the shifted hypercohomology $\mathbb{H}^{*-n}(P^\bullet_{L,M})$ as a substitute for the Lagrangian Floer cohomology $HF^*(L, M)$ in symplectic geometry. But $HF^*(L, M)$ is the morphisms in the derived Fukaya category $D^b\mathrm{Fuk}(S, \omega)$.

Problem

Given a complex symplectic manifold (S, ω), build a 'Fukaya category' with objects $(L, K_L^{1/2})$ for L a complex Lagrangian, and graded morphisms $\mathbb{H}^{*-n}(P^\bullet_{L,M})$.
Extend to **derived** *Lagrangians L in (S, ω).*
Work out the 'right' way to form a 'derived Fukaya category' for (S, ω) out of this, as a (Calabi–Yau?) triangulated category.
Show that (derived) Lagrangian correspondences induce functors between these derived Fukaya categories.

Question

Can we include complex coisotropic submanifolds as objects? Maybe using $\mathscr{D}$-modules?

The Conjecture in §7 is what we need to define composition of morphisms in this 'Fukaya category', as follows. If L, M, N are Lagrangians in (S, ω), then $M \cap L, N \cap M, L \cap N$ are -1-shifted symplectic / d-critical loci, and $L \cap M \cap N$ is Lagrangian in the product $(M \cap L) \times (N \cap M) \times (L \cap N)$ (lecture 1, §1). Applying the Conjecture to $L \cap M \cap N$ and rearranging gives a morphism of constructible complexes

$$\mu_{L,M,N} : P^\bullet_{L,M} \overset{L}{\otimes} P^\bullet_{M,N}[n] \longrightarrow P^\bullet_{L,N}.$$

Taking hypercohomology gives the multiplication $\mathrm{Hom}^*(L, M) \times \mathrm{Hom}^*(M, N) \to \mathrm{Hom}^*(L, N)$.