Motivic Donaldson-Thomas theory, Calabi-Yau 4-fold counting invariants, and future projects

Lecture 3 of 3

Dominic Joyce, Oxford University

January 2014

Based on: arXiv:1211.3259, arXiv:1305.6428, arXiv:1312.0090, and work in progress.
Joint work with Vittoria Bussi, Dennis Borisov, and Sven Meinhardt. Funded by the EPSRC.

Plan of talk:

9. Motives of d-critical loci

By similar (but easier) arguments to those used to construct the perverse sheaves $P^\bullet_{X,s}$ in lecture 2, §6, we prove:

Theorem (Bussi, Joyce and Meinhardt arXiv:1305.6428)

Let (X,s) be a finite type algebraic d-critical locus over $\mathbb{K}$, with an orientation $K^{1/2}_{X,s}$. Then we can construct a natural motive $MF_{X,s}$ in a certain ring of $\hat\mu$-equivariant motives $\bar{\mathcal{M}}^{\hat\mu}_X$ on X, such that if (X,s) is locally modelled on $\mathrm{Crit}(f:U\to\mathbb{A}^1)$, then $MF_{X,s}$ is locally modelled on $\mathbb{L}^{-\dim U/2}([X]-MF^{\mathrm{mot}}_{U,f})$, where $MF^{\mathrm{mot}}_{U,f}$ is the **motivic Milnor fibre** *or* **motivic nearby cycle** *of f.*

Torus localization of motives

Let (X,s) be an oriented, finite type d-critical locus, and $\rho:\mathbb{G}_m\times X\to X$ a $\mathbb{G}_m$-action on X preserving the orientation and scaling the d-critical structure by $\rho(\lambda)^\star(s)=\lambda^d s$ for some $d\in\mathbb{Z}$. If $d=0$ and ρ is 'good' and 'circle compact', Maulik (work in progress) proves a torus localization formula for the absolute motive $\pi_*(MF_{X,s})\in\bar{\mathcal{M}}^{\hat\mu}_{\mathbb{K}}$, with $\pi:X\to\mathrm{Spec}\,\mathbb{K}$ the projection

$$\pi_*(MF_{X,s})=\textstyle\sum_{i\in I}\mathbb{L}^{-\mathrm{ind}(X_i^{\mathbb{G}_m},X)/2}\odot\pi_*(MF_{X_i^{\mathbb{G}_m},s_i^{\mathbb{G}_m}}),$$

where $X^{\mathbb{G}_m}$ is the $\mathbb{G}_m$-fixed subscheme in X, and $X^{\mathbb{G}_m}=\coprod_{i\in I}X_i^{\mathbb{G}_m}$ its decomposition into connected components. It would be interesting to extend this to $d\neq 0$, and to consider torus localization for the perverse sheaves of lecture 2, §6.

Relation to motivic D–T invariants

The first corollary in lecture 1, §2 implies:

Corollary

Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a finite type classical moduli $\mathbb{K}$-scheme of (complexes of) coherent sheaves on Y, with (symmetric) obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$. Suppose we are given a square root $\det(\mathcal{E}^\bullet)^{1/2}$ for $\det(\mathcal{E}^\bullet)$ (i.e. **orientation data**, *K–S). Then we have a natural motive $MF^\bullet_{\mathcal{M},s}$ on $\mathcal{M}$.*

This motive $MF^\bullet_{\mathcal{M},s}$ is essentially the motivic Donaldson–Thomas invariant of $\mathcal{M}$ defined (partially conjecturally) by Kontsevich and Soibelman, arXiv:0811.2435. K–S work with motivic Milnor fibres of formal power series at each point of $\mathcal{M}$. Our results show the formal power series can be taken to be a regular function, and clarify how the motivic Milnor fibres vary in families over $\mathcal{M}$.

Motives of d-critical loci
D–T style invariants for Calabi-Yau 4-folds
Cohomological Hall Algebras
Gluing matrix factorization categories

Extension to Artin stacks

We can also generalize BJM to d-critical stacks:

Theorem (Ben-Bassat, Brav, Bussi, Joyce)

Let (X, s) be an oriented d-critical stack, of finite type and locally a global quotient. Then we can construct a natural motive $MF_{X,s}$ in a certain ring of $\hat{\mu}$-equivariant motives $\bar{\mathcal{M}}^{\mathrm{st},\hat{\mu}}_X$ on X, such that if $\varphi : U \to X$ is smooth and U is a scheme then

$$\varphi^*(MF_{X,s}) = \mathbb{L}^{\dim\varphi/2} \odot MF_{U,s(U,\varphi)},$$

where $MF_{U,s(U,\varphi)}$ for the scheme case is as in BJM above.

For CY3 moduli stacks, these $MF_{X,s}$ are basically Kontsevich–Soibelman's motivic Donaldson–Thomas invariants.

Note: all the rest of this lecture is either work in progress, or projects I hope to do soon, or things I'd like to prove but don't know how. A result in quotes ("Theorem", ...) means we haven't finished the proof yet.

10. D–T style invariants for Calabi-Yau 4-folds

If Y is a Calabi–Yau 3-fold (say over $\mathbb{C}$), then the *Donaldson-Thomas invariants* $DT^\alpha(\tau)$ in $\mathbb{Z}$ or $\mathbb{Q}$ 'count' τ-(semi)stable coherent sheaves on Y with Chern character $\alpha \in H^{\rm even}(Y, \mathbb{Q})$, for τ a (say Gieseker) stability condition. The $DT^\alpha(\tau)$ are unchanged under continuous deformations of Y, and transform by a wall-crossing formula under change of stability condition τ.
We have τ-(semi)stable moduli schemes $\mathcal{M}^\alpha_{\rm st}(\tau) \subseteq \mathcal{M}^\alpha_{\rm ss}(\tau)$, where $\mathcal{M}^\alpha_{\rm ss}(\tau)$ is proper, and $\mathcal{M}^\alpha_{\rm st}(\tau)$ has a symmetric obstruction theory.
The easy case (Thomas 1998) is when $\mathcal{M}^\alpha_{\rm ss}(\tau) = \mathcal{M}^\alpha_{\rm st}(\tau)$. Then $DT^\alpha(\tau) \in \mathbb{Z}$ is the *virtual cycle* (which has dimension zero) of the proper scheme with obstruction theory $\mathcal{M}^\alpha_{\rm st}(\tau)$.
Note that the derived moduli scheme $\boldsymbol{\mathcal{M}}^\alpha_{\rm st}(\tau)$ is -1-shifted symplectic by PTVV, and $\mathcal{M}^\alpha_{\rm st}(\tau)$ is a d-critical locus by BBJ.

Holomorphic Donaldson invariants?

In joint work with Dennis Borisov (in progress, preprint available on https://sites.google.com/site/dennisborisov/), I am developing a similar story for Calabi–Yau 4-folds. We want to define invariants 'counting' τ-(semi)stable coherent sheaves on Calabi–Yau 4-folds. If CY3 Donaldson–Thomas invariants are 'holomorphic Casson invariants', as in Thomas 1998, these should be thought of as 'holomorphic Donaldson invariants'.
The idea for doing this goes back to Donaldson–Thomas 1998, using gauge theory: one wants to 'count' moduli spaces of Spin(7)-instantons on a Calabi–Yau 4-fold (or more generally a Spin(7)-manifold). However, it has not gone very far, as compactifying such higher-dimensional gauge-theoretic moduli spaces in a nice way is too difficult. (See Cao arXiv:1309.4230.)

Virtual cycles using algebraic geometry?

Rather than using gauge theory, we stay within algebraic geometry, so we get compactness of moduli spaces more-or-less for free. So, suppose Y is a Calabi–Yau 4-fold, and $\alpha \in H^{\rm even}(Y,\mathbb{Q})$ such that $\mathcal{M}_{\rm ss}^\alpha(\tau)=\mathcal{M}_{\rm st}^\alpha(\tau)$ (the easy case). Then $\mathcal{M}_{\rm st}^\alpha(\tau)$ is proper, and the corresponding derived moduli scheme $\boldsymbol{\mathcal{M}}_{\rm st}^\alpha(\tau)$ is -2-shifted symplectic by PTVV. It need not have virtual dimension zero. Our task is to define a virtual cycle for $\boldsymbol{\mathcal{M}}_{\rm st}^\alpha(\tau)$, or more generally for any proper -2-shifted symplectic derived scheme $(\mathbf{X},\omega)$.
There is a natural obstruction theory $\phi:\mathcal{E}^\bullet\rightarrow\mathbb{L}_{\mathcal{M}}$ on $\mathcal{M}_{\rm st}^\alpha(\tau)$, but $\mathcal{E}^\bullet$ is perfect in $[-2,0]$ not $[-1,0]$, so the usual Behrend–Fantechi virtual cycles do not work.

Virtual cycles using d-manifolds

Here is the first part of what we want to prove:

"Theorem" (Borisov–Joyce, in progress, proof nearly finished)

Let $(\mathbf{X}, \omega)$ be a -2-shifted symplectic derived scheme over $\mathbb{C}$. Then one can construct a **d-manifold** *(derived smooth manifold) $X_{\rm dm}$ which has the same underlying topological space X as $(\mathbf{X}, \omega)$, with the complex analytic topology.*
The construction involves arbitrary choices, but $X_{\rm dm}$ is unique up to bordisms which fix the topological space X.
The (real) virtual dimension of $X_{\rm dm}$ is

$$\mathrm{vdim}_{\mathbb{R}} X_{\rm dm} = \mathrm{vdim}_{\mathbb{C}} \mathbf{X} = \tfrac{1}{2} \mathrm{vdim}_{\mathbb{R}} \mathbf{X},$$

which is half what one would have expected.

D-manifolds and bordisms

I haven't time to explain d-manifolds properly – see my webpage `people.maths.ox.ac.uk/~joyce/dmanifolds.html`, and arXiv:1206.4207, arXiv:1208.4948.
Two useful facts: firstly, a d-manifold $X_{\rm dm}$ is locally modelled by a 'Kuranishi neighbourhood' (V, E, s) of a real manifold V, real vector bundle $E \to V$ and smooth section $s : V \to E$, where the topological space of $X_{\rm dm}$ is locally homeomorphic to $s^{-1}(0) \subset V$.
Think of $X_{\rm dm}$ as locally the (homotopy) fibre product $V \times_{s,E,0} V$.
Secondly, any (compact) d-manifold $\mathbf{X}$ can be perturbed to a (compact) ordinary manifold $\tilde{X}$, which is unique up to bordism. In a Kuranishi neighbourhood (V, E, s), perturb s to a generic, transverse $\tilde{s} : V \to E$, so that $\tilde{s}^{-1}(0) \subset V$ is a manifold.
Thus, if $(\mathbf{X}, \omega)$ is a proper -2-shifted symplectic derived $\mathbb{C}$-scheme, the "Theorem" will give us a bordism class of (unoriented) compact manifolds $\tilde{X}$, which is basically a virtual cycle over $\mathbb{Z}_2$.

Orientations of -2-shifted symplectic derived schemes

To lift this to a virtual cycle over $\mathbb{Z}$, we need to include orientations of $(\mathbf{X},\omega)$ and $X_{\rm dm}$.
Recall that if $(\mathbf{X},\omega)$ is a -1-shifted symplectic derived scheme (the Calabi–Yau 3 case), an *orientation* of $(\mathbf{X},\omega)$ is a square root line bundle $\det(\mathbb{L}_{\mathbf{X}})^{1/2}$. These were introduced by Kontsevich and Soibelman, and are essential for motivic and categorified D–T theory. Here is the Calabi–Yau 4 analogue:

Definition

Let $(\mathbf{X},\omega)$ be a -2-shifted symplectic derived scheme. There is a natural isomorphism $\iota:\det(\mathbb{L}_{\mathbf{X}})^{\otimes^2}\longrightarrow\mathcal{O}_{\mathbf{X}}$. An *orientation* of $(\mathbf{X},\omega)$ is an isomorphism $\alpha:\det(\mathbb{L}_{\mathbf{X}})\longrightarrow\mathcal{O}_{\mathbf{X}}$ with $\alpha\otimes\alpha=\iota$.

Note that this is simpler, one categorical level down from the CY3 case: a morphism in a category, not an object in a category.

The next two results will be easy, given the "Theorem".

"Lemma"

In the "Theorem", there is a natural 1-1 *correspondence between orientations on* $(\mathbf{X},\omega)$ *and orientations on the d-manifold* $X_{\rm dm}$.

"Corollary"

Let $(\mathbf{X},\omega)$ *be a proper, oriented* -2*-shifted symplectic derived* $\mathbb{C}$*-scheme. Then we construct a bordism class* $[X_{\rm dm}]$ *of compact oriented manifolds. We consider this a* **virtual cycle** *for* $(\mathbf{X},\omega)$.

Observe that though all the input data is strictly complex algebraic, the 'virtual cycle' can have odd real dimension, which is weird, and very unlike Behrend–Fantechi style virtual cycles.

Sketch proof of "Theorem"

Let $(\mathbf{X},\omega)$ be a -2-shifted symplectic derived $\mathbb{C}$-scheme. Then the BBJ 'Darboux Theorem' gives local models for $(\mathbf{X},\omega)$ in the Zariski topology. As in lecture 1, §2, in the -2-shifted case, the local models reduce to the following data:

- A smooth $\mathbb{C}$-scheme U
- A vector bundle $E\to U$
- A section $s\in H^0(E)$
- A nondegenerate quadratic form Q on E with $Q(s,s)=0$.

The underlying topological space of $\mathbf{X}$ is $\{x\in U: s(x)=0\}$. The virtual dimension of $\mathbf{X}$ is $\mathrm{vdim}_{\mathbb{C}}\mathbf{X}=2\dim_{\mathbb{C}}U-\mathrm{rank}_{\mathbb{C}}E$. The cotangent complex $\mathbb{L}_{\mathbf{X}}|_X$ of $\mathbf{X}$ is

$$\bigl[\underset{-2}{TU}|_{s^{-1}(0)}\xrightarrow{Q\circ\mathrm{d}s}\underset{-1}{E^*}|_{s^{-1}(0)}\xrightarrow{\mathrm{d}s}\underset{0}{T^*U}|_{s^{-1}(0)}\bigr].$$

The local model for $X_{\rm dm}$

Here is how to build the d-manifold $X_{\rm dm}$ locally: regard $E\to U$ as a real vector bundle over the real manifold U. Choose a splitting $E=E_+\oplus E_-$, where $Q|_{E_+}$ is real and positive definite, and $E_-=iE_+$ so that $Q|_{E_-}$ is real and negative definite. Write $s=s_+\oplus s_-$ with $s_\pm\in C^\infty(E_\pm)$. Then $X_{\rm dm}$ is locally the derived fibre product $U\times_{0,E_+,s_+}U$, given by the 'Kuranishi neighbourhood' (U,E_+,s_+). It has virtual dimension

$$\dim_{\mathbb{R}}U-\mathrm{rank}_{\mathbb{R}}E_+=2\dim_{\mathbb{C}}U-\mathrm{rank}_{\mathbb{C}}E=\mathrm{vdim}_{\mathbb{C}}\mathbf{X}.$$

Observe that $Q(s,s)=0$ implies that $|s_+|^2=|s_-|^2$, where norms $|\,.\,|$ on E_+,E_- are defined using $\pm\mathrm{Re}\,Q$. Hence as sets we have

$$\{x\in U:s(x)=0\}=\{x\in U:s_+(x)=0\}\subseteq U.$$

This is why $\mathbf{X}$ and $X_{\rm dm}$ have the same topological space X. The difficult bit is to show we can choose compatible splittings $E=E_+\oplus E_-$ on an open cover of $\mathbf{X}$, and glue the local models to make a global d-manifold $X_{\rm dm}$.

Relation to the perverse sheaf picture

Let $(\boldsymbol{X},\omega)$ be an oriented -2-shifted symplectic derived scheme over $\mathbb{C}$, e.g. a Calabi–Yau 4-fold derived moduli scheme. Regard the point $*$ as an oriented -1-shifted symplectic derived scheme. Its perverse sheaf is the constant sheaf $\mathbb{Q}_*$. Then $\boldsymbol{\pi}:\boldsymbol{X}\to *$ is Lagrangian in $(*,\omega)$, so the Conjecture in lecture 2, §7 gives a morphism

$$\mu_{\boldsymbol{X}}:\mathbb{Q}_{\boldsymbol{X}}[\mathrm{vdim}\,\boldsymbol{X}]\longrightarrow \boldsymbol{\pi}^!(\mathbb{Q}_*)=\mathcal{D}_{\boldsymbol{X}}(\mathbb{Q}_{\boldsymbol{X}}).$$

Taking hypercohomology induces a linear map

$$H^{\mathrm{vdim}\,\boldsymbol{X}}_{\mathrm{cs}}(X,\mathbb{Q})\longrightarrow\mathbb{Q}.$$

If $\boldsymbol{X}$ is compact, this should be contraction with a class $[\boldsymbol{X}]_{\mathrm{virt}}\in H_{\mathrm{vdim}\,\boldsymbol{X}}(X,\mathbb{Q})$. I expect this to be the virtual cycle above.
Note that this also works over other fields $\mathbb{K}\neq\mathbb{C}$.
The perverse sheaf picture does not obviously explain why $[\boldsymbol{X}]_{\mathrm{virt}}$ should be deformation-invariant.

11. Cohomological Hall Algebras

Let Y be a Calabi–Yau 3-fold, and $\boldsymbol{\mathcal{M}}$ the derived moduli stack of coherent sheaves (or suitable complexes) on Y, with its -1-shifted symplectic structure ω. Then BBBJ makes the classical stack $\mathcal{M}$ into a d-critical stack $(\mathcal{M},s)$. Suppose we have 'orientation data' for Y, i.e. an orientation $K^{1/2}_{\mathcal{M},s}$, with compatibility condition on exact sequences. Then we have a perverse sheaf $P^\bullet_{\mathcal{M},s}$, with hypercohomology $\mathbb{H}^*(P^\bullet_{\mathcal{M},s})$.
We would like to define an associative multiplication on $\mathbb{H}^*(P^\bullet_{\mathcal{M},s})$, making it into a *Cohomological Hall Algebra*, in the style of Kontsevich and Soibelman (arXiv:1006.2706).

Let $\boldsymbol{\mathcal{E}}$**xact** be the derived stack of short exact sequences $0 \to F_1 \to F_2 \to F_3 \to 0$ in $\mathrm{coh}(Y)$ (or distinguished triangles in $D^b \mathrm{coh}(Y)$), with projections $\pi_1, \pi_2, \pi_3 : \boldsymbol{\mathcal{E}}\mathbf{xact} \to \boldsymbol{\mathcal{M}}$.

"Theorem" (Oren Ben-Bassat, work in progress.)

$\pi_1 \times \pi_2 \times \pi_3 : \boldsymbol{\mathcal{E}}\mathbf{xact} \to (\boldsymbol{\mathcal{M}}, \omega) \times (\boldsymbol{\mathcal{M}}, -\omega) \times (\boldsymbol{\mathcal{M}}, \omega)$ *is Lagrangian in* -1*-shifted symplectic.*

Then apply the stack version of the Conjecture in lecture 2, §7 to get COHA multiplication, as for the Fukaya category case.

12. Gluing matrix factorization categories

Suppose $f : U \to \mathbb{A}^1$ is a regular function on a smooth scheme U. The *matrix factorization category* $\mathrm{MF}(U, f)$ is a $\mathbb{Z}_2$-periodic triangulated category. It depends only on U, f in a neighbourhood of $\mathrm{Crit}(f)$, and we can think of it as a sheaf of triangulated categories on $\mathrm{Crit}(f)$. By BBJ, -1-shifted symplectic derived schemes $(X, \omega_{\mathbf{X}})$ are locally modelled on $\mathrm{Crit}(f : U \to \mathbb{A}^1)$.

Problem

Given a -1*-shifted symplectic derived scheme* $(X, \omega_{\mathbf{X}})$ *with extra data (orientation and 'spin structure'?), construct a sheaf of* $\mathbb{Z}_2$*-periodic triangulated categories* $\mathrm{MF}_{\mathbf{X},\omega_{\mathbf{X}}}$ *on* $\mathbf{X}$*, such that if* $(\mathbf{X}, \omega_{\mathbf{X}})$ *is locally modelled on* $\mathrm{Crit}(f : U \to \mathbb{A}^1)$*, then* $\mathrm{MF}_{\mathbf{X},\omega_{\mathbf{X}}}$ *is locally modelled on* $\mathrm{MF}(U, f)$*.*

Although d-critical loci (X, s) are also locally modelled on $\mathrm{Crit}(f : U \to \mathbb{A}^1)$, I do not expect the analogue for d-critical loci to work; $\mathrm{MF}_{\mathbf{X},\omega_{\mathbf{X}}}$ will encode derived data in $(\mathbf{X}, \omega_{\mathbf{X}})$ which is forgotten by the d-critical locus (X, s).

I expect that a Lagrangian $\mathbf{i} : \mathbf{L} \to \mathbf{X}$ (plus extra data) should define an object (global section of sheaf of objects) in $\mathrm{MF}_{\mathbf{X},\omega_{\mathbf{X}}}$, with nice properties.

It is conceivable that one could actually *define* $\mathrm{MF}_{\mathbf{X},\omega_{\mathbf{X}}}$ as a derived 'Fukaya category' of Lagrangians $\mathbf{i} : \mathbf{L} \to \mathbf{X}$ in $(\mathbf{X}, \omega_{\mathbf{X}})$.

Kapustin–Rozansky 2-categories for complex symplectic manifolds

Given a complex symplectic manifold (S, ω), Kapustin and Rozansky conjecture the existence of an interesting 2-category, with objects complex Lagrangians L with $K_L^{1/2}$, such that $\mathrm{Hom}(L, M)$ is a $\mathbb{Z}_2$-periodic triangulated category (or sheaf of such on $L \cap M$), and if $L \cap M$ is locally modelled on $\mathrm{Crit}(f : U \to \mathbb{A}^1)$ then $\mathrm{Hom}(L, M)$ is locally modelled on $\mathrm{MF}(U, f)$.

A lot of this K–R Conjecture would follow by combining lecture 2, §8, Fukaya categories and §12, Gluing matrix factorization categories above.

Seeing what the rest of the K–R Conjecture requires should tell us some interesting properties to expect of $\mathrm{MF}_{\mathbf{X},\omega_{\mathbf{X}}}$.

Categorifying Cohomological Hall Algebras?

Question

Do Cohomological Hall algebras of Calabi–Yau 3-folds Y admit a categorification using matrix factorization categories, in a similar way to the Kapustin–Rozansky conjectured categorification of the 'Fukaya categories' of complex symplectic manifolds?

Let $\boldsymbol{\mathcal{M}}$ be the derived moduli stack of coherent sheaves on Y, with its -1-shifted symplectic structure ω, and discrete extra data (orientation and 'spin structure'). One would expect to build such a categorification by writing $\mathcal{M}$ as a critical locus locally in the smooth topology, and then 'gluing' the associated matrix factorization categories.

Compare Kontsevich and Soibelman arXiv:1006.2706, §8.1, 'Categorification of critical COHA'.