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## Introduction to Kähler manifolds

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# INTRODUCTION TO KÄHLER MANIFOLDS
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## Introduction

These notes are intended to accompany the course Introduction to Complex, Hermitian and Kähler Manifolds at the ICTP Summer School on Hodge Theory. Their purpose is to review the concepts and results in the theory of Kähler manifolds that both motivate and are at the center of Hodge Theory. Although we have tried to clearly define the main objects of study we have often referred to the literature for proofs of the main results. We are fortunate in that there are several excellent books on this subject and we have freely drawn from them in the preparation of these notes, which make no claim of originality. The classical references remain the pioneer books by A. Weil [38], S. S. Chern [10], J. Morrow and K. Kodaira [22, 25], R. O. Wells [39], S. Kobayashi[21], and P. Griffiths and J. Harris [15]. In these notes we refer most often to two superb recent additions to the literature: the two-volume work by C. Voisin [34, 35] and D. Huybrechts' book [18].

We assume from the outset that the reader is familiar with the basic theory of smooth manifolds at the level of [2], [23], or [33]. Some of this material as well as the basics of cohomology theory will be reviewed in the course by Loring Tu. The book by R. Bott and L. Tu [3] is an excellent introduction to the algebraic topology of smooth manifolds.

These notes consist of five sections which correspond, roughly, to the five lectures in the course. There is also an Appendix which collects some results on the linear algebra of complex vector spaces, Hodge structures, nilpotent linear transformations, and representations of $\mathfrak{sl}(2,\mathbb{C})$. There are many exercises interspersed throughout the text, many of which ask the reader to prove, or complete the proof, of some result in the notes.

A final version of these notes will be posted after the conclusion of the Summer School.

## Contents

## 1. Complex Manifolds

1.1. **Definition and Examples.** Let $U \subset \mathbb{C}^n$ be an open subset and $f\colon U \to \mathbb{C}$. We say that $f$ is holomorphic if and only if it is holomorphic as a function of each variable separately; i.e. if we fix $z_\ell = a_\ell$, $\ell \neq j$ then the function $f(a_1, \dots, z_j, \dots, a_n)$ is a holomorphic function of $z_j$. A map $F = (f_1, \dots, f_n)\colon U \to \mathbb{C}^n$ is said to be *holomorphic* if each component $f_k = f_k(z_1, \dots, z_n)$ is holomorphic. If we identify $\mathbb{C}^n \cong \mathbb{R}^{2n}$, and set $z_j = x_j + iy_j$, $f_k = u_k + iv_k$, $j, k = 1, \dots, n$, then the functions $u_k$, $v_k$ are $C^\infty$ functions of the variables $x_1, y_1, \dots, x_n, y_n$ and satisfy the Cauchy-Riemann equations:

$$\frac{\partial u_k}{\partial x_j} = \frac{\partial v_k}{\partial y_j}\,; \quad \frac{\partial u_k}{\partial y_j} = -\frac{\partial v_k}{\partial x_j} \tag{1.1}$$

Conversely, if $(u_1, v_1, \dots, u_n, v_n)\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a $C^\infty$ map satisfying the Cauchy-Riemann equations (1.1) then the map $(u_1 + iv_1, \dots, u_n + iv_n)$ is holomorphic. In other words a $C^\infty$ map $F\colon U \subset \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ defines a holomorphic map $\mathbb{C}^n \to \mathbb{C}^n$ if and only if the differential of $F$, written in terms of the basis

$$\{\partial/\partial x_1, \dots, \partial/\partial x_n, \partial/\partial y_1, \dots, \partial/\partial y_n\} \tag{1.2}$$

of the tangent space $T_p(\mathbb{R}^{2n})$ and the basis $\{\partial/\partial u_1, \dots, \partial/\partial u_n, \partial/\partial v_1, \dots, \partial/\partial v_n\}$ of $T_{F(p)}(\mathbb{R}^{2n})$ is of the form:

$$DF(p) = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \tag{1.3}$$

for all $p \in U$. Thus, it follows from Exercise 36 in Appendix A.1 that $F$ is holomorphic if and only if $DF(p)$ defines a $\mathbb{C}$-linear map $\mathbb{C}^n \to \mathbb{C}^n$.

**Exercise 1.** Prove that a $(2n) \times (2n)$-matrix is of the form (1.3) if and only if it commutes with the matrix $J$:

$$J := \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \tag{1.4}$$

**Definition 1.1.** A complex structure on a topological manifold $M$ consists of a collection of *coordinate charts* $(U_\alpha, \phi_\alpha)$ such that:

- i) The sets $U_\alpha$ are an open covering of $M$.
- ii) $\phi_\alpha\colon U_\alpha \to \mathbb{C}^n$ is a homeomorphism of $U_\alpha$ onto an open subset of $\mathbb{C}^n$ for some fixed $n$. We call $n$ the complex dimension of $M$.
- iii) If $U_\alpha \cap U_\beta \neq \emptyset$, the map

$$\phi_\beta \circ \phi_\alpha^{-1}\colon \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta) \tag{1.5}$$

is holomorphic.

**Example 1.2.** The most basic example of a complex manifold is $\mathbb{C}^n$ or any open subset of $\mathbb{C}^n$. For any $p \in \mathbb{C}^n$, the tangent space $T_p(\mathbb{C}^n) \cong \mathbb{R}^{2n}$ which is identified, in the natural way, with $\mathbb{C}^n$ itself.

**Example 1.3.** Since $\mathrm{GL}(n, \mathbb{C})$, the set of non-singular $n \times n$ matrices with complex coefficients, is an open set in $\mathbb{C}^{n^2}$, we may view $\mathrm{GL}(n, \mathbb{C})$ as a complex manifold.

**Example 1.4.** The basic example of a compact complex manifold is complex projective space which we will simply denote by $\mathbb{P}^n$. Recall that:

$$\mathbb{P}^n \ := \ \left(\mathbb{C}^{n+1} \setminus \{0\}\right) / \sim ,$$

where $\sim$ is the equivalence relation: $z \sim z'$ if and only if there exists $\lambda \in \mathbb{C}^*$ such that $z' = \lambda z$; $z, z' \in \mathbb{C}^{n+1} \setminus \{0\}$. We denote the equivalence class of a point $z \in \mathbb{C}^{n+1} \setminus \{0\}$ by $[z]$. The sets

$$U_i \ := \{[z] \in \mathbb{P}^n : z_i \neq 0\} \tag{1.6}$$

are open and the maps

$$\phi_i \colon U_i \to \mathbb{C}^n \ ; \quad \phi_i([z]) = \left(\frac{z_0}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_n}{z_i}\right) \tag{1.7}$$

define local coordinates such that the maps

$$\phi_i \circ \phi_j^{-1} \colon \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j) \tag{1.8}$$

are holomorphic.

In particular, if $n = 1$, $\mathbb{P}^1$ is covered by two coordinate neighborhoods $(U_0, \phi_0)$, $(U_1, \phi_1)$ with $\phi_i(U_i) = \mathbb{C}$. The coordinate changes are given by the maps $\phi_1 \circ \phi_0^{-1} \colon \mathbb{C}^* \to \mathbb{C}^*$:

$$\phi_1 \circ \phi_0^{-1}(z) = \phi_1([1, z]) = 1/z.$$

Thus, this is the usual presentation of the sphere $S^2$ as the Riemann sphere. We usually identify $U_0 \cong \mathbb{C}$ and denote the point $[0, 1] = \infty$.

**Exercise 2.** Verify that the map (1.8) is holomorphic.

**Example 1.5.** To each point $[z] \in \mathbb{P}^n$ we may associate the line spanned by $z$ in $\mathbb{C}^{n+1}$; hence, we may regard $\mathbb{P}^n$ as the space of lines in $\mathbb{C}^{n+1}$. This construction may then be generalized by considering $k$-dimensional subspaces in $\mathbb{C}^n$. This is the so-called Grassmann manifold $G(k, n)$. To define the complex manifold structure on $G(k, n)$, we consider first of all the open set in $\mathbb{C}^{nk}$:

$$V(k, n) = \{W \in \mathcal{M}(n \times k, \mathbb{C}) : \operatorname{rank}(W) = k\}.$$

The Grassmann manifold $G(k, n)$ may then be viewed as the quotient space:

$$G(k, n) \ := \ V(k, n)/ \sim \ ,$$

where $W \sim W'$ if and only if there exists $M \in \mathrm{GL}(k, \mathbb{C})$ such that $W' = W \cdot M$; equivalently, $W \sim W'$ if and only if the column vectors of $W$ and $W'$ span the same $k$-dimensional subspace $\Omega \subset \mathbb{C}^n$.

Given an index set $I = \{1 \leq i_1 < \cdots < i_k \leq n\}$ and $W \in V(k, n)$, we consider the $k \times k$ matrix $W_I$ consisting of the $I$-rows of $W$ and note that if $W \sim W'$ then, for every index set $I$, $\det(W_I) \neq 0$ if and only if $\det(W'_I) \neq 0$. We then define:

$$U_I \ := \ \{[W] \in G(k, n) : \det(W_I) \neq 0\}$$

This is clearly an open set in $G(k, n)$ and the map:

$$\phi_I \colon U_I \to \mathbb{C}^{(n-k)k} \ ; \quad \phi_I([W]) = W_{I^c} \cdot W_I^{-1} ,$$

where $I^c$ denotes the $(n-k)$-tuple of indices complementary to $I$. The maps $\phi_I$ define coordinates in $U_I$ and one can easily verify that given index sets $I$ and $J$, the maps:

$$\phi_I \circ \phi_J^{-1} \colon \phi_J(U_I \cap U_J) \to \phi_I(U_I \cap U_J) \tag{1.9}$$

are holomorphic.

**Exercise 3.** Verify that the map (1.9) is holomorphic.

**Exercise 4.** Prove that both $\mathbb{P}^n$ and $G(k,n)$ are compact.

The notion of a holomorphic map between complex manifolds is defined in a way completely analogous to that of a smooth map between $C^\infty$ manifolds; i.e. if $M$ and $N$ are complex manifolds of dimension $m$ and $n$ respectively, a map $F \colon M \to N$ is said to be *holomorphic* if for each $p \in M$ there exists local coordinate systems $(U,\phi)$, $(V,\psi)$ around $p$ and $q = F(p)$, respectively, such that $F(U) \subset V$ and the map

$$\psi \circ F \circ \phi^{-1} \colon \phi(U) \subset \mathbb{C}^m \to \psi(V) \subset \mathbb{C}^n$$

is holomorphic. Given an open set $U \subset M$ we will denote by $\mathcal{O}(U)$ the ring of holomorphic functions $f \colon U \to \mathbb{C}$ and by $\mathcal{O}^*(U)$ the nowhere zero holomorphic functions on $U$. A map between complex manifolds is said to be *biholomorphic* if it is holomorphic and has a holomorphic inverse.

The following result shows a striking difference between $C^\infty$ and complex manifolds:

**Theorem 1.6.** *If $M$ is a compact, connected, complex manifold and $f \colon M \to \mathbb{C}$ is holomorphic, then $f$ is constant.*

*Proof.* The proof uses the fact that the Maximum Principle[†] holds for holomorphic functions of several complex variables (cf. [34, Theorem 1.21]) as well as the Principle of Analytic Continuation[‡] [34, Theorem 1.22]. ☐

Given a holomorphic map $F = (f_1, \ldots, f_n) \colon U \subset \mathbb{C}^n \to \mathbb{C}^n$ and $p \in U$, we can associate to $F$ the $\mathbb{C}$-linear map

$$DF(p) \colon \mathbb{C}^n \to \mathbb{C}^n \ ; \quad DF(p)(v) = \left(\frac{\partial f_i}{\partial z_j}(p)\right) \cdot v,$$

where $v = (v_1, \ldots, v_n)^T \in \mathbb{C}^n$. The Cauchy-Riemann equations imply that if we regard $F$ as a smooth map $\tilde{F} \colon U \subset \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ then the matrix of the differential $D\tilde{F}(p) \colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is of the form (1.3) and, clearly $DF(p)$ is non-singular if and only if $D\tilde{F}(p)$ is non-singular. In that case, by the Inverse Function Theorem, $\tilde{F}$ has a local inverse $\tilde{G}$ whose differential is given by $(D\tilde{F}(p))^{-1}$. By Exercise 1, the inverse of a non-singular matrix of the form (1.3) is of the same form. Hence, it follows that $\tilde{G}$ is holomorphic and, consequently, $F$ has a local holomorphic inverse. We then get:

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[†] If $f \in \mathcal{O}(U)$, where $U \subset \mathbb{C}^n$ is open, has a local maximum at $p \in U$, then $f$ is constant in a neighborhood of $p$

[‡] If $U \subset \mathbb{C}^n$ is a connected open subset and $f \in \mathcal{O}(U)$ is constant on an open subset $V \subset U$, then $f$ is constant on $U$.

**Theorem 1.7** (**Holomorphic Inverse Function Theorem**). *Let* $F\colon U \to V$ *be a holomorphic map between open subsets* $U, V \subset \mathbb{C}^n$*. If* $DF(p)$ *is non singular for* $p \in U$ *then there exists open sets* $U'$, $V'$*, such that* $p \in U' \subset U$ *and* $F(p) \in V' \subset V$ *and such that* $F\colon U' \to V'$ *is a biholomorphic map.*

The fact that we have a holomorphic version of the Inverse Function Theorem means that we may also extend the Implicit Function Theorem or, more generally, the Rank Theorem:

**Theorem 1.8** (**Rank Theorem**). *Let* $F\colon U \to V$ *be a holomorphic map between open subsets* $U \subset \mathbb{C}^n$ *and* $V \subset \mathbb{C}^m$*. If* $DF(q)$ *has rank* $k$ *for all* $q \in U$ *then, given* $p \in U$ *there exists open sets* $U'$, $V'$*, such that* $p \in U' \subset U$, $F(p) \in V' \subset V$, $F(U') \subset V'$*, and biholomorphic maps* $\phi\colon U' \to A$, $\psi\colon V' \to B$*, where* $A$ *and* $B$ *are open sets of the origin in* $\mathbb{C}^n$ *and* $\mathbb{C}^m$*, respectively, so that the composition*

$$\psi \circ F \circ \phi^{-1}\colon A \to B$$

*is the map* $(z_1, \ldots, z_n) \in A \mapsto (z_1, \ldots, z_k, 0, \ldots, 0)$*.*

*Proof.* We refer to [2, Theorem 7.1] or [33] for a proof in the $C^\infty$ case which can easily be generalized to the holomorphic case. ☐

Given a holomorphic map $F\colon M \to N$ between complex manifolds and $p \in M$, we may define the rank of $F$ at $p$ as

$$\operatorname{rank}_p(F) \ := \ \operatorname{rank}(D(\psi \circ F \circ \phi^{-1})(\phi(p)))\,, \tag{1.10}$$

for any local-coordinates expression of $F$ around $p$.

**Exercise 5.** Prove that $\operatorname{rank}_p(F)$ is well-defined by (1.10); i.e. it's independent of the choices of local coordinates.

We then have the following consequence of the Rank Theorem:

**Theorem 1.9.** *Let* $F\colon M \to N$ *be a holomorphic map, let* $q \in F(M)$ *and let* $X = F^{-1}(q)$*. Suppose* $rank_x(F) = k$ *for all* $x$ *in an open set* $U$ *containing* $X$*. Then,* $X$ *is a complex manifold and*

$$\operatorname{codim}(X) \ := \ \dim M - \dim X \ = \ k\,.$$

*Proof.* The Rank Theorem implies that given $p \in X$ there exist local coordinates $(U, \phi)$ and $(V, \psi)$ around $p$ and $q$, respectively such that $\psi(q) = 0$ and

$$\psi \circ F \circ \phi^{-1}(z_1, \ldots, z_n) = (z_1, \ldots, z_k, 0, \ldots, 0).$$

Hence

$$\phi(U \cap X) = \{z \in \phi(U) : z_1 = \cdots = z_k = 0\}.$$

Hence $(U \cap X, p \circ \phi)$, where $p$ denotes the projection onto the last $n-k$ coordinates in $\mathbb{C}^n$ defines local coordinates on $X$. It is easy to check that these coordinates are holomorphically compatible. ☐

**Definition 1.10.** We will say that $N \subset M$ is a *complex submanifold* if we may cover $M$ with coordinate patches $(U_\alpha, \phi_\alpha)$ such that

$$\phi_\alpha(X \cap U_\alpha) \ = \ \{z \in \phi_\alpha(U) : z_1 = \cdots = z_k = 0\}.$$

for some fixed $k$. In this case, as we saw above, $N$ has the structure of an $(n-k)$-dimensional complex manifold.

**Proposition 1.11.** *There are no compact complex submanifolds of* $\mathbb{C}^n$ *of dimension greater than zero.*

*Proof.* Suppose $M \subset \mathbb{C}^n$ is a submanifold. Then, each of the coordinate functions $z_i$ restricts to a holomorphic function on $M$. But, if $M$ is compact, it follows from Theorem 1.6 that $z_i$ must be locally constant. Hence, $\dim M = 0$. $\square$

*Remark* 1. The above result means that there is no chance for a Whitney Embedding Theorem in the holomorphic category. One of the major results of the theory of complex manifolds is the Kodaira Embedding Theorem which gives necessary and sufficient conditions for a compact complex manifold to embed in $\mathbb{P}^n$.

**Example 1.12.** Let $f\colon \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function and suppose $Z = f^{-1}(0) \neq \emptyset$. Then we say that 0 is a *regular value* for $f$ if $\operatorname{rank}_p(f) = 1$ for all $p \in Z$; i.e. for each $p \in X$ there exists some $i$, $i = 1, \ldots, n$ such that $\partial f/\partial z_i(p) \neq 0$. In this case, $Z$ is a complex submanifold of $\mathbb{C}^n$ and $\operatorname{codim}(Z) = 1$. We call $Z$ an *affine hypersurface.* More generally, given $F\colon \mathbb{C}^n \to \mathbb{C}^m$, we say that 0 is a *regular value* if $\operatorname{rank}_p(F) = m$ for all $p \in F^{-1}(0)$. In this case or $F^{-1}(0)$ is either empty or is a submanifold of $\mathbb{C}^n$ of codimension $m$.

**Example 1.13.** Let $P(z_0, \ldots, z_n)$ be a homogeneous polynomial of degree $d$. We set

$$X \ := \ \{[z] \in \mathbb{P}^n : P(z_0, \ldots, z_n) = 0\}.$$

We note that while $P$ does not define a function on $\mathbb{P}^n$, the zero locus $X$ is still well defined since $P$ is a homogeneous polynomial. We assume now that the following regularity condition holds:

$$\left\{ z \in \mathbb{C}^{n+1} : \frac{\partial P}{\partial z_0}(z) = \cdots = \frac{\partial P}{\partial z_n}(z) = 0 \right\} = \{0\}; \tag{1.11}$$

i.e. 0 is a regular value of the map $P|_{\mathbb{C}^{n+1}\{0\}}$. Then $X$ is a hypersurface in $\mathbb{P}^n$.

To prove this we note that the requirements of Definition 1.10 are local. Hence, it is enough to check that $X \cap U_i$ is a submanifold of $U_i$ for each $i$; indeed, an affine hypersurface. Consider the case $i = 0$ and let $f\colon U_0 \cong \mathbb{C}^n \to \mathbb{C}$ be the function $f(u_1, \ldots, u_n) = P(1, u_1, \ldots, u_n)$. Set $u = (u_1, \ldots, u_n)$ and $\tilde{u} = (1, u_1, \ldots, u_n)$. Suppose $[\tilde{u}] \in U_0 \cap X$ and

$$\frac{\partial f}{\partial u_1}(u) = \cdots = \frac{\partial f}{\partial u_n}(u) = 0$$

then

$$\frac{\partial P}{\partial z_1}(\tilde{u}) = \cdots = \frac{\partial P}{\partial z_n}(\tilde{u}) = 0$$

But, since $P$ is a homogeneous polynomial of degree $d$, it follows from the Euler identity that:

$$0 \ = \ d \cdot P(\tilde{u}) \ = \ \frac{\partial P}{\partial z_0}(\tilde{u}).$$

Hence, by (1.11), we would have $\tilde{u} = 0$, which is impossible. Hence 0 is a regular value of $f$ and $X \cap U_0$ is an affine hypersurface.

**Exercise 6.** Let $P_1(z_0,\dots,z_n),\dots,P_m(z_0,\dots,z_n)$ be homogeneous polynomials. Suppose that 0 is a regular value of the map

$$(P_1,\dots,P_m)\colon \mathbb{C}^{n+1}\{0\} \to \mathbb{C}^m.$$

Prove that

$$X = \{[z] \in \mathbb{P}^n : P_1([z]) = \cdots = P_m([z]) = 0\}$$

is a codimension $m$ submanifold of $\mathbb{P}^n$. $X$ is called a complete intersection submanifold.

**Example 1.14.** Consider the Grassmann manifold $G(k,n)$ and let $I_1,\dots,I_{\binom{n}{k}}$, denote all strictly increasing $k$-tuples $I \subset \{1,\dots,n\}$. We then define

$$\mathfrak{p}\colon G(k,n) \to \mathbb{P}^{N-1}\ ;\quad \mathfrak{p}([W]) = [\det(W_{I_1}),\dots,\det(W_{I_N})]$$

Note that the map $\mathfrak{p}$ is well defined, since $W \sim W'$ implies that $W' = W \cdot M$ with $M \in \mathrm{GL}(k,\mathbb{C})$ and then for any index set $I$, $\det(W'_I) = \det(M)\det(W_I)$. We leave it to the reader to verify that the map $\mathfrak{p}$, which is usually called the *Plücker map*, is holomorphic.

**Exercise 7.** Consider the Plücker map $\mathfrak{p}\colon G(2,4) \to \mathbb{P}^5$ and suppose that the index sets $I_1,\dots,I_6$ are ordered lexicographically. Show that $\mathfrak{p}$ is a 1 : 1 holomorphic map from $G(2,4)$ onto the subset

$$X = \{[z_0,\dots,z_6] : z_0z_5 - z_1z_4 + z_2z_3 = 0\}. \tag{1.12}$$

Prove that $X$ is a hypersurface in $\mathbb{P}^5$. Compute $\mathrm{rank}_{[W]}\mathfrak{p}$ for $[W] \in G(2,4)$.

**Example 1.15.** We may define complex Lie groups in a manner completely analogous to the real, smooth case. A *complex Lie group* is a complex manifold $G$ with a group structure such that the group operations are holomorphic. The basic example of a complex Lie group is $\mathrm{GL}(n,\mathbb{C})$. We have already observed that $\mathrm{GL}(n,\mathbb{C})$ is an open subset of $\mathbb{C}^{n^2}$ and the product of matrices is given by polynomial functions, while the inverse of a matrix is given by rational functions on the entries of the matrix. Other classical examples include the *special linear group* $\mathrm{SL}(n,\mathbb{C})$ and the *symplectic group* $\mathrm{Sp}(g,\mathbb{C})$. We recall the definition of the latter. Let $J$ be the matrix (1.4), then

$$\mathrm{Sp}(n,\mathbb{C}) \ :=\ \{X \in \mathrm{GL}(2n,\mathbb{C}) : X^T \cdot J \cdot X = J\}. \tag{1.13}$$

We set $\mathrm{Sp}(n,R) := \mathrm{Sp}(n,\mathbb{C}) \cap \mathrm{GL}(2n,R)$ for $R = \mathbb{Z},\mathbb{Q},\mathbb{R}$. More generally, let $V$ be a real vector space, $V_{\mathbb{C}}$ its complexification (cf. (A.1)), and $A$ a non-degenerate, alternating form on $V$. We denote by $\mathrm{Sp}(A,V)$ (resp. $\mathrm{Sp}(A,V_{\mathbb{C}})$) the group of automorphisms of $V$ (resp. $V_C$) that preserve $A$. Note that we must have $\dim_{\mathbb{R}}(V) = 2n$ and there exists a basis relative to which the matrix of $A$ is given by (1.4).

**Example 1.16.** Let $A$ be a non-degenerate, alternating form on a $2n$-dimensional, real vector space $V$. Consider the space:

$$M = \{\Omega \in G(n,V_{\mathbb{C}}) : A(u,v) = 0 \text{ for all } u,v \in \Omega\}.$$

Let $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2n}\}$ be a basis of $V$ in which the matrix of $A$ is as in (1.4). Then if

$$\Omega = [W] \; = \; \begin{bmatrix} W_1 \\ W_2 \end{bmatrix},$$

where $W_1$ and $W_2$ are $n \times n$ matrices, we have that $\Omega \in M$ if and only if

$$\left[W_1^T, W_2^T\right] \cdot \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \cdot \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \; = \; W_2^T \cdot W_1 - W_1^T \cdot W_2 \; = \; 0.$$

Set $I_0 = \{1, \dots, n\}$. Every element $\Omega \in M \cap U_{I_0}$ may be represented by a matrix of the form $\Omega = [I_n, Z]^T$ with $Z^T = Z$. It follows that $M \cap U_{I_0}$ is an $n(n+1)/2$-dimensional submanifold. Now, given an arbitrary $\Omega \in M$, there exists an element $X \in \mathrm{Sp}(A, V_{\mathbb{C}})$ such that $X \cdot \Omega = \Omega_0$, where $\Omega_0 = \mathrm{span}(e_1, \dots, e_n)$. Since the elements of $\mathrm{Sp}(A, V_{\mathbb{C}})$ act by biholomorphisms on $G(n, V_{\mathbb{C}})$ it follows that $M$ and $n(n+1)/2$ dimensional submanifold of $G(n, V_{\mathbb{C}})$. Moreover, since $M$ is a closed submanifold of the compact manifold $G(k, n)$, $M$ is also compact.

We will also be interested in considering the open set $D \subset M$ consisting of

$$D = D(V, A) \; := \; \{\Omega \in M : i\, A(w, \bar{w}) > 0 \text{ for all } 0 \neq w \in \Omega\}. \tag{1.14}$$

It follows that $\Omega \in D$ if and only if the hermitian matrix

$$i \cdot \left[\bar{W}_1^T, \bar{W}_2^T\right] \cdot \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \cdot \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \; = \; i(\bar{W}_2^T \cdot W_1 - \bar{W}_1^T \cdot W_2)$$

is positive definite. Note that in particular $D \subset U_{I_0}$ and that

$$D \; \cong \; \{Z \in \mathcal{M}(n, \mathbb{C}) : Z^T = Z \; ; \; \mathrm{Im}(Z) = (1/2i)(Z - \bar{Z}) > 0\}, \tag{1.15}$$

where $\mathcal{M}(n, \mathbb{C})$ denotes the $n \times n$ complex matrices. If $n = 1$ then $M \cong \mathbb{C}$ and $D$ is the upper-half plane. We will call $D$ the generalized *Siegel upper-half space.*

The elements of the complex lie group $\mathrm{Sp}(A, V_{\mathbb{C}}) \cong \mathrm{Sp}(n, \mathbb{C})$ define biholomorphisms of $G(n, V_{\mathbb{C}})$ preserving $M$. The subgroup

$$\mathrm{Sp}(A, V) = \mathrm{Sp}(A, V_{\mathbb{C}}) \cap \mathrm{GL}(V) \cong \mathrm{Sp}(n, \mathbb{R})$$

preserves $D$.

**Exercise 8.** Prove that relative to the description of $D$ as in (1.15) the action of $\mathrm{Sp}(A, V)$ is given by generalized fractional linear transformations:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z \; = \; (A \cdot Z + B) \cdot (C \cdot Z + D)^{-1}.$$

**Exercise 9.** Prove that the action of $\mathrm{Sp}(A, V)$ on $D$ is *transitive* in the sense that given any two points $\Omega, \Omega' \in D$ there exists $X \in \mathrm{Sp}(A, V)$ such that $X \cdot \Omega = \Omega'$.

**Exercise 10.** Compute the *isotropy subgroup*:

$$K \; := \; \{X \in \mathrm{Sp}(A, V) : X \cdot \Omega_0 = \Omega_0\},$$

where $\Omega_0 = [I_n, i\, I_n]^T$.

**Example 1.17.** Let $T_\Lambda := \mathbb{C}/\Lambda$, where $\Lambda \subset \mathbb{Z}^2$ is a rank-two lattice in $\mathbb{C}$; i.e.

$$\Lambda = \{m\,\omega_1 + n\,\omega_2\ ;\ m, n \in \mathbb{Z}\},$$

where $\omega_1, \omega_2$ are complex numbers linearly independent over $\mathbb{R}$. $T_\Lambda$ is locally diffeomorphic to $\mathbb{C}$ and since the translations by elements in $\Lambda$ are biholomorphisms of $\mathbb{C}$, $T_\Lambda$ inherits a complex structure relative to which the natural projection

$$\pi_\Lambda \colon \mathbb{C} \to T_\Lambda$$

is a local biholomorphic map.

It is natural to ask if, for different lattices $\Lambda$, $\Lambda'$, the *complex tori* $T_\Lambda$, $T_{\Lambda'}$ are biholomorphic. Suppose $F\colon T_\Lambda \to T_{\Lambda'}$ is a biholomorphism. Then, since $\mathbb{C}$ is the universal covering of $T_\Lambda$ there exists a map $\tilde{F}\colon \mathbb{C} \to \mathbb{C}$ such that the diagram:

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\tilde{F}"] \arrow[d, "\pi_\Lambda"'] & \mathbb{C} \arrow[d, "\pi_{\Lambda'}"] \\
\mathbb{C}/\Lambda \arrow[r, "F"] & \mathbb{C}/\Lambda'
\end{tikzcd}$$

commutes. In particular, given $z \in \mathbb{C}$, $\lambda \in \Lambda$, there exists $\lambda' \in \Lambda'$ such that

$$\tilde{F}(z + \lambda) \;=\; \tilde{F}(z) + \lambda'.$$

This means that the derivative $\tilde{F}$ must be $\Lambda$-periodic and, hence, it defines a holomorphic function on $\mathbb{C}/\Lambda$ which, by Theorem 1.6, must be constant. This implies that $\tilde{F}$ must be a linear map and, after translation if necessary, we may assume that $\tilde{F}(z) = \mu \cdot z$, $\mu = a + ib \in \mathbb{C}$. Conversely, any such linear map $\tilde{F}$ induces a biholomorphic map $\mathbb{C}/\Lambda \to \mathbb{C}/\tilde{F}(\Lambda)$. In particular, if $\{\omega_1, \omega_2\}$ is a $\mathbb{Z}$-basis of $\Lambda$ then $\mathrm{Im}(\omega_2/\omega_1) \neq 0$ and we may assume without loss of generality that $\mathrm{Im}(\omega_2/\omega_1) > 0$. Setting $\mu = \omega_2/\omega_1$ we see that $T_\Lambda$ is always biholomorphic to a torus $T_\tau$ associated with a lattice

$$\{m + n\tau\ ;\ m, n \in \mathbb{Z}\}$$

and where $\mathrm{Im}(\tau) > 0$.

Now, suppose the tori $T_\lambda$, $T_{\Lambda'}$ are biholomorphic and let $\{\omega_1, \omega_2\}$ (resp. $\{\omega_1', \omega_2'\}$) be a $\mathbb{Z}$-basis of $\Lambda$ (resp. $\Lambda'$) as above. We have

$$\mu \cdot \omega_1 = m_{11}\omega_1' + m_{21}\omega_2'\ ; \quad \mu \cdot \omega_2 = m_{12}\omega_1' + m_{22}\omega_2'\,, \quad m_{ij} \in \mathbb{Z}.$$

Moreover, $m_{11}m_{22} - m_{12}m_{21} = 1$, since $F$ is biholomorphic and therefore $\tilde{F}(\Lambda) = \Lambda'$. Hence

$$\tau \;=\; \frac{\omega_1}{\omega_2} \;=\; \frac{m_{11}\omega_1' + m_{21}\omega_2'}{m_{12}\omega_1' + m_{22}\omega_2'} \;=\; \frac{m_{11} + m_{21}\tau'}{m_{12} + m_{22}\tau'}$$

Consequently, $T_\tau \cong T_{\tau'}$ if and only if $\tau$ and $\tau'$ are points in the upper-half plane congruent under the action of the group $\mathrm{SL}(2, \mathbb{Z})$ by fractional linear transformations.

*Remark* 2. Note that while all differentiable structures on the torus $S^1 \times S^1$ are equivalent there is a continuous *moduli* of different complex structures. This is one of the key differences between real and complex geometry and one which we will study using Hodge Theory.

1.2. **Holomorphic Vector Bundles.** We may extend the notion of smooth vector bundle to complex manifolds and holomorphic maps:

**Definition 1.18.** A holomorphic vector bundle $E$ over a complex manifold $M$ is a complex manifold $E$ together with a holomorphic map $\pi\colon E \to M$ such that:

- i) For each $x \in M$, the *fiber* $E_x = \pi^{-1}(x)$ is a complex vector space of dimension $d$ (the *rank* of $E$).
- ii) There exists an open covering $\{U_\alpha\}$ of $M$ and biholomorphic maps
$$\Phi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^d$$
such that
  - (a) $p_1(\Phi_\alpha(x)) = x$ for all $x \in U$, where $p_1\colon U_\alpha \times \mathbb{C}^d \to U_\alpha$ denotes projection on the first factor, and
  - (b) For every $x \in U_\alpha$ the map $p_2 \circ \Phi|_{E_x}\colon E_x \to \mathbb{C}^d$ is an isomorphism of $\mathbb{C}$-vectorspaces.

$E$ is called the *total space* of the bundle and $M$ its *base.* The covering $\{U_\alpha\}$ is called a *trivializing cover* of $M$ and the biholomorphisms $\{\Phi_\alpha\}$ *local trivializations.* When $d = 1$ we often refer to $E$ as a *line bundle.*

We note that as in the case of smooth vector bundles, a holomorphic vector bundle may be described by *transition functions.* That is, by a covering of $M$ by open sets $U_\alpha$ together with holomorphic maps

$$g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(d, \mathbb{C})$$

such that

$$g_{\alpha\beta} \cdot g_{\beta\gamma} = g_{\alpha\gamma} \tag{1.16}$$

on $U_\alpha \cap U_\beta \cap U_\gamma$. The maps $g_{\alpha\beta}$ are defined by the commutative diagram:

$$\begin{tikzcd}
& \pi^{-1}(U_\alpha \cap U_\beta) \arrow[dl, "\Phi_\beta"'] \arrow[dr, "\Phi_\alpha"] & \\
(U_\alpha \cap U_\beta) \times \mathbb{R}^d \arrow[rr, "(\mathrm{id}, g_{\alpha\beta})"] & & (U_\alpha \cap U_\beta) \times \mathbb{R}^d
\end{tikzcd} \tag{1.17}$$

In particular, a holomorphic line bundle over $M$ is given by a collection $\{U_\alpha, g_{\alpha\beta}\}$, where $U_\alpha$ is an open cover of $M$ and the $\{g_{\alpha\beta}\}$ are nowhere-zero holomorphic functions defined on $U_\alpha \cap U_\beta$, i.e. $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ satisfying the *cocycle condition* (1.16).

**Example 1.19.** The product $M \times \mathbb{C}^d$ with the natural projection may be viewed as vector bundle of rank $d$ over the complex manifold $M$. It is called the *trivial bundle* over $M$.

**Example 1.20.** We consider the *tautological* line bundle over $\mathbb{P}^n$. This is the bundle whose fiber over a point in $\mathbb{P}^n$ is the line in $\mathbb{C}^{n+1}$ defined by that point. More precisely, let

$$\mathcal{T} \;:=\; \{([z], v) \in \mathbb{P}^n \times \mathbb{C}^{n+1} : v = \lambda z, \lambda \in \mathbb{C}\},$$

and let $\pi\colon \mathcal{T} \to \mathbb{P}^n$ be the projection to the first factor. Let $U_i$ be as in (1.6). Then we can define

$$\Phi_i\colon \pi^{-1}(U_i) \to U_i \times \mathbb{C}$$

by

$$\Phi_i([z], v) = v_i.$$

The transition functions $g_{ij}$ are defined by the diagram (1.17) and we have

$$\Phi_i \circ \Phi_j^{-1}([z], 1) = \Phi_i([z], (z_0/z_j, \ldots, 1, \ldots, z_n/z_j)) = ([z], z_i/z_j),$$

with the one in the $j$-th position. Hence,

$$g_{ij}\colon U_i \cap U_j \to \mathrm{GL}(1, \mathbb{C}) \cong \mathbb{C}^*$$

is the map $[z] \mapsto z_i/z_j$. It is common to denote the tautological bundle as $\mathcal{O}(-1)$.

**Exercise 11.** Generalize the construction of the tautological bundle over projective space to obtain the universal rank $k$ bundle over the Grassmann manifold $G(k, n)$. Consider the space:

$$\mathcal{U} \;:=\; \{(\Omega, v) \in G(k, n) \times \mathbb{C}^n : v \in \Omega\}, \tag{1.18}$$

where we regard $\Omega \in G(k, n)$ as a $k$-dimensional subspace of $\mathbb{C}^n$. Prove that $\mathcal{U}$ may be trivialized over the open sets $U_I$ defined in Example 1.5 and compute the transition functions relative to these trivializations.

Let $\pi\colon E \to M$ be a holomorphic vector bundle and suppose $F\colon N \to M$ is a holomorphic map. Given a trivializing cover $\{(U_\alpha, \Phi_a)\}$ of $E$ with transition functions $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(d, \mathbb{C})$, we define

$$h_{\alpha\beta}\colon F^{-1}(U_\alpha) \cap F^{-1}(U_\beta) \to \mathrm{GL}(d, \mathbb{C}) \;;\quad h_{\alpha\beta} \;:=\; g_{\alpha\beta} \circ F. \tag{1.19}$$

It is easy to check that the functions $h_{\alpha\beta}$ satisfy the *cocycle condition* (1.16) and, therefore, define a holomorphic vector bundle over $N$ denoted by $F^*(E)$, and called the *pullback* bundle. Note that we have a commutative diagram:

$$\begin{tikzcd} F^*(E) \arrow[r, "\bar{\mathrm{F}}"] \arrow[d, "\pi^*"'] & E \arrow[d, "\pi"] \\ N \arrow[r, "\mathrm{F}"] & M \end{tikzcd} \tag{1.20}$$

If $L$ and $L'$ are line bundles, and $g^L_{\alpha\beta}$, $g^{L'}_{\alpha\beta}$ are their transition functions relative to a common trivializing cover then the functions:

$$h_{\alpha\beta} = g^L_{\alpha\beta} \cdot g^{L'}_{\alpha\beta}$$

satisfy (1.16) and define a new line bundle which we denote by $L \otimes L'$.

Similarly, the functions

$$h_{\alpha\beta} = (g^L_{\alpha\beta})^{-1}$$

also satisfy (1.16) and define a bundle, called *the dual bundle* of $L$ and denoted by $L^*$ or $L^{-1}$. Clearly $L \otimes L^*$ is the trivial line bundle over $M$. The dual bundle of the tautological bundle is called the *hyperplane bundle* over $\mathbb{P}^n$ and denoted by $H$ or $\mathcal{O}(1)$. Note that the transition functions of $H$ are $g^H_{ij} \in \mathcal{O}^*(U_i \cap U_j)$ defined by

$$g^H_{ij}([z]) \;:=\; z_j/z_i. \tag{1.21}$$

We may also extend the notion of sections to holomorphic vector bundles:

**Definition 1.21.** A holomorphic section of a holomorphic vector bundle $\pi\colon E \to M$ over an open set $U \subset M$ is a holomorphic map:

$$\sigma\colon U \to E$$

such that

$$\pi \circ \sigma = \mathrm{id}|_U\,. \tag{1.22}$$

The sections of $E$ over $U$ are an $\mathcal{O}(U)$-module which will be denoted by $\mathcal{O}(U, E)$. Clearly, the local sections over $U$ of the trivial line bundle are the ring $\mathcal{O}(U)$.

If $\pi\colon L \to M$ is a line bundle and $g_{\alpha\beta}$ are the transition functions associated to a trivializing covering $(U_\alpha, \Phi_\alpha)$, then a section $\sigma\colon M \to L$ may be described by a collection of functions $f_\alpha \in \mathcal{O}(U_\alpha)$ defined by:

$$\sigma(x) = f_\alpha(x)\Phi_\alpha^{-1}(x, 1).$$

Hence, for $x \in U_\alpha \cap U_\beta$ we must have

$$f_\alpha(x) = g_{\alpha\beta}(x) \cdot f_\beta(x). \tag{1.23}$$

**Example 1.22.** Let $M = \mathbb{P}^n$ and let $U_i = \{[z] \in \mathbb{P}^n : z_i \neq 0\}$. Let $P \in \mathbb{C}[z_0, \dots, z_n]$ be a homogeneous polynomial of degree $d$. For each $i = 0, \dots, n$ define

$$f_i([z]) = \frac{P(z)}{z_i^d} \in \mathcal{O}(U_i).$$

We then have in $U_i \cap U_j$:

$$z_i^d \cdot f_i([z]) = P(z) = z_j^d \cdot f_j([z]),$$

and therefore

$$f_i([z]) \;=\; (z_j/z_i)^d \cdot f_j([z]).$$

This means that we can consider the polynomial $P(z)$ as defining a section of the line bundle over $\mathbb{P}^n$ with transition functions

$$g_{ij} \;=\; (z_j/z_i)^d\,,$$

that is of the bundle $H^d = \mathcal{O}(d)$. In fact, it is possible to prove that every global holomophic section of the bundle $\mathcal{O}(d)$ is defined, as above, by a homogeneous polynomial of degree $d$. The proof of this fact requires Hartogs' Theorem [18, Proposition 1.1.14] from the theory of holomorphic functions of several complex variables. We refer to [18, Proposition 2.4.1].

We note that, on the other hand, the tautological bundle has no non-trivial global holomorphic sections. Indeed, suppose $\sigma \in \mathcal{O}(\mathbb{P}^n, \mathcal{O}(-1))$ let $\ell$ denote the global section of $\mathcal{O}(1)$ associated to a non-zero linear form $\ell$. Then, the map

$$[z] \in \mathbb{P}^n \mapsto \ell([z])(\sigma([z])$$

defines a global holomorphic function on the compact complex manifold $\mathbb{P}^n$, hence it must be constant. If that constant is non-zero then both $\sigma$ and $\ell$ are nowhere-zero which would imply that both $\mathcal{O}(-1)$ and $\mathcal{O}(1)$ are trivial bundles. Hence $\sigma$ must be identically zero.

Note that given a section $\sigma\colon M \to E$ of a vector bundle $E$, the zero locus $\{x \in M : \sigma(x) = 0\}$ is a well defined subset of $M$. Thus, we may view the projective hypersurface defined in Example 1.13 by a homogeneous polynomial of degree $d$ as the zero-locus of a section of $\mathcal{O}(d)$.

*Remark* 3. The discussion above means that one should think of sections of line bundles as locally defined holomorphic functions satisfying a suitable compatibility condition. Given a compact, connected, complex manifold, global sections of holomorphic line bundles (when they exist) often play the role that global smooth functions play in the study of smooth manifolds. In particular, one uses sections of line bundles to define embeddings of compact complex manifolds into projective space. This vague observation will be made precise later in the course.

Given a holomorphic vector bundle $\pi\colon E \to M$ and a local trivialization

$$\Phi\colon \pi^{-1}(U) \to U \times \mathbb{C}^d$$

we may define a basis of local sections of $E$ over $U$ (*a local frame*) as follows. Let $e_1, \dots, e_d$ denote the standard basis of $\mathbb{C}^d$ and for $x \in U$ set:

$$\sigma_j(x) \ := \ \Phi^{-1}(x, e_j); \ j = 1, \dots, d$$

Then $\sigma_j(x) \in \mathcal{O}(U, E)$ and for each $x \in U$ the vectors $\sigma_1(x), \dots, \sigma_d(x)$ are a basis of the $d$-dimensional vector space $E_x$ (they are the image of the basis $e_1, \dots, e_d$ by a linear isomorphism). In particular, if $\tau\colon U \to M$ is a map satisfying (1.22) we can write:

$$\tau(x) \ = \ \sum_{j=1}^{d} f_j(x)\sigma_j(x)$$

and $\tau$ is holomorphic (resp. smooth) if and only if the functions $f_j \in \mathcal{O}(U)$ (resp. $f_j \in C^\infty(U)$).

Conversely, suppose $U \subset M$ is an open set and let $\sigma_1, \dots, \sigma_d \in \mathcal{O}(U, E)$ be a *local frame*; i.e. holomorphic sections such that for each $x \in U$, $\sigma_1(x), \dots, \sigma_d(x)$ is a basis of $E_x$, then we may define a local trivialization

$$\Phi\colon \pi^{-1}(U) \to U \times \mathbb{C}^d$$

by

$$\Phi(v) \ := \ (\pi(v), (\lambda_1, \dots, \lambda_d)),$$

where $v \in \pi^{-1}(U)$ and

$$v \ = \ \sum_{j=1}^{d} \lambda_j \, \sigma_j(\pi(v)).$$

## 2. Differential Forms on Complex Manifolds

2.1. **Almost Complex Manifolds.** Given a complex manifold $M$ and a coordinate atlas $(U_\alpha, \phi_\alpha)$ covering $M$, the fact that the change-of-coordinate maps (1.5) are holomorphic implies that the matrix of the differential $D(\phi_\beta \circ \phi_\alpha^{-1})$ is of the form (1.3). This means that the map

$$J_p\colon T_p(M) \to T_p(M)$$

defined by

$$J\left(\frac{\partial}{\partial x_j}\right) := \frac{\partial}{\partial y_j}\ ; \quad J\left(\frac{\partial}{\partial y_j}\right) := -\frac{\partial}{\partial x_j} \tag{2.1}$$

is well defined, provided that the functions $z_j = x_j + iy_j$, $j = 1, \ldots, n$, define local holomorphic coordinates near the point $p$. We note that $J$ is a $(1,1)$ smooth tensor on $M$ such that $J^2 = -\operatorname{id}$ and that, for each $p \in M$, $J_p$ defines a complex structure on the real vector space $T_p(M)$ (cf.(A.8)).

**Definition 2.1.** An *almost complex structure* on a $C^\infty$ (real) manifold $M$ is a $(1,1)$ tensor $J$ such that $J^2 = -\operatorname{id}$. An *almost complex manifold* is a pair $(M, J)$ where $J$ is an almost complex structure on $M$. The almost complex structure $J$ is said to be *integrable* if $M$ has a complex structure inducing $J$.

If $(M, J)$ is an almost complex manifold then $J_p$ is a complex structure on $M$ and therefore by Proposition A.1, $M$ must be even-dimensional. We also have:

**Exercise 12.** If $M$ has an almost complex structure then $M$ is orientable.

**Exercise 13.** Let $M$ be an orientable (and oriented) two-dimensional manifold and let $\langle\ ,\ \rangle$ be a Riemannian metric on $M$. Given $p \in M$ let $v_1, v_2 \in T_p(M)$ be a positively oriented orthonormal basis. Prove that $J_p\colon T_p(M) \to T_p(M)$ defined by:

$$J_p(v_1) = v_2\ ; \quad J_p(v_2) = -v_1,$$

defines an almost complex structure on $M$. Show, moreover, that if $\langle\langle\ ,\ \rangle\rangle$ is a Riemannian metric conformally equivalent to $\langle\ ,\ \rangle$ then the two metrics define the same almost complex structure.

The discussion above shows that if $M$ is a complex manifold then the operator (2.1) defines an almost complex structure. It is natural to ask about the converse of this statement; i.e. when does an almost complex structure $J$ on a manifold arise from a complex structure? The answer, which is provided by the Newlander-Nirenberg Theorem, may be stated in terms of the *Nijenhuis* torsion of $J$:

**Exercise 14.** Let $J$ be an almost complex structure on $M$. Prove that

$$N(X, Y) = [JX, JY] - [X, Y] - J[X, JY] - J[JX, Y] \tag{2.2}$$

is a $(1,2)$-tensor satisfying $N(X, Y) = -N(Y, X)$. $N$ is called the *torsion* of $J$.

**Exercise 15.** Let $J$ be an almost complex structure on a two-dimensional manifold $M$. Prove that $N(X, Y) = 0$ for all vector fields $X$ and $Y$ on $M$.

**Theorem 2.2 (Newlander-Nirenberg).** *Let $(M, J)$ be an almost complex manifold, then $M$ has a complex structure inducing the almost complex structure $J$ if and only if $N(X, Y) = 0$ for all vector fields $X$ and $Y$ on $M$.*

*Proof.* We refer to [38, Proposition 2], [34, §2.2.3] for a proof in the case when $M$ is a real analytic manifold. □

*Remark* 4. Note that assuming the Newlander-Nirenberg Theorem, it follows from Exercise 15 that the almost complex structure constructed in Exercise 13 is integrable. We may explicitly construct the complex structure on $M$ by using local isothermal coordinates. Thus, a complex structure on an oriented, two-dimensional manifold $M$ is equivalent to a Riemannian metric up to conformal equivalence.

In what follows we will be interested in studying complex manifolds; however, the notion of almost complex structures gives a very convenient way to distinguish those properties of complex manifolds that depend only on having a (smoothly varying) complex structure on each tangent space. Thus, we will not explore in depth the theory of almost complex manifolds except to note that there are many examples of almost complex structures which are not integrable, that is, do not come from a complex structure. One may also ask which even-dimensional orientable manifolds admit almost complex structures. For example, in the case of a sphere $S^{2n}$ it was shown by Borel and Serre that only $S^2$ and $S^6$ admit almost complex structures. This is related to the fact that $S^1$, $S^3$ and $S^7$ are the only parallelizable spheres. We point out that while it is easy to show that $S^6$ has a non-integrable almost complex structure it is still unknown whether $S^6$ has a complex structure.

2.2. **Tangent and Cotangent space.** Let $(M, J)$ be an almost complex manifold and $p \in M$. Let $T_p(M)$ denote the tangent space of $M$. Then $J_p$ defines a complex structure on $T_p(M)$ and therefore, by Proposition A.1, the complexification: $T_{p,\mathbb{C}}(M) := T_p(M) \otimes_{\mathbb{R}} \mathbb{C}$, decomposes as

$$T_{p,\mathbb{C}}(M) = T_p'(M) \oplus T_p''(M)$$

where $T_p''(M) = \overline{T_p'(M)}$ and $T_p'(M)$ is the $i$-eigenspace of $J$ acting on $T_{p,\mathbb{C}}(M)$. Moreover, by Proposition A.2, the map $v \in T_p(M) \mapsto v - iJ_p(v)$ defines an isomorphism of complex vector spaces $(T_p(M), J_p) \cong T_p'(M)$.

If $J$ is integrable, then given holomorphic local coordinates $\{z_1, \ldots, z_n\}$ around $p$, we may consider the local coordinate frame (1.2) and, given (2.1), we have that the above isomorphism maps:

$$\partial/\partial x_j \mapsto \partial/\partial x_j - i\,\partial/\partial y_j.$$

We set

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\,\frac{\partial}{\partial y_j}\right) \ ; \quad \frac{\partial}{\partial \bar{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\,\frac{\partial}{\partial y_j}\right) . \tag{2.3}$$

Then, the vectors $\partial/\partial z_j \in T_p'(M)$ are a basis of the complex subspace $T_p'(M)$.

*Remark* 5. Given local coordinates $(U, \{z_1, \ldots, z_n\})$ on $M$, a function $f\colon U \to \mathbb{C}$ is holomorphic if the local coordinates expression $f(z_1, \ldots, z_n)$ satisfies the Cauchy-Riemann equations. This is equivalent to the condition

$$\frac{\partial}{\partial \bar{z}_j}(f) = 0,$$

for all $j$. Moreover, in this case $\frac{\partial}{\partial z_j}(f)$ coincides with the partial derivative of $f$ with respect to $z_j$. This justifies the choice of notation. However, we point out that it makes sense to consider $\frac{\partial}{\partial z_j}(f)$ even if the function $f$ is only a $C^\infty$ function.

We will refer to $T'_p(M)$ as the holomorphic tangent space[†] of $M$ at $p$. We note that if $\{z_1, \dots, z_n\}$ and $\{w_1, \dots, w_n\}$ are local complex coordinates around $p$ then the change of basis matrix from the basis $\{\partial/\partial z_j\}$ to the basis $\{\partial/\partial w_k\}$ is given by the matrix of holomorphic functions

$$\left(\frac{\partial w_k}{\partial z_j}\right).$$

Thus, the complex vector spaces $T'_p(M)$ define a holomorphic vector bundle $T^h(M)$ over $M$, the *holomorphic tangent bundle.*

**Example 2.3.** Let $M$ be an oriented real surface with a Riemannian metric. Let $(U, (x, y))$ be positively-oriented, local isothermal coordinates on $M$; i.e. the coordinate vector fields $\partial/\partial x$, $\partial/\partial y$ are orthogonal and of the same length. Then $z = x + iy$ defines complex coordinates on $M$ and the vector field $\partial/\partial z = \partial/\partial x - i\,\partial/\partial y$ is a local holomorphic section of the holomorphic tangent bundle of $M$.

We can now characterize the holomorphic tangent bundle of $\mathbb{P}^n$:

**Theorem 2.4.** *The holomorphic tangent bundle $T^h\mathbb{P}^n$ is equivalent to the bundle*

$$\mathrm{Hom}(\mathcal{T}, E/\mathcal{T}),$$

*where $E = \mathbb{P}^n \times \mathbb{C}^{n+1}$ is the trivial bundle of rank $n+1$ on $\mathbb{P}^n$.*

*Proof.* Consider the projection $\pi\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$. Given $\lambda \in \mathbb{C}^*$, let $M_\lambda\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}^{n+1} \setminus \{0\}$ be the map "multiplication by $\lambda$". Then, for every $v \in \mathbb{C}^{n+1} \setminus \{0\}$ we may identify $T'(\mathbb{C}^{n+1} \setminus \{0\}) \cong \mathbb{C}^{n+1}$ and we have the following commutative diagram

$$\begin{tikzcd}
\mathbb{C}^{n+1} \arrow[rr, "M_\lambda"] \arrow[dr, "\pi_{*,v}"'] & & \mathbb{C}^{n+1} \arrow[dl, "\pi_{*,\lambda v}"] \\
& \mathcal{T}_{[v]}(\mathbb{P}^n) &
\end{tikzcd}$$

Now, the map $\pi_{*,v}\colon \mathbb{C}^{n+1} \to T'_{[v]}(\mathbb{P}^n)$ is surjective and its kernel is the line $L = \mathbb{C} \cdot v$. Hence we get a family of linear isomorphisms

$$\mathfrak{p}_v\colon \mathbb{C}^{n+1}/L \to T'_{[v]}(\mathbb{P}^n); \quad v \in L,\ v \neq 0$$

with the relation

$$\mathfrak{p}_v \;=\; \lambda\, \mathfrak{p}_{\lambda v}$$

We can now define a map

$$\Theta : \mathrm{Hom}(\mathcal{T}, E/\mathcal{T}) \to T^h\mathbb{P}^n.$$

Let

$$\xi \in \mathrm{Hom}(\mathcal{T}, E/\mathcal{T})_{[z]} = \mathrm{Hom}(\mathcal{T}_{[z]}, (E/\mathcal{T})_{[z]}) \cong \mathrm{Hom}(L, \mathbb{C}^{n+1}/L)$$

then we set

$$\Theta(\xi) \;:=\; \mathfrak{p}_v(\xi(v)) \text{ for any } v \in L, v \neq 0.$$

---

[†] This construction makes sense even if $J$ is not integrable. In that case we may replace the coordinate frame (1.2) by a local frame $\{X_1, \dots, X_n, Y_1, \dots, Y_n\}$ such that $J(X_j) = Y_j$ and $J(Y_j) = -X_j$.

Note that this is well defined since

$$\mathfrak{p}_{\lambda v}(\xi(v)) \;=\; \lambda^{-1}\mathfrak{p}_v(\lambda\,\xi(v)) \;=\; \mathfrak{p}_v(\xi(v)).$$

Alternatively one may define

$$\Theta(\xi) \;=\; \frac{d}{dt}|_{t=0}(\gamma(t)),$$

where $\gamma(t)$ is the holomorphic curve through $[z]$ in $\mathbb{P}^n$ defined by

$$\gamma(t) \;:=\; [v + t\xi(v)].$$

One then has to show that this map is well defined. It is straightforward, though tedious, to verify that $\Theta$ is an isomorphism of vector bundles. ☐

**Exercise 16.** Prove that

$$T^h(G(k,n)) \;\cong\; \mathrm{Hom}(\mathcal{U}, E/\mathcal{U}),$$

where $\mathcal{U}$ is the universal bundle over $G(k,n)$ defined in Exercise 1.18 and $E$ is the trivial bundle $E = G(k,n) \times \mathbb{C}^n$.

As seen in Appendix A, a complex structure on a vector space induces a dual complex structure on the dual vector space. Thus, the complexification of the cotangent space $T^*_{p,\mathbb{C}}(M)$ decomposes as

$$T^*_{p,\mathbb{C}}(M) \;:=\; T^{1,0}_p(M) \oplus T^{0,1}_p(M)\;; \quad T^{0,1}_p(M) = \overline{T^{1,0}_p(M)}.$$

Given local holomorphic coordinates $\{z_1,\ldots,z_n\}$, $z_j = x_j + iy_j$, the one-forms $dz_j := dx_j + idy_j$, $d\bar{z}_j = dx_j - idy_j$, are the dual coframe to $\partial/\partial z_1,\ldots,\partial/\partial \bar{z}_n$ and, consequently $dz_1,\ldots,dz_n$ are a local holomorphic frame of the holomorphic bundle $T^{1,0}(M)$.

The complex structure on $T^*_p(M)$ induces a decomposition of the $k$-th exterior product (cf. (A.19)):

$$\Lambda^k(T^*_{p,\mathbb{C}}(M)) \;=\; \bigoplus_{a+b=k} \Lambda^{a,b}_p(M),$$

where

$$\Lambda^{a,b}_p(M) = \overbrace{T^{1,0}_p(M)\wedge\ldots\wedge T^{1,0}_p(M)}^{a \text{ times}} \wedge \overbrace{T^{0,1}_p(M)\wedge\ldots\wedge T^{0,1}_p(M)}^{b \text{ times}}. \tag{2.4}$$

In this way, the smooth vector bundle $\Lambda^k(T^*_{\mathbb{C}}(M))$ decomposes as a direct sum of $C^\infty$ vector bundles

$$\Lambda^k(T^*_{\mathbb{C}}(M)) \;=\; \bigoplus_{a+b=k} \Lambda^{a,b}(M). \tag{2.5}$$

We will denote by $\mathcal{A}^k(U)$ (resp. $\mathcal{A}^{a,b}(U)$) the $C^\infty(U)$ module of local sections of the bundle $\Lambda^k(T^*_{\mathbb{C}}(M))$ (resp. $\Lambda^{a,b}(M)$) over $U$. We then have

$$\mathcal{A}^k(U) \;==\; \bigoplus_{a+b=k} \mathcal{A}^{a,b}(U). \tag{2.6}$$

Note that, given holomorphic coordinates $\{z_1,\ldots,z_n\}$, the local differential forms

$$dz_I \wedge d\bar{z}_J \;:=\; dz_{i_1}\wedge\cdots dz_{i_a}\wedge d\bar{z}_{j_1}\wedge\cdots d\bar{z}_{j_b}\,,$$

where $I$ (resp. $J$) runs over all strictly increasing index sets $1 \leq i_1 < \cdots < i_a \leq n$ of length $a$ (resp. $1 \leq j_1 < \cdots < j_b \leq n$ of length $b$) are a local frame for the bundle $\Lambda^{a,b}(M)$.

We note that the bundles $\Lambda^{k,0}(M)$ are holomorphic bundles of rank $\binom{n}{k}$. We denote them by $\Omega^k_M$ to emphasize that we are viewing them as holomorphic, rather than smooth, bundles. We denote the $\mathcal{O}(U)$-module of holomorphic sections by $\Omega^k(U)$. In particular $\Lambda^{n,0}(M)$ is a holomorphic line bundle over $M$ called the *canonical bundle* and usually denoted by $K_M$.

**Example 2.5.** Let $M = \mathbb{P}^1$. Then as we saw in Example 1.4, $M$ is covered by two coordinate neighborhoods $(U_0, \phi_0)$, $(U_1, \phi_1)$. The coordinate changes are given by the maps $\phi_1 \circ \phi_0^{-1} \colon \mathbb{C}^* \to \mathbb{C}^*$:

$$w = \phi_1 \circ \phi_0^{-1}(z) = \phi_1([1, z]) = 1/z.$$

This means that the local sections $dz$, $dw$ of the holomorphic cotangent bundle are related by

$$dz \ = (-1/z)^2 \, dw.$$

It follows from (1.17) that $g_{01}[z_0, z_1] \ = \ -(z_0/z_1)^2$. Hence $K_{\mathbb{P}^1} \cong \mathcal{O}(-2) = \mathcal{T}^2$.

**Exercise 17.** Find the transition functions for the holomorphic cotangent bundle of $\mathbb{P}^n$. Prove that $K_{\mathbb{P}^n} \cong \mathcal{O}(-n-1) = \mathcal{T}^{n+1}$.

2.3. **de Rham and Dolbeault Cohomologies.** [†] We recall that if $U \subset M$ is an open set in a smooth manifold $M$ and $\mathcal{A}^k(U)$ denotes the space of $\mathbb{C}$-valued differential $k$-forms on $U$, then there exists a unique operator, *the exterior differential*:

$$d \colon \mathcal{A}^k(U) \to \mathcal{A}^{k+1}(U) \ ; \quad k \geq 0$$

satisfying the following properties:

i) $d$ is $\mathbb{C}$-linear;

ii) For $f \in \mathcal{A}^0(U) = C^\infty(U)$, $df$ is the one-form on $U$ which acts on a vector field $X$ by $df(X) := X(f)$.

iii) Given $\alpha \in \mathcal{A}^r(U)$, $\beta \in \mathcal{A}^s(U)$, the *Leibniz property* holds:

$$d(\alpha \wedge \beta) \ = \ d\alpha \wedge \beta + (-1)^r \, \alpha \wedge d\beta; \tag{2.7}$$

iv) $d \circ d = 0$.

It follows from (ii) above that if $\{X_1, \ldots, X_m\}$ is a local frame on $U \subset M$ and $\xi_1, \ldots, \xi_m \in \mathcal{A}^1(U)$ is the dual coframe, then given $f \in C^\infty(U)$ we have

$$df \ = \ \sum_{i=1}^m X_i(f) \xi_i \, .$$

In particular, if $M$ is a complex manifold and $(U, z_1, \ldots, z_n)$ are local coordinates then for $f \in C^\infty(U)$ we have

$$df \ = \ \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j + \sum_{j=1}^n \frac{\partial f}{\partial y_j} dy_j \ = \ \sum_{j=1}^n \frac{\partial f}{\partial z_j} dz_j + \sum_{j=1}^n \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j \tag{2.8}$$

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[†] Many of the topics in this section will be studied in greater depth in the courses by L. Tu and F. Elzein.

The properties of the operator $d$ imply that for each open set $U$ in $M$ we have a complex:

$$\mathbb{C} \hookrightarrow C^\infty(U) \xrightarrow{d} \mathcal{A}^1(U) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{A}^{2n-1}(U) \xrightarrow{d} \mathcal{A}^{2n}(U) \tag{2.9}$$

The quotients:

$$H^k_{dR}(U, \mathbb{C}) \;:=\; \frac{\ker\{d\colon \mathcal{A}^k(U) \to \mathcal{A}^{k+1}(U)\}}{d(\mathcal{A}^{k-1}(U))} \tag{2.10}$$

are called the *de Rham cohomology groups* of $U$. The elements in

$$\mathcal{Z}^k(U) := \ker\{d\colon \mathcal{A}^k(U) \to \mathcal{A}^{k+1}(U)\}$$

are called *closed $k$-forms* and the elements in $\mathcal{B}^k(U) := d(\mathcal{A}^{k-1}(U))$ *exact $k$-forms.* We note that if $U$ is connected then $H^0_{dR}(U, \mathbb{C}) \cong \mathbb{C}$. Unless there is possibility of confusion we will drop the subscript since, in this notes, we will only consider de Rham cohomology.

**Exercise 18.** Prove that the closed forms are a subring of the ring of differential forms and that the exact forms are an ideal in the ring of closed forms. Deduce that the de Rham cohomology

$$H^*(U, \mathbb{C}) \;:=\; \bigoplus_{k \geq 0} H^k(U, \mathbb{C}), \tag{2.11}$$

inherits a ring structure:

$$[\alpha] \cup [\beta] \;:=\; [\alpha \wedge \beta]. \tag{2.12}$$

This is called the *cup product* on cohomology.

If $F\colon M \to N$ is a smooth map, then given an open set $V \subset N$, $F$ induces maps

$$F^*\colon \mathcal{A}^k(V) \to \mathcal{A}^k(F^{-1}(V))$$

which commute with the exterior differential; i.e. $F^*$ is a map of complexes. This implies that $F^*$ defines a map between de Rham cohomology groups:

$$F^*\colon H^k(V, \mathbb{C}) \to H^k(F^{-1}(V), \mathbb{C})$$

which satisfies the *chain rule* $(F \circ G)^* = G^* \circ F^*$. Since $(\mathrm{id})^* = \mathrm{id}$ it follows that if $F\colon M \to N$ is a diffeomorphism then $F^*\colon H^k(N, \mathbb{C}) \to H^k(M, \mathbb{C})$ is an isomorphism. In fact, the de Rham cohomology groups are a (smooth) homotopy invariant:

**Definition 2.6.** Let $f_0, f_1\colon M \to N$ be smooth maps. We say that $f_0$ is (smoothly) homotopic to $f_1$ if there exists a smooth map

$$H\colon \mathbb{R} \times M \to N$$

such that $H(0, x) = f_0(x)$ and $H(1, x) = f_1(x)$ for all $x \in M$.

**Theorem 2.7.** *Let $f_0, f_1\colon M \to N$ be smoothly homotopic maps. Then*

$$f_0^* = f_1^*\colon H^k(N, \mathbb{C}) \to H^k(M, \mathbb{C}).$$

*Proof.* We refer to [3, §4] for a proof of this important result. □

**Corollary 2.8 (Poincaré Lemma).** *Let $U \subset M$ be a contractible open subset then $H^k(U, \mathbb{C}) = \{0\}$ for all $k \geq 1$.*

*Proof.* The result follows from Theorem 2.7 since in a contractible open set the identity map is homotopic to a constant map. □

Hence, if $U$ is contractible, the sequence

$$0 \to \mathbb{C} \hookrightarrow C^\infty(U) \xrightarrow{d} \mathcal{A}^1(U) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{A}^{2n-1}(U) \xrightarrow{d} \mathcal{A}^{2n}(U) \to 0 \tag{2.13}$$

is exact.

The exterior differential operator is not of "pure bidegree" relative to the decomposition (2.6). Indeed, it follows from (2.8) that

$$d(\mathcal{A}^{a,b}(U)) \subset \mathcal{A}^{a+1,b}(U) \oplus \mathcal{A}^{a,b+1}(U).$$

We write $d = \partial + \bar\partial$, where $\partial$ (resp. $\bar\partial$) is the component of $d$ of bidegree $(1,0)$ (resp. $(0,1)$). From $d^2 = 0$ we obtain:

$$\partial^2 \;=\; \bar\partial^2 \;=\; 0 \;\;;\;\; \partial \circ \bar\partial + \bar\partial \circ \partial \;=\; 0. \tag{2.14}$$

**Exercise 19.** Generalize the Leibniz property to the operators $\partial$ and $\bar\partial$.

It follows from (2.14) that, for each $p$, $0 \le p \le n$, we get a complex

$$\mathcal{A}^{p,0}(U) \xrightarrow{\bar\partial} \mathcal{A}^{p,1}(U) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathcal{A}^{p,n-1}(U) \xrightarrow{\bar\partial} \mathcal{A}^{p,n}(U) \tag{2.15}$$

called the *Dolbeault complex.* Its cohomology spaces are denoted by $H^{p,q}_{\bar\partial}(U)$ and called the *Dolbeault cohomology groups.*

**Exercise 20.** Let $\alpha \in \mathcal{A}^{p,q}(U)$. Prove that $\overline{\partial\alpha} \;=\; \bar\partial\bar\alpha$. Deduce that a form $\alpha$ is $\partial$-closed if and only if $\bar\alpha$ is $\bar\partial$-closed. Similarly for $\partial$-exact forms. Conclude that via conjugation the study of $\partial$-cohomology reduces to the study of Dolbeault cohomology.

**Proposition 2.9.** *Let $M$ be a complex manifold, then $H^{p,0}_{\bar\partial}(M) \cong \Omega^p(M)$.*

*Proof.* Let $\alpha \in \mathcal{A}^{p,0}(M)$ and suppose $\bar\partial\alpha = 0$. Let $(U, \{z_1, \ldots, z_n\}$ be local coordinates in $M$ and let $\alpha|_U \;=\; \sum_I f_I\, dz_I$ , where $I$ runs over all increasing index sets $\{1 \le i_1 < \cdots < i_p \le n\}$. Then

$$\bar\partial\alpha|_U \;=\; \sum_I \sum_{j=1}^n \frac{\partial f_I}{\partial \bar z_j} d\bar z_j \wedge dz_I \;=\; 0\,,$$

This implies that $\partial f_I/\partial \bar z_j = 0$ for all $I$ and all $j$. Hence $f_I \in \mathcal{O}(U)$ for all $I$ and $\alpha$ is a holomorphic $p$-form. □

Given $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in (\mathbb{R}_{>0} \cup \infty)^n$ we denote by

$$\Delta_\varepsilon(a) = \{z \in \mathbb{C}^n : |z_i - a_i| < \varepsilon_i\}$$

the $n$-dimesional *polydisk.* For $n = 1$, $a = 0$, $\varepsilon = 1$ we set $\Delta = \Delta_1(0)$, the *unit disk,* and $\Delta^* = \Delta \setminus \{0\}$ the *punctured unit disk.* The following result is known as the $\bar\partial$-Poincaré Lemma:

**Theorem 2.10.** *If $q \ge 1$ and $\alpha$ is a $(p,q)$, $\bar\partial$-closed form on a polydisk $\Delta_\varepsilon(a)$, then $\alpha$ is $\bar\partial$-exact; i.e.*

$$H^{p,q}_{\bar\partial}(\Delta_\varepsilon(a)) \;=\; 0 \;;\; q \ge 1\,.$$

*Proof.* We refer to [15, Chapter 0] or [18, Corollary 1.3.9] for a proof. □

Hence, if $U = \Delta_\varepsilon(a)$ is a polydisk we have exact sequences:

$$0 \to \Omega^p(U) \hookrightarrow \mathcal{A}^{p,0}(U) \xrightarrow{\bar\partial} \mathcal{A}^{p,1}(U) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathcal{A}^{p,n}(U) \to 0 \tag{2.16}$$

*Remark* 6. It will be shown in the course by L. Tu that:

$$H^k(\mathbb{P}^n, \mathbb{C}) \cong \begin{cases} \mathbb{C} & \text{if } 0 \le k = 2\ell \le 2n \\ 0 & \text{otherwise.} \end{cases} \tag{2.17}$$

$$H^{p,q}_{\bar\partial}(\mathbb{P}^n) \cong \begin{cases} \mathbb{C} & \text{if } 0 \le p = q \le n \\ 0 & \text{otherwise.} \end{cases} \tag{2.18}$$

Also in that course you will see how to realize the de Rham and Dolbeault cohomology groups as sheaf cohomology groups. This will show, in particular, that even though our definition of the de Rham cohomology uses the differentiable structure they are, in fact, topological invariants. On the other hand, the Dolbeault cohomology groups depend essentially on the complex structure. This observation is at the core of Hodge Theory.

## 3. Hermitian and Kähler metrics

**Definition 3.1.** Let $M$ be a complex manifold and $J$ its complex structure. A Riemannian metric $g$ on $M$ is said to be a *Hermitian metric* if and only if for each $p \in M$, the bilinear form $g_p$ on the tangent space $T_p(M)$ is compatible with the complex structure $J_p$ (cf. (A.22).

We recall from (A.23) in the Appendix that given a bilinear form compatible with the complex structure we may define a $J$-invariant alternating form. Thus, given a Hermitian metric on $M$ we may define a differential two-form $\omega \in \mathcal{A}^2(M, \mathbb{C})$ by:

$$\omega(X, Y) \ := \ g(JX, Y), \tag{3.1}$$

where we also denote by $g$ the bilinear extension of $g$ to the complexified tangent space. By (A.24), we have

$$\omega \in \mathcal{A}^{1,1}(M) \quad \text{and} \quad \bar\omega = \omega. \tag{3.2}$$

We also recall that by Theorem A.8 every form $\omega$ as in (3.2) defines a symmetric $(1,1)$ tensor on $M$ compatible with $J$ and a Hermitian form $H$ on the complex vector space $(T_p(M), J)$.

We express these objects in local coordinates: let $(U, \{z_1, \ldots, z_n\})$ be local complex coordinates on $M$, then (3.2) implies that we may write

$$\omega \ := \ \frac{i}{2} \sum_{j,k=1}^{n} h_{jk}\, dz_j \wedge d\bar z_k \ ; \quad h_{kj} = \bar h_{jk}. \tag{3.3}$$

Hence $\omega(\partial/\partial z_j, \partial/\partial \bar z_k) = (i/2)\, h_{jk}$ from which it follows that

$$\omega(\partial/\partial x_j, \partial/\partial x_k) \ = \ -\mathrm{Im}(h_{jk}).$$

Moreover, it follows from (A.25) that

$$g(\partial/\partial x_j, \partial/\partial x_k) = \omega(\partial/\partial x_j, \partial/\partial y_k) \ = \ \mathrm{Re}(h_{jk}).$$

Thus,

$$H(\partial/\partial x_j, \partial/\partial x_k) = h_{jk},$$

where $H$ is the Hermitian form on $(T_p(M), J_p)$ defined as in (A.26). Hence $g$, and consequently $H$, is positive definite if and only if the Hermitian matrix $(h_{jk})$ is positive definite. We may then restate Definition 3.1 by saying that a Hermitian structure is a $(1,1)$ real form $\omega$ as in (3.3) such that the matrix $(h_{jk})$ is positive definite. By abuse of notation we will say that, in this case, the two-form $\omega$ is positive.

### 3.1. **Kähler Manifolds.**

**Definition 3.2.** A Hermitian structure on a manifold $M$ is said to be a *Kähler metric* if and only if the two-form $\omega$ is closed. We will say that a complex manifold is Kähler if and only if it admits a Kähler structure and refer to $\omega$ as a Kähler form.

**Exercise 21.** Let $(M, \omega)$ be a Kähler manifold. Prove that there exist local coframes $\chi_1, \dots, \chi_n \in \mathcal{A}^{1,0}(U)$ such that

$$\omega \ = \ (i/2) \sum_{j=1}^{n} \chi_j \wedge \bar{\chi}_j.$$

**Proposition 3.3.** *Every Kähler manifold $M$ is symplectic.*

*Proof.* Recall that every complex manifold is orientable and that if $\{z_1, \dots, z_n\}$ are local coordinates on $M$ then we may assume that the frame

$$\{\partial/\partial x_1, \partial/\partial y_1, \dots, \partial/\partial x_n, \partial/\partial y_n\}$$

is positively oriented.

Now, if $\omega$ is a Kähler form on $M$ then

$$\begin{aligned} \omega^n \quad &= \quad n! \left(\frac{i}{2}\right)^n \det((h_{ij})) \bigwedge_{j=1}^{n} (dz_j \wedge d\bar{z}_j) \\ &= \quad n! \det((h_{ij})) \bigwedge_{j=1}^{n} (dx_j \wedge dy_j), \end{aligned}$$

since $dz_j \wedge d\bar{z}_j = (2/i) dx_j \wedge dy_j$. Therefore, $\omega^n$ is never zero. □

**Exercise 22.** Prove that $\omega^n/n!$ is the volume element of the Riemannian metric $g$ defined by the Kähler form $\omega$.

Proposition 3.3 immediately gives a necessary condition for a compact complex manifold to be Kähler:

**Proposition 3.4.** *If $M$ is a compact Kähler manifold then*

$$\dim H^{2k}(M, \mathbb{R}) > 0,$$

*for all $k = 0, \dots, n$.*

*Proof.* Indeed, this is true of all compact symplectic manifolds as the forms $\omega^k$, $k = 1, \ldots, n$, induce non-zero de Rham cohomology classes. Suppose, otherwise, that $\omega^k = d\alpha$, then

$$\omega^n = d(\omega^{n-k} \wedge \alpha).$$

But then it would follow from Stokes' Theorem that

$$\int_M \omega^n \;=\; 0$$

which contradicts the fact that $\omega^n$ is a non-zero multiple of the volume element. □

*Remark* 7. As we will see below, the existence of a Kähler metric on a manifold imposes many other topological restrictions beyond those satisfied by symplectic manifolds. The earliest examples of compact symplectic manifolds with no Kähler structure are due to Thurston [30]. We refer to [36] for further details.

**Example 3.5.** Affine space $\mathbb{C}^n$ with the Kähler form:

$$\omega \;=\; \frac{i}{2}\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j$$

is a Kähler manifold. The form $\omega$ gives the usual symplectic structure on $\mathbb{R}^{2n}$.

The following theorem states that, locally, a Kähler metric agrees with the Euclidean metric up to order two.

**Theorem 3.6.** *Let $M$ be a complex manifold and $g$ a Kähler metric on $M$. Then, given $p \in M$ there exist local coordinates $(U, \{z_1, \ldots, z_n\})$ around $p$ such that $z_j(p) = 0$ and*

$$\omega \;=\; \frac{i}{2}\sum_{j=1}^{n} h_{jk} dz_j \wedge d\bar{z}_k\,,$$

*where the coefficients $h_{jk}$ are of the form*

$$h_{jk}(z) \;=\; \delta_{jk} + O(||z||^2). \tag{3.4}$$

*Proof.* We refer to [34, Proposition 3.14] for a proof. □

**Example 3.7.** We will now construct a Kähler form on $\mathbb{P}^n$. We will do this by exhibiting a positive, real, closed $(1,1)$-form on $\mathbb{P}^n$, the resulting metric is called the Fubini-Study metric on $\mathbb{P}^n$.

Given $z \in \mathbb{C}^{n+1}$ we denote by

$$||z||^2 \;=\; |z_0|^2 + \cdots + |z_n|^2.$$

Let $U_j \subset \mathbb{P}^n$ be the open set (1.6) and let $\rho_j \in C^\infty(U_j)$ be the positive function

$$\rho_j([z]) \;:=\; \frac{||z||^2}{|z_j|^2}, \tag{3.5}$$

and define $\omega_i \in \mathcal{A}^{1,1}(U_j)$ by

$$\omega_j \;:=\; \frac{-1}{2\pi i}\,\partial\bar{\partial}\log(\rho_j). \tag{3.6}$$

Clearly, $\omega_j$ is a real two-form. Moreover, on $U_j \cap U_k$, we have

$$\log(\rho_j) - \log(\rho_k) = \log|z_k|^2 - \log|z_j|^2 = \log(z_k\bar{z}_k) - \log(z_j\bar{z}_j).$$

Hence, since $\partial\bar{\partial}(\log(z_j\bar{z}_j)) = 0$, we have that $\omega_j = \omega_k$ on $U_j \cap U_k$. Thus, the forms $\omega_j$ piece together to give us a global, real, $(1,1)$-form on $\mathbb{P}^n$:

$$\omega \;=\; \frac{-1}{2\pi i}\,\partial\bar{\partial}\log(||z||^2). \tag{3.7}$$

Moreover, it is clear from the definition of $\omega$ that $d\omega = 0$.

It remains to show that $\omega$ is positive. We observe first of all that the expression (3.7) shows that if $A$ is a unitary matrix and $\mu_A\colon \mathbb{P}^n \to \mathbb{P}^n$ is the biholomorphic map $\mu_A([z]) := [A \cdot z]$ then $\mu_A^*(\omega) = \omega$. Hence, since given any two points $[z], [z'] \in \mathbb{P}^n$ there exists a unitary matrix such that $\mu_a([z]) = [z']$, it suffices to prove that $\omega$ is positive definite at just one point, say $[1, 0, \ldots, 0] \in U_0$. In the coordinates $\{u_1, \ldots, u_n\}$ in $U_0$, we have $\rho_0(u) = 1 + ||u||^2$ and therefore:

$$\bar{\partial}(\log\rho_0(u)) \;=\; \rho_0^{-1}(u)\sum_{k=1}^n u_k\,\bar{\partial}\bar{u}_k \;=\; \rho_0^{-1}(u)\sum_{k=1}^n u_k\,d\bar{u}_k\,,$$

$$\omega \;=\; \frac{i}{2\pi}\,\rho_0^{-2}(u)\left(\rho_0(u)\sum_{j=1}^n du_j\wedge d\bar{u}_j + \left(\sum_{j=1}^n \bar{u}_j du_j\right)\wedge\left(\sum_{j=1}^n u_k d\bar{u}_k\right)\right).$$

Hence, at the origin, we have

$$\omega = \frac{i}{2\pi}\sum_{j=1}^n du_j\wedge d\bar{u}_j.$$

which is a positive form.

The function $\log(\rho_j)$ in the above proof is called a Kähler *potential.* As the following result shows, every Kähler metric may be described by a (local) potential.

**Proposition 3.8.** *Let $M$ be a complex manifold and $\omega$ a Kähler form on $M$. Then for every $p \in M$ there exists an open set $U \subset M$ and a real function $v \in C^\infty(U)$ such that $\omega = i\,\partial\bar{\partial}(v)$.*

*Proof.* Since $d\omega = 0$, it follows from the Poincaré Lemma that in a neighborhood $U'$ of $p$, $\omega = d\alpha$, where $\alpha \in \mathcal{A}^1(U', \mathbb{R})$. Hence, we maw write $\alpha = \beta + \bar{\beta}$, where $\beta \in \mathcal{A}^{1,0}(U', \mathbb{R})$. Now, we can write:

$$\omega \;=\; d\alpha \;=\; \partial\beta + \bar{\partial}\beta + \partial\bar{\beta} + \bar{\partial}\bar{\beta}\,,$$

but, since $\omega$ is of type $(1,1)$ it follows that

$$\omega \;=\; \bar{\partial}\beta + \partial\bar{\beta} \quad\text{and}\quad \partial\beta \;=\; \bar{\partial}\bar{\beta} \;=\; 0.$$

We may now apply the $\bar{\partial}$-Poincaré Lemma to conclude that there exists a neighborhood $U \subset U'$ around $p$ where $\bar{\beta} = \bar{\partial}f$ for some ($\mathbb{C}$-valued) $C^\infty$ function $f$ on $U$. Hence:

$$\omega \;=\; \bar{\partial}\partial\bar{f} + \partial\bar{\partial}f \;=\; \partial\bar{\partial}(f - \bar{f}) \;=\; 2i\,\partial\bar{\partial}(\mathrm{Im}(f)).$$

□

**Theorem 3.9.** *Let $(M,\omega)$ be a Kähler manifold and suppose $N\subset M$ is a complex submanifold, then $(N,\omega|_N)$ is a Kähler manifold.*

*Proof.* Let $g$ be the $J$-invariant Riemannian metric on $M$ associated with $\omega$. Then $g$ restricts to an invariant Riemannian metric on $N$ whose associated two-form is $\omega|_N$. Since $d(\omega|_N)=(d\omega)|_N=0$ it follows that $N$ is a Kähler manifold as well. □

It follows from Theorem 3.9 that a necessary condition for a compact complex manifold $M$ to have an embedding in $\mathbb{P}^n$ is that there exists a Kähler metric on $M$. Moreover, as we shall see below, for a submanifold of projective space there exists a Kähler metric whose associated cohomology satisfies a suitable integrality condition.

3.2. **The Chern Class of a Holomorphic Line Bundle.** The construction of the Kähler metric in $\mathbb{P}^n$ may be further understood in the context of hermitian metrics on (line) bundles. We recall that a Hermitian metric on a $\mathbb{C}$-vector bundle $\pi\colon E\to M$ is given by a positive definite Hermitian form

$$H_p\colon E_p\times E_p\to\mathbb{C}$$

on each fiber $E_p$ which is smooth in the sense that given sections $\sigma,\tau\in\Gamma(U,E)$, the function

$$H(\sigma,\tau)(p)\ :=\ H_p(\sigma(p),\tau(p))$$

is $C^\infty$ on $U$. In particular, a Hermitian metric on a complex manifold is equivalent to a Hermitian metric on the holomorphic tangent bundle (or on the complexified tangent bundle). Using partitions of unity one can prove that every smooth vector bundle $E$ has a Hermitian metric $H$.

In the case of a line bundle $L$, the Hermitian form $H_p$ is completely determined by the value $H_p(v,v)$ on a non-zero element $v\in L_p$. In particular, if $\{(U_\alpha,\Phi_\alpha)\}$ is a cover of $M$ by trivializing neighborhoods of $L$ and $\sigma_\alpha\in\mathcal{O}(U_\alpha,L)$ is the local frame

$$\sigma_\alpha(x)\ =\ \Phi_\alpha^{-1}(x,1)\ ;\quad x\in U_\alpha,$$

then a Hermitian metric $H$ on $L$ is determined by the collection of positive functions:

$$\rho_\alpha\ :=\ H(\sigma_\alpha,\sigma_\alpha)\in C^\infty(U_\alpha).$$

we note that if $U_\alpha\cap U_\beta\neq\emptyset$ then we have $\sigma_\beta=g_{\alpha\beta}\cdot\sigma_\alpha$ and, consequently, the functions $\rho_\alpha$ satisfy the compatibility condition:

$$\rho_\beta\ =\ |g_{\alpha\beta}|^2\rho_\alpha. \tag{3.8}$$

In particular, if $L$ is a holomorphic line bundle then the transition functions $g_{\alpha\beta}$ are holomorphic and we have, as in Example 3.7 that

$$\partial\bar\partial\log(\rho_\alpha)\ =\ \partial\bar\partial\log(\rho_\beta)$$

on $U_\alpha\cap U_\beta$ and therefore the form

$$\frac{1}{2\pi i}\,\partial\bar\partial\log(\rho_\alpha) \tag{3.9}$$

is a global, real, $(1,1)$ form on $M$. The cohomology class

$$[(1/2\pi i)\,\partial\bar\partial\log(\rho_\alpha)]\in H^2(M,\mathbb{R}) \tag{3.10}$$

is called the *Chern class* of the vector bundle $L$ and denoted by $c(L)$. In fact, the factor $1/2\pi$ factor is chosen so that the Chern class is actually an integral cohomology class:

$$c(L) \in H^2(M, \mathbb{Z}). \tag{3.11}$$

Recall that if $g_{\alpha\beta}$ are the transition functions for a bundle $L$ then the functions $g_{\alpha\beta}^{-1}$ are the transition functions of the dual bundle $L^*$. In particular, if $\rho_\alpha$ are a collection of positive $C^\infty$ functions defining a Hermitian metric on $L$ then the functions $\rho_\alpha^{-1}$ define a Hermitian metric, $H^*$ on $L^*$. We call $H^*$ the dual Hermitian metric. We then have:

$$c(L^*) \;=\; -c(L). \tag{3.12}$$

**Definition 3.10.** A holomorphic line bundle $L \to M$ over a compact Kähler manifold is said to be *positive* if and only if there exists a Hermitian metric $H$ on $L$ for which the $(1,1)$ form (3.10) is positive. We say that $L$ is *negative* if its dual bundle $L^*$ is positive.

We note that in Example 3.7 we have:

$$|z_k|^2\,\rho_k([z]) \;=\; |z_j|^2\,\rho_j([z])$$

on $U_j \cap U_k$. Hence

$$\rho_k([z]) \;=\; \left|\frac{z_j}{z_k}\right|^2 \rho_j([z])$$

and, by (3.8), it follows that the functions $\rho_j$ define a Hermitian metric on the tautological bundle $\mathcal{O}(-1)$. Hence taking into account the sign change in (3.6) it follows that for the Kähler class of the Fubini-study metric agrees with the Chern class of the hyperplane bundle $\mathcal{O}(1)$.

$$c(\mathcal{O}(1)) \;=\; [\omega] \;=\; \left[\frac{i}{2\pi}\,\partial\bar\partial\log(||z||^2)\right]. \tag{3.13}$$

Hence, the hyperplane bundle $\mathcal{O}(1)$ is a positive line bundle. Moreover, if $M \subset \mathbb{P}^n$ is a complex submanifold then the restriction of $\mathcal{O}(1)$ to $M$ is a positive line bundle over $M$. We can now state:

**Theorem 3.11 (Kodaira Embedding Theorem).** *A compact complex manifold $M$ may be embedded in $\mathbb{P}^n$ if and only if it there exists a positive holomorphic line bundle $\pi\colon L \to M$.*

We refer to [34, Theorem 7.11], [25, Theorem 8.1], [15], [39, Theorem 4.1], and [18, §5.3] for various proofs of this theorem.

*Remark* 8. The existence of a positive holomorphic line bundle $\pi\colon L \to M$ implies that $M$ admits a Kähler metric whose Kähler class is integral. Conversely, any integral cohomology class represented by a closed $(1,1)$ form is the Chern class of a line bundle (cf. [10, §6]), hence a compact complex manifold $M$ may be embedded in $\mathbb{P}^n$ if and only if it admits a Kähler metric whose Kähler class is integral.

When the Kodaira Embedding Theorem is combined with Chow's Theorem that asserts that every analytic subvariety of $\mathbb{P}^n$ is algebraic we obtain a characterization of complex projective varieties as those compact Kähler manifolds admitting a Kähler metric whose Kähler class is integral.

## 4. Harmonic Forms - Hodge Theorem

4.1. **Compact Real Manifolds.** Throughout §4.1 we will let $M$ denote a compact, oriented, real, $n$-dimensional manifold with a Riemannian metric $g$. We recall that the metric on the tangent bundle $TM$ induces a dual metric on the cotangent bundle $T^*M$ by the condition that if $X_1, \dots, X_n \in \Gamma(U, TM)$ are a local orthonormal frame on $U$ then the dual frame $\xi_1, \dots, \xi_n \in \mathcal{A}^1(U)$ is orthonormal as well. We will denote the dual inner product by $\langle\ ,\ \rangle$.

**Exercise 23.** Verify that this metric on $T^*M$ is well-defined. That is, it is independent of the choice of local orthonormal frames.

We extend the inner product to the exterior bundles $\Lambda^r(T{*}M)$ by the specification that the local frame

$$\xi_I \ := \ \xi_{i_1} \wedge \cdots \xi_{i_r},$$

where $I$ runs over all strictly increasing index sets $\{1 \leq i_1 < \cdots < i_r \leq n\}$, is orthonormal.

**Exercise 24.** Verify that this metric on $T^*M$ is well-defined: i.e., it is independent of the choice of local orthonormal frames, by proving that:

$$\langle \alpha_1 \wedge \cdots \wedge \alpha_r, \beta_1 \wedge \cdots \wedge \beta_r \rangle \ = \ \det(\langle \alpha_i, \beta_j \rangle),$$

where $\alpha_i, \beta_j \in \mathcal{A}^1(U)$.
**Hint:** Use the Cauchy-Binet formula for determinants.

Recall that given an oriented Riemannian manifold, the volume element is defined as the unique $n$-form $\Omega \in \mathcal{A}^n(M)$ such that

$$\Omega(p)(v_1, \dots, v_n) \ = \ 1$$

for any positively oriented ortonormal basis $\{v_1, \dots, v_n\}$ of $T_p(M)$. If $\xi_1, \dots, \xi_n \in \mathcal{A}^1(U)$ is a positively oriented orthonormal coframe then

$$\Omega_U \ = \ \xi_1 \wedge \cdots \wedge \xi_n.$$

**Exercise 25.** Prove that in local coordinates, the volume element may be written as

$$\Omega \ = \ \sqrt{G}\ dx_1 \wedge \cdots \wedge dx_n\,,$$

where $G = \det(g_{ij})$ and

$$g_{ij} \ := \ g(\partial/\partial x_i, \partial/\partial x_j).$$

We now define the *Hodge $*$-operator*: Let $\alpha_1, \dots, \alpha_n \in T_p^*(M)$ be a positively oriented orthonormal basis, $I = \{1 \leq i_1 < \cdots < i_r \leq n\}$, and $I^c$ the complementary index set. Then we set

$$*(\alpha_I) = \operatorname{sign}(I, I^c)\ \alpha_{I^c}\,, \tag{4.1}$$

where $\operatorname{sign}(I, I^c)$ is the sign of the permutation $\{I, I^c\}$, and extend it linearly to an operator:

$$* \colon \Lambda^r(T_p^*(M)) \to \Lambda^{n-r}(T_p^*(M)) \tag{4.2}$$

Note that since $*$ maps an orthonormal basis to an orthonormal basis, it must preserve the inner product.

**Exercise 26.** Verify that $*$ is well defined by proving that for $\alpha, \beta \in \Lambda^r(T_p^*(M))$:

$$\alpha \wedge *\beta \;=\; \langle \alpha, \beta \rangle \, \Omega(p). \tag{4.3}$$

**Exercise 27.** Prove that $*$ is an isomorphism and that $*^2$ acting on $\Lambda^r(T_p^*(M))$ equals $(-1)^{r(n-r)}$ id.

Suppose now that $M$ is compact. We can then define an $L^2$-inner product on the space of $r$-forms on $M$ by:

$$(\alpha, \beta) \;:=\; \int_M \alpha \wedge *\beta \;=\; \int_M \langle \alpha(p), \beta(p) \rangle \; \Omega \; ; \quad \alpha, \beta \in \mathcal{A}^r(M). \tag{4.4}$$

**Proposition 4.1.** *The bilinear form* $(\ ,\ )$ *is a positive definite inner product on* $\mathcal{A}^r(M)$*.*

*Proof.* First of all we check that $(\ ,\ )$ is symmetric:

$$(\beta, \alpha) = \int_M \beta \wedge *\alpha = (-1)^{r(n-r)} \int_M *(*\beta) \wedge *\alpha = \int_M *\alpha \wedge *(*\beta) = (\alpha, \beta).$$

Now, given $0 \neq \alpha \in \mathcal{A}^r(M)$, we have

$$(\alpha, \alpha) \;=\; \int_M \alpha \wedge *\alpha \;=\; \int_M \langle \alpha(p), \alpha(p) \rangle \; \Omega > 0$$

since $\langle \alpha(p), \alpha(p) \rangle$ is a non-negative function which is not identically zero. □

**Proposition 4.2.** *The operator* $\delta \colon \mathcal{A}^{r+1}(M) \to \mathcal{A}^r(M)$ *defined by:*

$$\delta \;:=\; (-1)^{nr+1} * d\, * \tag{4.5}$$

*is the formal adjoint of d, that is:*

$$(d\alpha, \beta) \;=\; (\alpha, \delta\beta) \; ; \quad \textit{for all} \;\; \alpha \in \mathcal{A}^r(M), \beta \in \mathcal{A}^{r+1}(M). \tag{4.6}$$

*Proof.*

$$\begin{aligned} (d\alpha, \beta) \;&=\; \int_M d\alpha \wedge *\beta \;=\; \int_M d(\alpha \wedge *\beta) - (-1)^r \int_M \alpha \wedge d * \beta \\ &=\; -(-1)^r(-1)^{r(n-r)} \int_M \alpha \wedge *(* \, d * \beta) \;=\; \int_M \alpha \wedge *\delta\beta \\ &=\; (\alpha, \delta\beta). \end{aligned}$$

□

*Remark* 9. Note that if $\dim M$ is even then $\delta = - * d\, *$ independently of the degree of the form. Since we will be interested in applying these results in the case of complex manifolds which, as real manifolds, are even-dimensional we will make that assumption from now on.

We now define the *Laplace-Beltrami operator* of $(M,g)$ by

$$\Delta\colon \mathcal{A}^r(M) \to \mathcal{A}^r(M)\ ; \quad \Delta(\alpha)\ :=\ d\delta\alpha + \delta d\alpha.$$

**Proposition 4.3.** *The operators $d$, $\delta$, $*$ and $\Delta$ satisfy the following properties:*

i) $\Delta$ *is self-adjoint; i.e.* $(\Delta\alpha,\beta) = (\alpha,\Delta\beta)$.
ii) $[\Delta,d]\ =\ [\Delta,\delta]\ =\ [\Delta,*]\ =\ 0.$
iii) $\Delta(\alpha) = 0$ *if and only if* $d\alpha\ =\ \delta\alpha\ =\ 0.$

*Proof.* We leave the first two items as exercises. Note that given operators $D_1, D_2$, the bracket $[D_1,D_2] = D_1 \circ D_2 - D_2 \circ D_1$. Thus, ii) states that the Laplacian $\Delta$ commutes with $d,\delta$, and $*$.

Clearly, if $d\alpha = \delta\alpha = 0$ we have $\Delta\alpha = 0$. Conversely, uppose $\alpha \in \mathcal{A}^r(M)$, and $\Delta\alpha = 0$ then

$$0\ =\ (\Delta\alpha,\alpha)\ =\ (d\delta\alpha + \delta d\alpha,\alpha) = (\delta\alpha,\delta\alpha) + (da,da).$$

Hence $d\alpha = \delta\alpha = 0$. □

**Definition 4.4.** A form $\alpha \in \mathcal{A}^r(M)$ is said to be *harmonic* if $\Delta\alpha = 0$ or, equivalently, if $\alpha$ is closed and co-closed, i.e. $\delta\alpha = 0$.

**Exercise 28.** Let $M$ be a compact, connected, oriented, Riemannian manifold. Show that the only harmonic functions on $M$ are the constant functions.

**Exercise 29.** Let $\alpha \in \mathcal{A}^r(M)$ be closed. Show that $*\alpha$ is closed if and only if $\alpha$ is harmonic.

The following result shows that harmonic forms are very special within a given de Rham cohomology class:

**Proposition 4.5.** *A closed $r$-form $\alpha$ is harmonic if and only if $||\alpha||^2$ is a local minimum within the de Rham cohomology class of $\alpha$. Moreover, in any given de Rham cohomology class there is at most one harmonic form.*

*Proof.* Let $\alpha \in \mathcal{A}^r(M)$ be such that $||\alpha||^2$ is a local minimum within the de Rham cohomology class of $\alpha$. Then, for every $\beta \in \mathcal{A}^{r-1}(M)$, the function $\nu(t)\ :=\ ||\alpha + t\,d\beta||^2$ has a local minimum at $t = 0$. In particular,

$$\nu'(0) = 2(a,d\beta) = 2(\delta\alpha,\beta) = 0 \quad \text{for all} \quad \beta \in \mathcal{A}^{r-1}(M).$$

Hence, $\delta\alpha = 0$ and $\alpha$ is harmonic. Now, if $\alpha$ is harmonic, then

$$||\alpha +\ d\beta||^2 = ||\alpha||^2 + ||d\beta||^2 + 2(\alpha,d\beta) = ||\alpha||^2 + ||d\beta||^2 \geq ||\alpha||^2$$

and equality holds only if $d\beta = 0$. This proves the uniqueness statement. □

Hodge's theorem asserts that, in fact, every de Rham cohomology class contains a (unique) harmonic form. More precisely,

**Theorem 4.6** (**Hodge Theorem**). *Let $\mathcal{H}^r(M)$ denote the vector space of harmonic $r$-forms on $M$. Then:*

i) $\mathcal{H}^r(M)$ *is finite dimensional for all $r$.*

ii) *We have the following decomposition of the space of r forms:*

$$\begin{aligned}\mathcal{A}^r(M) &= \Delta(\mathcal{A}^r(M)) \oplus \mathcal{H}^r(M)\\ &= d\delta(\mathcal{A}^r(M)) \oplus \delta d(\mathcal{A}^r(M)) \oplus \mathcal{H}^r(M)\\ &= d(\mathcal{A}^{r-1}(M)) \oplus \delta(\mathcal{A}^{r+1}(M)) \oplus \mathcal{H}^r(M).\end{aligned}$$

*Proof.* The proof of this theorem involves the theory of elliptic differential operators on a manifold. We refer to [15, Chapter 0], [37, Chapter 6] and [39, Chapter 4] for proofs of this important result. □

Since $d$ and $\delta$ are formal adjoints of each other it follows that

$$(\ker(d), \operatorname{Im}(\delta)) = (\ker(\delta), \operatorname{Im}(d)) = 0$$

and, consequently, if $\alpha \in \mathcal{Z}^r(M)$ and we write

$$\alpha = d\beta + \delta\gamma + \mu \ ; \quad \beta \in \mathcal{A}^{r-1}(M), \gamma \in \mathcal{A}^{r+1}(M), \mu \in \mathcal{H}^r(M),$$

then

$$0 = (\alpha, \delta\gamma) = (\delta\gamma, \delta\gamma)$$

and therefore $\delta\gamma = 0$. Hence, $[\alpha] = [\mu]$. By the uniqueness statement in Proposition 4.5 we get:

$$H^r(M, \mathbb{R}) \cong \mathcal{H}^r(M). \tag{4.7}$$

**Corollary 4.7.** *Let $M$ be a compact, oriented, $n$-dimensional manifold. Then $H^r(M, \mathbb{R})$ is finite-dimensional for all $r$.*

**Corollary 4.8 (Poincaré Duality).** *Let $M$ be a compact, oriented, $n$-dimensional manifold. Then the bilinear pairing:*

$$\int_M : H^r(M, \mathbb{R}) \times H^{n-r}(M, \mathbb{R}) \to \mathbb{R} \tag{4.8}$$

*that maps $(\alpha, \beta) \mapsto \int_M \alpha \wedge \beta$ is non-degenerate. Hence*

$$\left(H^{n-r}(M, \mathbb{R})\right)^* \cong H^r(M, \mathbb{R}).$$

*Proof.* We may assume without loss of generality that $M$ is a Riemannian manifold. Then, the Hodge star operator commutes with the Laplacian and defines an isomorphism:

$$\mathcal{H}^r(M) \cong \mathcal{H}^{n-r}(M).$$

Hence if $0 \neq \alpha \in \mathcal{H}^r(M)$ we have $*\alpha \in \mathcal{H}^{n-r}(M)$ and

$$\int_M \alpha \wedge *\alpha = (\alpha, \alpha) \neq 0.$$

□

**Exercise 30.** Prove that the pairing (4.8) is well-defined.

4.2. **Compact Hermitian Manifolds.** We now consider the case of a compact, complex, $n$-dimensional manifold $M$ with a Hermitian metric. First of all, we will be interested in considering complex valued forms $\mathcal{A}^r(M,\mathbb{C})$. We extend the $*$-operator by linearity to $\Lambda^r(T_p^*(M)\otimes_{\mathbb{R}}\mathbb{C})$ and extending the inner product $\langle\ ,\ \rangle$ on $\Lambda^r(T_p^*(M)$ to the complexification as a Hermitian inner product $\langle\ ,\ \rangle^h$ we get:

$$\alpha\wedge *\bar{\beta} = \langle\alpha,\beta\rangle^h\ \Omega(p)$$

for $\alpha,\beta\in\Lambda^r(T_p^*(M)\otimes_{\mathbb{R}}\mathbb{C})$. Hence we get a positive definite Hermitian inner product on $\mathcal{A}^k(M,\mathbb{C})$ by

$$(\alpha,\beta)^h\ :=\ \int_M\langle\alpha,\beta\rangle^h\ \Omega\ =\ \int_M\alpha\wedge *\bar{\beta}. \tag{4.9}$$

**Exercise 31.** Prove that $*$ maps $(p,q)$-forms to forms of bidegree $(n-q,n-p)$.

**Exercise 32.** Let $M$ be a compact, complex, $n$-dimensional manifold and $\alpha\in\mathcal{A}^{2n-1}(M,\mathbb{C})$. Prove that

$$\int_M\partial\alpha\ =\ \int_M\bar{\partial}\alpha\ =\ 0.$$

**Proposition 4.9.** *The operator* $\partial^*\ :=\ -*\bar{\partial}*$ *(resp.* $\bar{\partial}^*\ :=\ -*\partial*$*) is the formal adjoint of* $\partial$ *(resp.* $\bar{\partial}$*) relative to the Hermitian inner product* $(\ ,\ )^h$*. The operator* $\partial^*$ *(resp.* $\bar{\partial}^*$*) is of bidegree* $(-1,0)$ *(resp.* $(0,-1)$*).*

*Proof.* Given Exercise 32 and the Leibniz property for the operators $\partial$, $\bar{\partial}$, the proof of the first statement is analogous to that of Proposition 4.2. The second statement follows from Exercise 31. The details are left as an exercise. ☐

We can now define Laplace-Beltrami operators:

$$\Delta_\partial = \partial\partial^* + \partial^*\partial\ ;\quad \Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} \tag{4.10}$$

Note that the operators $\Delta_\partial$ and $\Delta_{\bar{\partial}}$ are of bidegree $(0,0)$; i.e. they map forms of bidegree $(p,q)$ to forms of the same bidegree. In particular, if $\alpha\in\mathcal{A}^k(U)$ is decomposed according to (2.6) as

$$\alpha\ =\ \alpha^{k,0}+\alpha^{k-1,1}+\cdots+\alpha^{0,k}\,,$$

then $\Delta_{\bar{\partial}}(\alpha)=0$ if and only if $\Delta_{\bar{\partial}}(\alpha^{p,q})=0$ for all $p,q$.

The operators $\Delta_\partial$ and $\Delta_{\bar{\partial}}$ are elliptic and, consequently, the Hodge Theorem remains valid for them. Thus if we set:

$$\mathcal{H}^{p,q}_{\bar{\partial}}(M)\ :=\ \{\alpha\in\mathcal{A}^{p,q}(M):\Delta_{\bar{\partial}}(\alpha)=0\}, \tag{4.11}$$

we have

$$H^{p,q}_{\bar{\partial}}(M)\ \cong\ \mathcal{H}^{p,q}(M). \tag{4.12}$$

## 5. The Hodge Decompositon Theorem

In this section we will show that on a compact Kähler manifold the Laplacians $\Delta$ and $\Delta_{\bar\partial}$ are multiples of each other. Indeed, we have:

**Theorem 5.1.** *Let $M$ be a compact Kähler manifold. Then:*

$$\Delta \;=\; 2\Delta_{\bar\partial}. \tag{5.1}$$

This theorem has a remarkable consequence: Suppose $\alpha \in \mathcal{H}^k(M)$ is decomposed according to (2.6) as

$$\alpha \;=\; \alpha^{k,0} + \alpha^{k-1,1} + \cdots + \alpha^{0,k}\,,$$

then since $\Delta \;=\; 2\Delta_{\bar\partial}$, the form $\alpha$ is $\Delta_{\bar\partial}$-harmonic and, consequently, the components $\alpha^{p,q}$ are $\Delta_{\bar\partial}$-harmonic and, hence, $\Delta$-harmonic as well. Therefore, if we set for $p + q = k$:

$$\mathcal{H}^{p,q}(M) \;:=\; \mathcal{H}^k(M,\mathbb{C}) \cap \mathcal{A}^{p,q}(M), \tag{5.2}$$

we get

$$\mathcal{H}^k(M,\mathbb{C}) \cong \bigoplus_{p+q=k} \mathcal{H}^{p,q}(M). \tag{5.3}$$

Moreover, since $\Delta$ is a real operator, it follows that

$$\mathcal{H}^{q,p}(M) \;=\; \overline{\mathcal{H}^{p,q}(M)}. \tag{5.4}$$

If we combine these results with the Hodge Theorem we get:

**Theorem 5.2** (**Hodge Decomposition Theorem**). *Let $M$ be a compact Kähler manifold and let $H^{p,q}(M)$ be the space of de Rham cohomology classes in $H^{p,q}(M,\mathbb{C})$ that have a representative of bidegree $(p,q)$. Then,*

$$H^{p,q}(M) \;\cong\; H^{p,q}_{\bar\partial}(M) \;\cong\; \mathcal{H}^{p,q}(M), \tag{5.5}$$

*and*

$$H^k(M,\mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(M). \tag{5.6}$$

*Moreover, $H^{q,p}(M) = \overline{H^{p,q}(M)}$.*

*Remark* 10. In view of Definition A.3, Theorem 5.2 may be restated as: The de Rham cohomology groups $H^k(M,\mathbb{R})$ have a Hodge structure of weight $k$ with $(H(M,\mathbb{C}))^{p,q} \;\cong\; H^{p,q}_{\bar\partial}(M)$.

We will denote by $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}(M)$. These are the so-called *Hodge numbers* of $M$. Note that the *Betti numbers* $b^k$, that is, the dimension of the $k$-th cohohomology space are given by:

$$b^k \;=\; \sum_{p+q=k} h^{p,q}. \tag{5.7}$$

In particular, the Hodge Decomposition Theorem implies a new restriction on the cohomology of a compact Kähler manifold:

**Corollary 5.3.** *The odd Betti numbers of a compact Kähler manifold are even.*

*Proof.* This assertion follows from (5.7) together with the fact that $h^{p,q} = h^{q,p}$. □

*Remark* 11. Thurston's examples of complex symplectic manifolds with no Kähler structure are manifolds which do not satisfy Corollary 5.3.

*Remark* 12. As pointed out in Example 18, the de Rham cohomology $H^*(M,\mathbb{C})$ is an algebra under the cup product. We note that the Hodge decomposition (5.6) is compatible with the algebra structure in the sense that

$$H^{p,q} \cup H^{p',q'} \subset H^{p+p',q+q'}. \tag{5.8}$$

This additional topological restriction for a compact, complex, symplectic manifold to have a Kähler metric has been successfully exploited by C. Voisin [36] to obtain remarkable examples of non-Kähler, symplectic manifolds.

Let $M$ be a compact, $n$-dimensional Kähler manifold and $X \subset M$ a complex submanifold of codimension $k$. We may define a linear map:

$$\int_X : H^{2(n-k)}(M,\mathbb{C}) \to \mathbb{C}\ ; \quad [\alpha] \mapsto \int_X \alpha|_X. \tag{5.9}$$

This map defines an element in $(H^{2(n-k)}(M,\mathbb{C}))^*$ and, therefore, by Corollary 4.8 a cohomology class $\eta_X \in H^{2k}(M,\mathbb{C})$ defined by the property that for all $[\alpha] \in H^{2(n-k)}(M,\mathbb{C})$:

$$\int_M \alpha \wedge \eta_X \ = \ \int_X \alpha|_X. \tag{5.10}$$

The class $\eta_X$ is called the *Poincaré dual* of $X$ and one can show that:

$$\eta_X \in H^{k,k}(M) \cap H^{2k}(M,\mathbb{Z}). \tag{5.11}$$

One can also prove that the construction of the Poincaré dual may be extended to singular analytic subvarieties (cf. [15, 18]).

**Conjecture 5.4** (**Hodge Conjecture**). *Let $M$ be a smooth, projective manifold. Then*

$$H^{k,k}(M,\mathbb{Q}) \ := \ H^{k,k}(M) \cap H^{2k}(M,\mathbb{Q})$$

*is generated, as a $\mathbb{Q}$-vector space, by the Poincaré duals of analytic subvarieties of $M$.*

The Hodge Conjecture is one of the remaining six Clay Millenium Problems [11].

### 5.1. **Kähler identities.**

**Definition 5.5.** Let $(M,\omega)$ be an $n$-dimensional, compact, Kähler manifold. We define:

$$L_\omega\colon \mathcal{A}^k(M,\mathbb{C}) \to \mathcal{A}^{k+2}(M,\mathbb{C})\ ; \quad L_\omega(\alpha) = \omega \wedge \alpha\,, \tag{5.12}$$

$Y\colon \mathcal{A}^*(M,\mathbb{C}) \to \mathcal{A}^*(M,\mathbb{C})$, where $\mathcal{A}^*(M,\mathbb{C}) = \bigoplus_{k=0}^{2n} \mathcal{A}^k(M,\mathbb{C})$ by

$$Y(\alpha) \ = \ (n-k)\alpha \quad \text{for} \quad \alpha \in \mathcal{A}^k(M,\mathbb{C})\,, \tag{5.13}$$

and we let $N_+$ denote the formal adjoint of $L_\omega$ with respect to the Hermitian form $(\ ,\ )^h$; i.e. if $\alpha \in \mathcal{A}^k(M,\mathbb{C})$,

$$N_+(\alpha) = (-1)^k * L_\omega * \alpha.$$

*Remark* 13. If there is no chance of confusion we will write $L$ for $L_\omega$. It is clear, however, that this operator, as well as its formal adjoint depend on the choice of a Kähler form $\omega$.

**Exercise 33.** Prove that if $\alpha \in \mathcal{A}^k(M,\mathbb{C})$ and $\beta \in \mathcal{A}^{k+2}(M,\mathbb{C})$, then

$$(L(\alpha),\beta)^h \;=\; (-1)^k(\alpha, *L*\beta)^h$$

**Proposition 5.6.** *The operators $L$, $Y$, $N_+$ satisfy the following properties.*

i) *$L$, $Y$, and $N_+$ are real.*

ii) *$L$ is an operator of bidegree $(-1,-1)$; i.e.*

$$L(\mathcal{A}^{p,q}(M)) \subset \mathcal{A}^{p-1,q-1}(M).$$

iii) *$Y$ is an operator of bidegree $(0,0)$ and $N_+$ has bidegree $(1,1)$.*

iv) *They satisfy the commutativity relations:*

$$[Y,L] = -2L \ ; \quad [Y,N_+] = 2N_+ \ ; \quad [N_+,L] = Y\,.$$

*Proof.* All of the listed properties, with the exception of the last commutation relation, are easy to verify. We note that the three operators are $C^\infty(U)$ linear and hence are defined pointwise. Thus, the verification of the last identity is a purely linear algebra statement. We refer to [18, Proposition 1.2.26] for the details. ☐

As noted above, the operators $L$, $Y$, $N_+$ are defined pointwise and, because of iv) in Proposition 5.6 they define an $\mathfrak{sl}_2$-triple on the exterior algebra $\Lambda^*(T_p^*(M))$. Thus, their action on $\Lambda^*(T_p^*(M))$ is described by Theorem A.15 and Proposition A.17. Although the vector space $\mathcal{A}^*(M,\mathbb{C})$ is infinite dimensional we still obtain the Lefschetz decomposition from its validity at each point $p \in M$:

**Theorem 5.7.** *Let $(M,\omega)$ be an $n$-dimensional Kähler manifold and let $\ell$ be such that $0 \le \ell \le n$. Set*

$$\mathcal{P}^\ell(M,\mathbb{C}) \;:=\; \{\alpha \in \mathcal{A}^\ell(M,\mathbb{C}) : L^{n-\ell+1}(\alpha) = 0\}. \tag{5.14}$$

*Then*

i) $\mathcal{P}^\ell(M,\mathbb{C}) \;=\; \{\alpha \in \mathcal{A}^\ell(M,\mathbb{C}) : N_+(\alpha) = 0\}.$

ii) $\mathcal{A}^\ell(M,\mathbb{C}) \;=\; \mathcal{P}^\ell(M,\mathbb{C}) \oplus L(\mathcal{A}^{\ell-2}(M,\mathbb{C})).$

The following result describe the Kähler identities which describe the commutativity relations among the differential operators $d, \partial, \bar\partial$ and the Lefschetz operators $L, Y, N_+$.

**Theorem 5.8 (Kähler identities).** *Let $(M,\omega)$ be a compact, Kähler manifold. Then the following identities hold:*

i) $[\partial, L] \;=\; [\bar\partial, L] \;=\; [\partial^*, N_+] \;=\; [\bar\partial^*, N_+] \;=\; 0.$

ii) $[\bar\partial^*, L] = i\partial \ ; \quad [\partial^*, L] = -i\bar\partial \ ; \quad [\bar\partial, N_+] = i\partial^* \ ; \quad [\partial, N_+] = -i\bar\partial^*$

*Proof.* We refer to [18, Proposition 3.1.12] for a full proof. It is clear that it suffices to prove one of the identities in each item since the others follow from conjugation or the adjoint property of these operators. We have

$$[\partial, L]\alpha = \partial(L\alpha) - L(\partial\alpha) = \partial(\omega \wedge \alpha) - \omega \wedge \partial\alpha = (\partial\omega) \wedge \alpha = 0,$$

since $0 = d\omega = \partial\omega + \bar{\partial}\omega$ which implies that each summand vanishes.

The second item is harder to prove and follows from a very clever use of the Lefschetz decomposition and the fact that $\{L, Y, N_+\}$ are an $\mathfrak{sl}_2$-triple. ☐

*Remark* 14. An alternative way of proving the second set of identities makes use of the fact hat these identities are local and only involve the coefficients of the Kähler metric up to first order. On the other hand, Theorem 3.6 asserts that a Kähler metric agrees with the standard Hermitian metric on $\mathbb{C}^n$. Thus it suffices to verify the identities in that case. This is done by a direct computation. This is the approach in [15] and [34, Proposition 6.5].

We now show how Theorem 5.1 follows from the Kähler identities: Note first of all that ii) in Theorem 5.8 yields:

$$i(\partial\bar{\partial}^* + \bar{\partial}^*\partial) \;=\; \partial[N_+,\partial] + [N_+,\partial]\partial \;=\; \partial N_+\partial - \partial N_+\partial \;=\; 0.$$

Therefore,

$$\begin{array}{rcl}
\Delta_\partial & = & \partial\partial^* + \partial^*\partial \;=\; i\,\partial[N_+,\bar{\partial}] + i\,[N_+,\bar{\partial}]\partial \\
& = & i\,(\partial N_+\bar{\partial} - \partial\bar{\partial}N_+ + N_+\bar{\partial}\partial - \bar{\partial}N_+\partial) \\
& = & i\,\big(([\partial,N_+]\bar{\partial} + N_+\partial\bar{\partial}) - \partial\bar{\partial}N_+ + N_+\bar{\partial}\partial - (\bar{\partial}[N_+,\partial] + \bar{\partial}\partial N_+)\big) \\
& = & i\,\big(N_+(\partial\bar{\partial} + \bar{\partial}\partial) + (\partial\bar{\partial} + \bar{\partial}\partial)N_+ - i\,(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})\big) \\
& = & \Delta_{\bar{\partial}}.
\end{array}$$

These two identities together give the assertion of Theorem 5.1, that is:

$$\Delta = 2\Delta_\partial = 2\Delta_{\bar{\partial}}.$$

5.2. **Lefschetz Theorems.** We will now show how the Kähler identities imply that the Laplace-Beltrami operator $\Delta$ commutes with the $\mathfrak{sl}_2$-representation and consequently, we get a (finite-dimensional) $\mathfrak{sl}_2$-representation on the space of harmonic forms $\mathcal{H}^*(M)$.

**Theorem 5.9.** *Let $(M,\omega)$ be a Kähler manifold. Then, $\Delta$ commutes with $L$, $N_+$ and $Y$.*

*Proof.* Clearly $[\Delta, L] = 0$ if and only if $[\Delta_\partial, L] = 0$. We have:

$$\begin{array}{rcl}
[\Delta_\partial, L] & = & [\partial\partial^* + \partial^*\partial, L] \\
& = & \partial\partial^*L - L\partial\partial^* + \partial^*\partial L - L\partial^*\partial \\
& = & \partial\,([\partial^*,L] + L\partial^*) - L\partial\partial^* + ([\partial^*,L] + L\partial^*)\,\partial - L\partial^*\partial \\
& = & -i\partial\bar{\partial} - i\bar{\partial}\partial \\
& = & 0
\end{array}$$

A similar argument yields that $[\Delta, N_+] = 0$. Clearly, $[\Delta, Y] = 0$. ☐

We can now define an $\mathfrak{sl}_2$-representation on the de Rham cohomology of a compact Kähler manifold:

**Theorem 5.10.** *The operators $L$, $Y$, and $N_+$ define a real representation of $\mathfrak{sl}(2,\mathbb{C})$ on the de Rham cohomology $H^*(M,\mathbb{C})$. Moreover, these operators commute with the Weil operators of the Hodge structures on the subspaces $H^k(M,\mathbb{R})$.*

*Proof.* This is a direct consequence of Theorem 5.9. The last statement follows from the fact that $L$, $Y$, and $N_+$ are of bidegree $(1,1)$, $(0,0)$ and $(-1,-1)$, respectively. ☐

**Corollary 5.11 (Hard Lefschetz Theorem).** *Let $(M,\omega)$ be an $n$-dimensional, compact Kähler manifold. For each $k \leq n$ the map*

$$L_\omega^k \colon H^{n-k}(M,\mathbb{C}) \to H^{n+k}(M,\mathbb{C}) \tag{5.15}$$

*is an isomorphism.*

*Proof.* This follows from the results in A.5. Indeed, we know from (A.34) that the weight filtration of $L_\omega$ is given by:

$$W_k \;=\; W_k(L_\omega) \;=\; \bigoplus_{j\leq k} E_j(Y).$$

Hence, we have using (5.13):

$$Gr_k^W \cong E_k(Y) \;=\; H^{n-k}(X,\mathbb{C}),$$

and the result follows from the definition of the weight filtration in Proposition A.12. ☐

We note, in particular that for $j \leq k \leq n$, the maps

$$L^j \colon H^{n-k}(M,\mathbb{C}) \to H^{n-k+2j}(M,\mathbb{C})$$

are injective. This observation together with the Hard Lefschetz Theorem imply further cohomological restrictions on a compact Kähler manifold:

**Theorem 5.12.** *The Betti and Hodge numbers of a compact Kähler manifold satisfy:*

i) $b^{n-k} = b^{n+k}$; $h^{p,q} = h^{q,p} = h^{n-q,n-p} = h^{n-p,n-q}$.
ii) $b^0 \leq b^2 \leq b^4 \leq \cdots$
iii) $b^1 \leq b^3 \leq b^5 \leq \cdots$

*In both cases the inequalities continue up to, at most, the middle degree.*

**Definition 5.13.** Let $(M,\omega)$ be a compact, $n$-dimensional Kähler manifold. For each $k = p+q \leq n$, we define the *primitive cohomology* spaces:

$$H_0^{p,q}(M) \;:=\; \ker\{L_\omega^{n-k+1} \colon H^{p,q}(M) \to H^{n-q+1,n-p+1}(M)\} \tag{5.16}$$

$$H_0^k(M) \;:=\; \bigoplus_{p+q=k} H_0^{p,q}(M). \tag{5.17}$$

We now have

**Theorem 5.14 (Lefschetz Decomposition).** *Let $(M,\omega)$ be an $n$-dimensional, compact Kähler manifold. For each $k = p+q \leq n$, we have*

$$H^{p,q}(M) \;=\; H_0^{p,q}(M) \oplus L_\omega(H^{p-1,q-1}(M)), \tag{5.18}$$

$$H^k(M,\mathbb{C}) \;=\; H_0^k(M,\mathbb{C}) \oplus L_\omega(H^{k-2}(M,\mathbb{C})). \tag{5.19}$$

5.3. **Hodge-Riemann Bilinear Relations.** The following result, whose proof may be found in [18, Proposition 1.2.31] relates the Hodge star operator with the $\mathfrak{sl}_2$-action.

**Proposition 5.15.** *Let* $\alpha \in \mathcal{P}^k(M,\mathbb{C})$*, then:*

$$*L^j(\alpha) \;=\; (-1)^{k(k+1)/2}\,\frac{j!}{(n-k-j)!}\cdot L^{n-k-j}(C(\alpha)), \tag{5.20}$$

*where* $C$ *is the Weil operator in* $\mathcal{A}^k(M,\mathbb{C})$*.*

**Definition 5.16.** Let $(M,\omega)$ be an $n$-dimensional, compact, Kähler manifold. Let $k$ be such that $0 \le k \le n$. We define a bilinear form

$$Q_k = Q\colon H^k(M,\mathbb{C}) \times H^k(M,\mathbb{C}) \to \mathbb{C},$$

$$Q_k(\alpha,\beta) \;:=\; (-1)^{k(k+1)/2}\int_M \alpha\wedge\beta\wedge\omega^{n-k}. \tag{5.21}$$

**Exercise 34.** Prove that $Q$ is well defined independent of our choice of representative in the cohomology class. This justifies our using simply $\alpha$ to denote the cohomology class $[\alpha]$.

**Theorem 5.17.** *The bilinear form* $Q$ *satisfies the following properties:*

i) $Q_k$ *is symmetric if* $k$ *is even and skew-symmetric if* $k$ *is odd.*
ii) $Q(L_\omega\alpha,\beta) + Q(\alpha, L_\omega\beta) = 0$*; we say that* $L_\omega$ *is an infinitesimal isomorphism of* $Q$*.*
iii) $Q(H^{p,q}(M), H^{p',q'}(M)) = 0$ *unless* $p' = q$ *and* $q' = p$*.*
iv) *If* $0 \neq \alpha \in H_0^{p,q}(M)$ *then*

$$Q(C\alpha,\bar{\alpha}) > 0. \tag{5.22}$$

*Proof.* The first statement is clear. For the second note that the difference between the two terms is the preceding sign which changes as we switch from $k+2$ to $k$. The third assertion follows from the fact that the integral vanishes unless the bidegree of the integrand is $(n,n)$ and, for that to happen, we must have $p' = q$ and $q' = p$.

Therefore, we only need to show the positivity condition iv). Let $\alpha \in H_0^{p,q}(M)$. It follows from Proposition 5.15 that

$$(-1)^{k(k+1)/2}\,\omega^{n-k}\wedge\bar{\alpha} = *^{-1}(n-k)!\,C(\bar{\alpha}).$$

On the other hand, on $H^k(M)$, we have $C^2 = (-1)^k\mathrm{id} = *^2$ and therefore:

$$Q(C\alpha,\bar{\alpha}) \;=\; \int_M \alpha\wedge *\bar{\alpha} \;=\; (\alpha,\alpha)^h > 0.$$

□

Properties iii) and iv) in Theorem 5.17 are called the first and second *Hodge-Riemann bilinear relations.* In view of Definition A.9 we may say that the Hodge-Riemann bilinear relations amount to the statement that the Hodge structure in the primitive cohomology $H_0^k(M,\mathbb{R})$ is polarized by the *"intersection"* form $Q$ defined by (5.21).

**Example 5.18.** Let $X = X_g$ denote a compact Riemann surface of genus $g$. Then we know that $H^1(X, \mathbb{Z}) \cong \mathbb{Z}^{2g}$. The Hodge decomposition in degree 1 is of the form:

$$H^1(X, \mathbb{C}) \ = \ H^{1,0}(X) \oplus \overline{H^{1,0}(X)},$$

where $H^{1,0}(X)$ consists of the one-forms on $X$ which, locally, are of the form $f(z)\, dz$, with $f(z)$ is holomorphic. The form $Q$ on $H^1(X, \mathbb{C})$ is alternating and given by:

$$Q(\alpha, \beta) \ = \ \int \alpha \wedge \beta.$$

The Hodge-Riemann bilinear relations then take the form: $Q(H^{1,0}(X), H^{1,0}(X)) = 0$ and, since $H_0^{1,0}(X) = H^{1,0}(X)$,

$$iQ(\alpha, \bar{a}) \ = \ i \int_X \alpha \wedge \bar{\alpha} > 0$$

if $\alpha$ is a non-zero form in $H^{1,0}(X)$. Note that, locally,

$$i\alpha \wedge \bar{\alpha} \ = \ i|f(z)|^2 dz \wedge d\bar{z} \ = \ 2|f(z)|^2 dx \wedge dy,$$

so both bilinear relations are clear in this case. We note that it follows that $H^{1,0}(X)$ defines a point in the complex manifold $D = D(H^1(X, \mathbb{R}), Q)$ defined in Example 1.16.

**Example 5.19.** Suppose now that $(M, \omega)$ is a compact, connected, Kähler surface and let us consider the Hodge structure in the middle cohomology $H^2(X, \mathbb{R})$. We have the Hodge decomposition:

$$H^2(X, \mathbb{C}) = \ H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X) \ ; \quad H^{0,2}(X) = \overline{H^{0,2}(X)}.$$

Moreover, $H_0^{2,0}(X) = H^{2,0}(X)$ while

$$H^{1,1}(X) \ = \ H_0^{1,1}(X) \oplus L_\omega H^{0,0}(X) \ = \ H_0^{1,1}(X) \oplus \mathbb{C} \cdot \omega,$$

and

$$H_0^{1,1}(X) \ = \ \{\alpha \in H^{1,1}(X) : [\omega \wedge \alpha] = 0\}.$$

The intersection form on $H^2(X, \mathbb{R})$ is given by

$$Q(\alpha, \beta) \ = \ -\int_X \alpha \wedge \beta,$$

and the second Hodge-Riemann bilinear relation become:

$$\int_X \alpha \wedge \bar{\alpha} > 0 \text{ if } \quad 0 \neq \alpha \in H^{2,0}(X),$$

$$\int_X \omega^2 > 0,$$

$$\int_X \beta \wedge \bar{\beta} < 0, \text{ if } \quad 0 \neq \beta \in H_0^{1,1}(X).$$

We note that the first two statements are easy to verify, but that is not the case with the last one. We point out that the integration form $\mathcal{I}(\alpha, \beta) = -Q(\alpha, \beta)$ has index $(+, \cdots, +, -)$ in $H^{1,1}(X) \cap H^2(X, \mathbb{R})$; i.e. $\mathcal{I}$ is a hyperbolic symmetric bilinear form. Such forms satisfy the reverse Cauchy-Schwarz inequality:

$$\mathcal{I}(\alpha, \beta)^2 \geq \mathcal{I}(\alpha, \alpha) \cdot \mathcal{I}(\beta, \beta), \tag{5.23}$$

provided that $\mathcal{I}(\alpha, \alpha) > 0$.

The inequality (5.23) is called Hodge's inequality and plays a central role in the study of algebraic surfaces. Via Poincaré duals it may be interpreted as an inequality between intersection indexes of curves in an algebraic surface or, in other words, about the number of intersection points between two curves. If the ambient surface is an *algebraic torus,* $X = \mathbb{C}^* \times \mathbb{C}^*$, then a curve zero-locus of a Laurent polynomial in two variables and a classical result of Bernstein-Kushnirenko-Khovanskii shows that, generically on the coefficients of the polynomials, the intersection indexes may be computed combinatorially from the Newton polytope of the defining polynomials (cf. Khovanskii's Appendix in [6] for a full account of this circle of ideas). This relationship between the Hodge inequality and combinatorics led Khovanskii and Teissier [29] to give (independent) proofs of the classical Alexandrov-Fenchel inequality for mixed volumes of polytopes using the Hodge inequality and set the basis for a fruitful interaction between algebraic geometry and combinatorics. In particular, motivated by problems in convex geometry, Gromov [16] stated a generalization of the Hard Lefschetz Theorem, Lefschetz decomposition and Hodge-Riemann bilinear relation to the case of *"mixed"* Kähler forms. We give a precise statement in the case of the Hard Lefschetz Theorem and refer to [31, 32, 13, 9] for further details.

A Kähler class is a real, $(1,1)$ form satisfying a positivity condition. Those forms define a cone $\mathcal{K} \subset H^{1,1}(M) \cap H^2(M, \mathbb{R})$. We have:

**Theorem 5.20** (**Mixed Hard Lefschetz Theorem**). *Let $M$ be a compact, $n$-dimensional, Kähler manifold. Let $\omega_1, \dots, \omega_k \in \mathcal{K}$, $1 \leq k \leq n$. Then the map*

$$L_{\omega_1} \cdots L_{\omega_k} \colon H^{n-k}(M, \mathbb{C}) \to H^{n+k}(M, \mathbb{C})$$

*is an isomorphism.*

As mentioned above this result was originally formulated by Gromov who proved it in the $(1,1)$ (note that the operators involved preserve the Hodge decomposition). Later, Timorin [31, 32] proved it in the linear algebra case and in the case of simplicial toric varieties. Dinh and Nguyen [13] proved it in the form stated above. In [9] the author gave a proof in the context of variations of Hodge structure which unifies those previous results as well as similar results in other contexts [19, 4].

## Appendix A. Linear Algebra

A.1. **Real and Complex Vector Spaces.** Here we will review some basic facts about finite-dimensional real and complex vector spaces that are used throughout these notes.

We begin by recalling the notion of *"complexification"* of a real vector space. Given a vector space $V$ over $\mathbb{R}$ we denote by

$$V_{\mathbb{C}} \ := \ V \otimes_{\mathbb{R}} \mathbb{C}. \tag{A.1}$$

We can formally write $v \otimes (a + ib) = av + b(v \otimes i)$, $a, b \in \mathbb{R}$, and setting $iv := v \otimes i$ we may write $V_{\mathbb{C}} = V \oplus iV$. Scalar multiplication by complex numbers is then given by:

$$(a + ib)(v_1 + iv_2) = (av_1 - bv_2) + i(av_2 + bv_1) \ ; \quad v_1, v_2 \in V; \ a, b \in \mathbb{R}.$$

Note that $\dim_{\mathbb{R}} V = \dim_{\mathbb{C}} V_{\mathbb{C}}$ and, in fact, if $\{e_1, \ldots, e_n\}$ is a basis of $V$ (over $\mathbb{R}$) then $\{e_1, \ldots, e_n\}$ is also a basis of $V_{\mathbb{C}}$ (over $\mathbb{C}$). Clearly $(\mathbb{R}^n)_{\mathbb{C}} \cong \mathbb{C}^n$.

The usual conjugation of complex numbers induces a conjugation operator on $V_{\mathbb{C}}$:

$$\sigma(v \otimes \alpha) \ := \ v \otimes \bar{\alpha} \ ; \quad v \in V, \ \alpha \in \mathbb{C},$$

or, formally, $\sigma(v_1 + iv_2) = v_1 - iv_2$, $v_1, v_2 \in V$. Clearly for $w \in V_C$, we have that $w \in V$ if and only if $\sigma(w) = w$. If there is no possibility of confusion we will write $\sigma(w) = \bar{w}$, $w \in V_{\mathbb{C}}$.

Conversely, if $W$ is a complex vector space over $\mathbb{C}$, then $W = V_{\mathbb{C}}$ for a real vector space $V$ if and only if $W$ has a conjugation $\sigma$; i.e. a map $\sigma \colon W \to W$ such that $\sigma^2$ is the identity, $\sigma$ is additive-linear, and

$$\sigma(\alpha w) \ = \ \bar{\alpha}\sigma(w) \ ; \quad w \in W; \ \alpha \in \mathbb{C}.$$

The set of fixed points $V := \{w \in W : \sigma(w) = w\}$ is a real vector space and $W = V_{\mathbb{C}}$. We call $V$ a *"real form"* of $W$.

If $V, V'$ are real vector spaces we denote by $\mathrm{Hom}_{\mathbb{R}}(V, V')$ the vector space of $\mathbb{R}$-linear maps from $V$ to $V'$. It is easy to check that

$$\big(\mathrm{Hom}_{\mathbb{R}}(V, V')\big)_{\mathbb{C}} \ \cong \mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, V'_{\mathbb{C}}) \tag{A.2}$$

and that if $\sigma$, $\sigma'$ are the conjugation operators on $V_{\mathbb{C}}$ and $V'_{\mathbb{C}}$ respectively, then the conjugation operator on $\mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, V'_{\mathbb{C}})$ is given by:

$$\sigma_{\mathrm{Hom}}(T) \ = \ \sigma' \circ T \circ \sigma \ ; \quad T \in \big(\mathrm{Hom}_{\mathbb{R}}(V, V')\big)_{\mathbb{C}}\,, \tag{A.3}$$

or in more traditional notation:

$$\bar{T}(w) \ = \ \overline{T(\bar{w})} \ ; \quad w \in V_{\mathbb{C}}. \tag{A.4}$$

Thus, the group of real automorphisms of $V$ may be viewed as the subgroup:

$$\mathrm{GL}(V) = \{T \in \mathrm{GL}(V_{\mathbb{C}}) : \sigma \circ T = T \circ \sigma\} \subset \mathrm{GL}(V_{\mathbb{C}}).$$

If we choose $V' = \mathbb{R}$, then (A.2) becomes

$$(V^*)_{\mathbb{C}} \ \cong \ (V_{\mathbb{C}})^*, \tag{A.5}$$

where, as always, $V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ and $(V_{\mathbb{C}})^* = \mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C})$ are the dual vector spaces. Thus, we may drop the parenthesis and write simply $V_{\mathbb{C}}^*$. Note that for $\alpha \in V_C^*$, its conjugate $\bar{\alpha}$ is defined by

$$\bar{\alpha}(w) = \overline{\alpha(\bar{w})}\ ; \quad w \in V_{\mathbb{C}}.$$

We may similarly extend the notion of complexification to the tensor products

$$T^{a,b}(V) \ := \ \overbrace{V \otimes \cdots \otimes V}^{a \text{ times}} \otimes \overbrace{V^* \otimes \cdots \otimes V^*}^{b \text{ times}}, \tag{A.6}$$

and to the exterior algebra: $\Lambda^r(V^*)$ and we have

$$\Big(T^{a,b}(V)\Big)_{\mathbb{C}} \ \cong \ T^{a,b}(V_{\mathbb{C}})\ ; \quad (\Lambda^r(V^*))_{\mathbb{C}} \ \cong \ \Lambda^r(V_{\mathbb{C}}^*).$$

In particular, a tensor $B \in T^{0,2}(V)$, which defines a bilinear form

$$B\colon V \times V \to \mathbb{R}$$

may be viewed as an element in $T^{0,2}(V_{\mathbb{C}})$ and defines a bilinear form

$$B\colon V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$$

satisfying $\bar{B} = B$. Explicitly, given $v_1, v_2 \in V$ we set:

$$B(iv_1, v_2) \ = \ B(v_1, iv_2) \ = \ iB(v_1, v_2)$$

and extend linearly. A bilinear form $B\colon V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$ is real if and only if

$$B(w, w') = \overline{B(\bar{w}, \bar{w}')},$$

for all $w, w' \in V_{\mathbb{C}}$. Similarly, thinking of elements $\alpha \in \Lambda^r(V^*)$ as alternating multilinear maps

$$\alpha\colon \overbrace{V \times \cdots \times V}^{r \text{ times}} \to \mathbb{R}$$

we may view them as alternating multilinear maps

$$\alpha\colon V_{\mathbb{C}} \times \cdots \times V_{\mathbb{C}} \to \mathbb{R}$$

satisfying

$$\alpha(w_1, \ldots, w_r) \ = \overline{\alpha(\bar{w}_1, \ldots, \bar{w}_r)},$$

for all $w_1, \ldots, w_r \in V_{\mathbb{C}}$.

On the other hand, given a $\mathbb{C}$-vector space $W$ we may think of it as a real vector space simply by *"forgetting"* that we are allowed to multiply by complex numbers and restricting ourselves to multiplication by real numbers (this procedure is called *"restriction of scalars"*). To remind ourselves that we are only able to multiply by real numbers we write $W^{\mathbb{R}}$ when we are thinking of $W$ as a real vector space. Note that

$$\dim_{\mathbb{R}}(W^{\mathbb{R}}) \ = \ 2\,\dim_{\mathbb{C}}(W),$$

and that if $\{e_1, \ldots, e_n\}$ is a $\mathbb{C}$-basis of $W$ then $\{e_1, \ldots, e_n, ie_1, \ldots, ie_n\}$ is a basis of $W^{\mathbb{R}}$.

It is now natural to ask when a real vector space $V$ is obtained from a complex vector space $W$ by restriction of scalars. Clearly, a necessary condition is that $\dim_{\mathbb{R}}(V)$ be even. But, there is additional structure on $V = W^{\mathbb{R}}$ coming from the fact that $W$ is a $\mathbb{C}$-vector space. Indeed, multiplication by $i$ in $W$ induces an $\mathbb{R}$-linear map:

$$J\colon W^{\mathbb{R}} \to W^{\mathbb{R}}\ ; \quad J(w) \ := \ i\,w, \tag{A.7}$$

satisfying $J^2 = -I$, where $I$ denotes the identity map.

Conversely, let $V$ be a $2n$-dimensional real vector space and $J\colon V \to V$ a linear map such that $J^2 = -I$. Then we may define a $\mathbb{C}$-vector space structure on $V$ by:

$$(a+ib) * v \ := \ a\,v + b\,J(v). \tag{A.8}$$

We say that $J$ is a *complex structure* on $V$ and we will often denote by $(V, J)$ the complex vector space consisting of the points in $V$ endowed with the scalar multiplication[†] (A.8).

**Exercise 35.** Let $V$ be a real vector space and $J\colon V \to V$ a linear map such that $J^2 = -I$. Prove that there exists a basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ of $V$ such that the matrix of $J$ in this basis is of the form:

$$J \ = \ \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \tag{A.9}$$

where $I_n$ denotes the $(n \times n)$-identity matrix.

**Proposition A.1.** *Let $V$ be a real vector space. Then the following are equivalent:*

i) *$V$ has a complex structure $J$.*

ii) *The complexification $V_{\mathbb{C}}$ admits a decomposition*

$$V_{\mathbb{C}} \ = \ W_+ \oplus W_- \tag{A.10}$$

*where $W_\pm \subset V_{\mathbb{C}}$ are complex subspaces such that $\overline{W_\pm} = W_\mp$.*

*Proof.* Suppose $J\colon V \to V$ is a linear map such that $J^2 = -I$. Then we may extend $J$ to a map $J\colon V_{\mathbb{C}} \to V_{\mathbb{C}}$. Now, endomorphisms of complex vector spaces are diagonalizable and since $J^2 = -I$, the only possible eigenvalues for $J$ are $\pm i$. Let $W_\pm$ denote the $\pm i$-eigenspace of $J$. We then have:

$$V_{\mathbb{C}} \ = W_+ \oplus W_- \,.$$

Suppose now that $w \in W_\pm$, then since $J$ is a real map we have

$$J(\bar{w}) \ = \ \overline{J(w)} \ = \ \overline{\pm i\,w} \ = \ \mp i\,w.$$

Hence $W_\pm = \overline{W_\mp}$ and we obtain the decomposition (A.10).

Conversely, given the decomposition (A.10) we define a linear map $J\colon V_{\mathbb{C}} \to V_{\mathbb{C}}$ by the requirement that $J(w) = \pm iw$ if $w \in W_\pm$. It is easy to check that $J^2 = -I$ and that the assumption that $\overline{W_\pm} = W_\mp$ implies that $J$ is a real map; i.e. $\bar{J} = J$. ☐

**Proposition A.2.** *Let $V$ be a real vector space with a complex structure $J$. Then, the map $\phi : (V, J) \to W_+$ defined by $\phi(v) = v - iJ(v)$ is an isomorphism of complex vector spaces.*

---

[†] We will use $*$ to denote complex multiplication in $(V, J)$ to distinguish from the notation $\lambda v$, $\lambda \in \mathbb{C}$ which is traditionally used to represent the point $(v \otimes \lambda) \in V_{\mathbb{C}}$. We will most often identify $(V, J)$ with a complex subspace of $V_{\mathbb{C}}$ as in Proposition A.1 and therefore there will be no chance of confusion.

*Proof.* We verify first of all that $\phi(v) \in W_+$; that is $J(\phi(v)) = i\phi(v)$:

$$J(\phi(v)) \;=\; J(v - iJ(v)) \;=\; J(v) - iJ^2(v) \;=\; J(v) + iv \;=\; i(v - iJ(v)) \;=\; i\phi(v).$$

Next we check that the map is $\mathbb{C}$-linear. Let $a, b \in \mathbb{R}$, $v \in V$:

$$\begin{aligned}\phi((a+ib) * v) &= \phi(av + bJ(v)) \;=\; (av + bJ(v)) - iJ(av + bJ(v)) \\ &= (av + bJ(v)) + i(bv - aJ(v)) \;=\; (a+ib)(v - iJ(v)) \\ &= (a+ib)\phi(v).\end{aligned}$$

We leave it to the reader to verify that if $w \in W_+$ then $w = \phi(\frac{1}{2}(w + \bar{w}))$ and, therefore, $\phi$ is an isomorphism. □

Suppose now that $(V, J)$ is a $2n$-dimensional real vector space with a complex structure $J$ and let $T \in \mathrm{GL}(V)$. Then $T$ is a complex linear map if and only if $T(iv) = iT(v)$, i.e. if and only if:

$$T \circ J \;=\; J \circ T. \tag{A.11}$$

**Exercise 36.** Let $V$ be a real vector space and $J\colon V \to V$ a complex structure on $V$. Prove an $\mathbb{R}$-linear map $T\colon V \to V$ is $\mathbb{C}$-linear if and only if the matrix of $T$, written in terms of a basis as in Exercise 35, is of the form:

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix}, \tag{A.12}$$

where $A, B$ are $(n \times n)$-real matrices.

If $T \in \mathrm{GL}(V)$ satisfies (A.11) then the extension of $T$ to the complexification $V_{\mathbb{C}}$ and continues to satisfy the commutation relation (A.11). In particular, such a map $T$ must preserve the eigenspaces of $J\colon V_{\mathbb{C}} \to V_{\mathbb{C}}$. Now, if $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ is a a basis of $V$ as in Exercise 35, then $w_i = \frac{1}{2}(e_i - iJe_i) = \frac{1}{2}(e_i - if_i)$, $i = 1, \dots, n$, are a basis of $W_+$ and the conjugates $\bar{w}_i = \frac{1}{2}(e_i + iJe_i) = \frac{1}{2}(e_i - if_i)$ $i = 1, \dots, n$, are a basis of $W_-$. In this basis, the extension of $T$ to $V_{\mathbb{C}}$ is written as:

$$\begin{pmatrix} A + iB & 0 \\ 0 & A - iB \end{pmatrix}. \tag{A.13}$$

We note in particular that if $T \in \mathrm{GL}(V)$ satisfies (A.11) then $\det(T) > 0$. Indeed, the determinant is unchanged after complexification and in terms of the basis $\{w_1, \dots, w_n, \bar{w}_1, \dots, \bar{w}_n\}$ the matrix of $T$ is as in (A.13) and we have

$$\det(T) = |\det(A + iB)|^2. \tag{A.14}$$

If $J$ is a complex structure on the real vector space $V$ then the dual map $J^*\colon V^* \to V^*$ is a complex structure on the dual space $V^*$. The corresponding decomposition (A.10) on the complexification $V^*_{\mathbb{C}}$ is given by

$$V^*_{\mathbb{C}} \;=\; W_+^{\perp} \oplus W_-^{\perp}\,, \tag{A.15}$$

where $W_\pm^{\perp} := \{\alpha \in V_C^* : \alpha|_{W_\pm} = 0\}$. Indeed, if $\alpha \in V^*_{\mathbb{C}}$ is such that $J^*(\alpha) = i\alpha$ and $w \in W_-$ then we have:

$$i\alpha(w) = (J^*(\alpha))(w) = \alpha(J(w)) = \alpha(-iw) = -i\alpha(w)$$

which implies that $\alpha(w) = 0$. The statement now follows from dimensional reasons.

A.2. **Hodge structures.** The decomposition of the complexification of a real vector space defined by a complex structure is a simple but important example of a Hodge structure.

**Definition A.3.** A (real) Hodge structure of weight $k \in \mathbb{Z}$ consists of:

i) A finite-dimensional real vector space $V$.
ii) A decomposition of the complexification $V_{\mathbb{C}}$ as:

$$V_{\mathbb{C}} = \bigoplus_{p+q=k} V^{p,q} \; ; \quad V^{q,p} = \overline{V^{p,q}}. \tag{A.16}$$

We say that the Hodge structure is rational (resp. integral) if there exists a rational vector space $V_{\mathbb{Q}}$ (resp. a lattice $V_{\mathbb{Z}}$) such that $V = V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$ (resp. $V = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$).

The following statement is valid for Hodge structures defined over $\mathbb{R}$, $\mathbb{Q}$, or $\mathbb{Z}$.

**Proposition A.4.** *Let $V$ and $W$ be vector spaces with Hodge structures of weight $k$, $\ell$ respectively. Then* $\mathrm{Hom}(V, W)$ *has a Hodge structure of weight $\ell - k$.*

*Proof.* We set

$$\mathrm{Hom}(V, W)^{a,b} := \{X \in \mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, W_{\mathbb{C}}) : X(V^{p,q}) \subset W^{p+a,q+b}\}. \tag{A.17}$$

Then $\mathrm{Hom}(V, W)^{a,b} = 0$ unless $p + a + q + b = \ell$, that is unless $a + b = \ell - k$. The rest of the verifications are left to the reader. □

In particular, if we choose $W = \mathbb{R}$ with the Hodge structure $W_{\mathbb{C}} = W^{0,0}$, we see that if $V$ has a Hodge structure of weight $k$, then $V^*$ has a Hodge structure of weight $-k$. Similarly, if $V$ and $W$ have Hodge structures of weight $k$, $\ell$ respectively, then $V \otimes_{\mathbb{R}} W \cong \mathrm{Hom}_{\mathbb{R}}(V^*, W)$ has a Hodge structure of weight $k + \ell$ and

$$(V \otimes W)^{a,b} \; = \bigoplus_{\substack{p+r=a \\ q+s=b}} V^{p,q} \otimes_{\mathbb{C}} W^{r,s}. \tag{A.18}$$

Needless to say, we could take (A.18) as our starting point rather than (A.17). Note that if $V$ has a Hodge structure of weight $k$ then the tensor product $T^{a,b}(V)$ defined in (A.6) has a Hodge structure of weight $k(a - b)$.

**Example A.5.** Let $V$ be a real vector space with a complex structure $J$ and consider the complex structure $J^*$ on the dual vector space $V^*$. We may define a Hodge structure of weight 1 on $V^*$ by setting $(V^*)^{1,0} = W_+^{\perp}$ and $(V^*)^{0,1} = W_-^{\perp}$ in the decomposition (A.15). The exterior algebra $\Lambda^k(V^*)$ now inherits a Hodge structure of weight $k$, where for $p + q = k$:

$$\Lambda^{p,q}(V^*) = (\Lambda^k(V^*_{\mathbb{C}}))^{p,q} = \overbrace{(V^*)^{1,0} \wedge \ldots \wedge (V^*)^{1,0}}^{p \text{ times}} \wedge \overbrace{(V^*)^{0,1} \wedge \ldots \wedge (V^*)^{0,1}}^{q \text{ times}}. \tag{A.19}$$

There are two alternative ways of describing a Hodge structure on a vector space $V$ that will be very useful to us.

**Definition A.6.** A real (resp. rational, integral) Hodge structure of weight $k \in \mathbb{Z}$ consists of a real vector space $V$ (resp. a rational vector space $V_{\mathbb{Q}}$, a lattice $V_{\mathbb{Z}}$) and a decreasing filtration

$$\cdots F^p \supset F^{p-1} \cdots$$

of the complex vector space $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ (resp. $V_{\mathbb{C}} = V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$, $V_{\mathbb{C}} = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$) such that:

$$V_{\mathbb{C}} \ = \ F^p \oplus \overline{F^{k-p+1}}\,. \tag{A.20}$$

The equivalence of Definitions A.3 and A.6 is easy to verify. Indeed given a decomposition as in (A.16) we set:

$$F^p \ = \ \bigoplus_{a\geq p} H^{a,k-a}\,,$$

while given a filtration of $V_{\mathbb{C}}$ satisfying (A.20) the subspaces

$$H^{p,q} = F^p \cap \overline{F^q}\ ; \quad p+q=k.$$

define a decomposition of $V_{\mathbb{C}}$ satisfying (A.16).

In order to state the third definition of a Hodge structure, we need to recall some basic notions from representation theory. Let us denote by $\mathbb{S}(\mathbb{R})$ the real algebraic group:

$$\mathbb{S}(\mathbb{R}) \ := \ \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R}) \right\}.$$

Then $\mathbb{C}^* \cong \mathbb{S}(\mathbb{R})$ via the identification

$$z = a+ib \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

The circle group $S^1 = \{z \in C^* : |z| = 1\}$ is then identified with the group of rotations:

$$\left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \ ; \ \theta \in \mathbb{R} \right\}.$$

Recall that a representation of an algebraic group $G$ defined over the field $F = \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$, on a $F$-vector space $V_F$ is a group homomorphism $\varphi\colon G \to \mathrm{GL}(V_F)$.

Now, if $V$ is a real vector space with a Hodge structure of weight $k$, then we may define a representation of $\mathbb{S}(\mathbb{R})$ on $V_{\mathbb{C}}$ by:

$$\varphi(z)(v) \ := \ \sum_{p+q=k} z^p\,\bar{z}^q\,v_{p,q}\,,$$

where $v = \sum v_{p,q}$ is the decomposition of $v$ according to (A.16). We verify that $\varphi(z) \in \mathrm{GL}(V)$, i.e. that $\overline{\varphi(z)} = \varphi(z)$:

$$\overline{\varphi(z)}(v) \ = \ \overline{\varphi(z)(\bar{v})} \ = \ \overline{\sum_{p+q=k} z^p\,\bar{z}^q\,\overline{v_{q,p}}}$$

since $(\bar{v})_{p,q} = \overline{v_{q,p}}$. Hence $\overline{\varphi(z)}(v) = \varphi(z)(v)$. We note also that for $\lambda \in \mathbb{R}^* \subset \mathbb{C}^*$, $\varphi(\lambda)(v) = \lambda^k\,v$ for all $v \in V$.

Conversely, it follows from the representation theory of $\mathbb{S}(\mathbb{R})$ that every every finite dimensional representation of $\mathbb{S}(\mathbb{R})$ on a complex vector space splits as a direct sum of one-dimensional representations where $z \in \mathbb{S}(\mathbb{R})$ acts as multiplication by $z^p\bar{z}^q$, with $p,q \in \mathbb{Z}$. Hence, a representation $\varphi\colon \mathbb{S}(\mathbb{R}) \to \mathrm{GL}(V_{\mathbb{C}})$ defined over $\mathbb{R}$ (i.e. $\bar{\varphi} = \varphi$) decomposes $V_{\mathbb{C}}$ into subspaces $V^{p,q}$

$$V_{\mathbb{C}} \ = \ \bigoplus V^{p,q}\ ; \quad V^{q,p} \ = \ \overline{V^{p,q}}, \tag{A.21}$$

where $\varphi(z)$ acts as multiplication by $z^p\bar{z}^q$. Note, moreover, that if $\lambda \in \mathbb{R}^* \subset \mathbb{S}(\mathbb{R})$ then $\varphi(\lambda)$ acts on $V^{p,q}$ as multiplication by $\lambda^{p+q}$. Thus, the following definition is equivalent to Definitions A.3 and A.6.

**Definition A.7.** A real Hodge structure of weight $k \in \mathbb{Z}$ consists of a real vector space $V$ and a representation $\varphi\colon \mathbb{S}(\mathbb{R}) \to \mathrm{GL}(V)$ such that $\varphi(\lambda)(v) = \lambda^k v$ for all $v \in V$ and all $\lambda \in \mathbb{R}^* \subset \mathbb{S}(\mathbb{R})$.

Given a Hodge structure $\varphi$ of weight $k$ on $V$, the linear operator $\varphi(i)\colon V_{\mathbb{C}} \to V_{\mathbb{C}}$ is called the *Weil operator* and denoted by $C$. Note that on $V^{p,q}$, the Weil operator acts as multiplication by $i^{p-q}$ and, consequently, if $J$ is a complex structure on $V$ then for the Hodge structure of weight $k$ defined on the exterior product $\Lambda^k(V^*)$, the Weil operator agrees with the natural extension of $J^*$ to $\Lambda^k(V^*)$.

**Exercise 37.** Prove that if $(V,\varphi)$, $(V',\varphi')$ are Hodge structures of weight $k$ and $k'$ respectively, then $(V^*,\varphi^*)$ and $(V\otimes V', \varphi\otimes\varphi')$ are the natural Hodge structures on $V^*$ and $V\otimes V'$ defined above. Here $\varphi^*\colon \mathbb{S}(\mathbb{R}) \to \mathrm{GL}(V^*)$ is the representation $\varphi^*(z) := (\varphi(z))^*$ and, similarly, $(\varphi\otimes\varphi')(z) := \varphi(z)\otimes\varphi'(z)$.

A.3. **Symmetric and Hermitian Forms.** If $V$ is a real vector space with a complex structure $J$ and $B\colon V\times V \to \mathbb{R}$ is a bilinear form on $V$ then we say that $B$ is *compatible* with $J$ if and only if

$$B(Ju,Jv) = B(u,v) \quad \text{for all } u,v \in V. \tag{A.22}$$

We shall also denote by $B$ the bilinear extension of $B$ to the complexification $V_{\mathbb{C}}$. If $B$ is symmetric then the bilinear form on $V_{\mathbb{C}}$:

$$\omega(u,v) \;:=\; B(Ju,v) \tag{A.23}$$

is alternating; i.e. $\omega \in \Lambda^2(V_{\mathbb{C}}^*)$. Indeed:

$$\omega(u,v) = B(Ju,v) = B(J^2u,Jv) = -B(u,Jv) = -\omega(v,u).$$

We note that since $B$ is real then so is $\omega$ and that, in fact,

$$\omega \in \Lambda^{1,1}(V^*) \cap \Lambda^2(V_{\mathbb{R}}). \tag{A.24}$$

This follows from considering from the fact that the Weil operator on the Hodge structure of $\Lambda^2(V^*)$ agrees with the operator defined by the complex structure $J$:

$$(C\omega)(u,v) = \omega(Ju,Jv) = B(J^2u,Jv) = -B(u,Jv) = -\omega(v,u) = \omega(u,v).$$

Conversely, given an element $\omega \in \Lambda^{1,1}(V^*) \cap \Lambda^2(V_{\mathbb{R}})$, we have $\omega(Ju,Jv) = \omega(u,v)$ and we may define a bilinear symmetric form compatible with J by

$$B(u,v) = \omega(u,Jv). \tag{A.25}$$

We shall also be interested in Hermitian forms on a $\mathbb{C}$-vector space $W$:

$$H\colon W\times W \to \mathbb{C}\,.$$

Recall that for such a form,

$$H(\lambda u + \mu u', v) = \lambda H(u,v) + \mu H(u',v)\ ; \quad u,u',v \in W\ ;\ \lambda,\mu \in \mathbb{C}, \text{ and}$$

$$H(v,u) = \overline{H(u,v)}.$$

Now, if $H$ is a Hermitian form and we write:

$$H(u,v) = S(u,v) - iA(u,v)\ ; \quad S(u,v), A(u,v) \in \mathbb{R}.$$

it is clear that $S$ is a symmetric form and $A$ is an alternating form.

Given a real vector space with a complex structure $J$ and a compatible symmetric bilinear form $B\colon V \times V \to \mathbb{R}$, we may define a Hermitian form on the complex vector space $(V, J)$ by:

$$H(u,v) = B(u,v) - i\omega(u,v), \tag{A.26}$$

where $\omega$ is as in (A.23). Indeed, we only need to verify that $H$ is $\mathbb{C}$-linear on the first argument but we have:

$$H(iu,v) = H(Ju,v) = B(Ju,v) - i\omega(Ju,v) = \omega(u,v) + iB(u,v) = iH(u,v).$$

We collect these observations in the following:

**Theorem A.8.** *Let $V$ be a real vector space with a complex structure $J$ then the following data are equivalent:*

i) *A symmetric bilinear form $B$ on $V$ compatible with $J$.*
ii) *An element $\omega \in \Lambda^{1,1}(V_{\mathbb{C}}^*) \cap \Lambda^2(V^*)$.*
iii) *A Hermitian form $H$ on the complex vector space $(V, J)$*

*Moreover, $H$ is positive definite if and only if $B$ is positive definite.*

### A.4. **Polarized Hodge Structures.**

**Definition A.9.** Let $(V, \varphi)$ be a real Hodge structure of weight $k$. A polarization of $(V, \varphi)$ is a real[†] bilinear form $Q\colon V \times V \to \mathbb{R}$ such that

i) $Q(u,v) = (-1)^k Q(v,u)$; i.e., $Q$ is symmetric or skew-symmetric depending on whether $k$ is even or odd.
ii) The Hodge decomposition is orthogonal relative to the Hermitian form $H\colon V_C \times V_{\mathbb{C}} \to \mathbb{C}$ defined by

$$H(w_1, w_2) \ := \ Q(C\, w_1, \bar{w}_2), \tag{A.27}$$

where $C = \varphi(i)$ is the Weil operator.
iii) $H$ is positive definite.

*Remark* 15. Note that if $k$ is even then the Weil operator acts on $V^{p,q}$ as multiplication by ±1 and then it is clear that the form $H$ defined by (A.27) is Hermitian. Similarly if $k$ is odd since in this case $C$ acts on $V^{p,q}$ as multiplication by $\pm i$. We also note that (ii) and (iii) above may be restated as follows:

ii') $Q(V^{p,q}, V^{p',q'}) = 0$ if $p' \neq k - p$.
iii') $i^{p-q} Q(w, \bar{w}) > 0$ for all $0 \neq w \in V^{p,q}$.

The statements ii') and iii') correspond to the *Hodge-Riemann bilinear relations* in Theorem 5.17.

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[†] If $(V, \varphi)$ is a rational (resp. integral) Hodge structure then we require $Q$ to be defined over $\mathbb{Q}$ (resp. over $\mathbb{Z}$).

**Example A.10.** Let $(V, J)$ be a real vector space with a complex structure and let $B$ be a positive definite symmetric bilinear form on $V$ compatible with $J$. Let $V_{\mathbb{C}} = W_+ \oplus W_-$ be the induced decomposition and consider the Hodge structure of weight one on $V$ defined by $V^{1,0} = W_+$. The form $Q(u, v) = B(u, Jv)$ is alternating and for $w = u - iJu, w' = u' - iJu' \in W_+$ we have:

$$Q(u - iJu, u' - iJu') = Q(u, u') - Q(Ju, Ju') - iQ(u, Ju') - iQ(Ju, u') = 0.$$

Similarly, for $0 \neq u \in V$:

$$\begin{aligned} iQ(u - iJu, u + iJu) &= i\left(Q(u, u) + Q(Ju, Ju) + iQ(u, Ju) - iQ(Ju, u)\right) \\ &= -2Q(u, Ju) \;=\; 2B(u, u) > 0\,. \end{aligned}$$

Hence $Q$ polarizes the Hodge structure defined by $J$.

A.5. **The Weight filtration of a nilpotent transformation.** In this section we will construct a filtration canonically attached to a nilpotent linear transformation and study its relationship with representations of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$.

Throughout this section $N\colon V \to V$ will be a nilpotent linear transformation of *nilpotency index* $k$; i.e. $k$ the first positive integer such that $N^{k+1} = 0$. Given an integer $\ell \leq k$, we will say that a subspace $A_\ell \subset V$ is a *Jordan block* of weight $m$ if $A_\ell$ has a basis $\{f_0^A, f_1^A, \ldots, f_m^A\}$ such that $N(f_i^A) = f_{i+1}^A$, where we set $f_{m+1}^A = 0$. It is convenient to reindex the basis as

$$e_{m-2j}^A \;:=\; f_j^A.$$

Note that if $m$ is even then the index of $e^A$ takes even values from $m$ to $-m$, while if $m$ is odd then it takes odd values from $m$ to $-m$.

It is known from the study of the Jordan Normal Form for a nilpotent transformation $N\colon V \to V$ of a nilpotency index $k$, that we may decompose $V$ as a direct sum:

$$V = \bigoplus_{m=0}^{k} U_m\,,$$

where $U_m$ is the direct sum of all Jordan blocks of weight $m$. In fact, these subspaces $U_m$ are unique. Clearly, each $U_m$ decomposes further as

$$U_m = \bigoplus_{j=0}^{m} U_{m,m-2j}\,,$$

where $U_{m,m-2j}$ is the subspace spanned by all basis vectors $e_{m-2j}^A$ as $A$ runs over all Jordan blocks of weight $m$. We now define

$$E_\ell = E_\ell(N) = \bigoplus_{m=0}^{k} U_{m,\ell} \tag{A.28}$$

**Theorem A.11.** *The decomposition* (A.28) *satisfies:*

i) $N(E_\ell) \subset E_{\ell-2}$.

ii) *For* $\ell \geq 0$, $N^\ell\colon E_\ell \to E_{-\ell}$ *is an isomorphism.*

*Proof.* The first statement is clear since for any Jordan block $A$, $N(f_j^A) = f_{j+1}^A$ which implies the assertion. Suppose now that $\ell$ is even. Then, $E_\ell$ is spanned by all basis vectors of the form $e_\ell^A = f_{(m-\ell)/2}^A$ where $A$ runs over all Jordan blocks of even weight $m \geq \ell$. □

**Proposition A.12.** *Let $N$ be a nilpotent transformation with nilpotency index $k$. Then there exists a unique increasing filtration $W = W(N)$:*

$$\{0\} \subset W_k \subset W_{-k+1} \subset \cdots \subset W_{k-1} \subset W_k = V\,, \tag{A.29}$$

*with the following properties:*

- i) $N(W_\ell) \subset W_{\ell-2}$,
- ii) *For* $\ell \geq 0 : N^\ell \colon \mathrm{Gr}_\ell^W \to \mathrm{Gr}_{-\ell}^W$, *where* $\mathrm{Gr}_\ell^W := W_\ell / W_\ell$, *is an isomorphism.*

*Moreover, the subspaces $W_\ell$ may be expressed in terms of* $\ker(N^a)$, $\mathrm{Im}(N^b)$ *and hence are defined over $\mathbb{Q}$ (resp. over $\mathbb{R}$) if $N$ is.*

*Proof.* The existence of $W(N)$ follows from Theorem A.11 while the uniqueness is a consequence of the uniqueness properties of the Jordan decomposition. Alternatively one may give an inductive construction of $W_\ell(N)$ as in [27, Lemma 6.4]. For an explicit construction involving kernels and images of powers of $N$ we refer to [28]. □

**Example A.13.** Suppose $k = 1$, then the weight filtration is of the form:

$$\{0\} \subset W_{-1} \subset W_0 \subset W_1 = V.$$

Since $N \colon V/W_0 \to W_{-1}$ is an isomorphism it follows that

$$W_{-1}(N) = \mathrm{Im}(N)\ ; \quad W_0(N) = \ker(N)\,.$$

**Exercise 38.** Prove that if $k = 2$ the weight filtration:

$$\{0\} \subset W_{-2} \subset W_{-1} \subset W_0 \subset W_1 \subset W_2 = V$$

is given by:

$$\{0\} \subset \mathrm{Im}(N^2) \subset \mathrm{Im}(N) \cap \ker(N) \subset \mathrm{Im}(N) + \ker(N) \subset \ker(N^2) \subset V$$

A.6. **Representations of $\mathfrak{sl}(2,\mathbb{C})$.** We recall that the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ consists of all $2 \times 2$-complex matrices of trace zero. It has a basis consisting of

$$\mathbf{n}_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ ; \quad \mathbf{n}_- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ ; \quad \mathbf{y} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.30}$$

This basis satisfies the commutativity relations:

$$[\mathbf{y}, \mathbf{n}_+] = 2\mathbf{n}_+ \ ; \quad [\mathbf{y}, \mathbf{n}_-] = -2\mathbf{n}_- \ ; \quad [\mathbf{n}_+, \mathbf{n}_-] = \mathbf{y} \ ; \tag{A.31}$$

A *representation* $\rho$ of $\mathfrak{sl}(2,\mathbb{C})$ on a complex vector space $V_\mathbb{C}$ is a Lie algebra homomorphism

$$\rho \colon \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{gl}(V_\mathbb{C}).$$

We denote the image of the generators of $\mathfrak{sl}(2,\mathbb{C})$ by $N_+, N_-$ and $Y$. These elements satisfy commutativity relations analogous to (A.31). Conversely, given elements $\{N_+, N_-, Y\} \subset \mathfrak{gl}(V)$ satisfying the commutation relations from (A.31) we can define a representation $\rho \colon \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{gl}(V_\mathbb{C})$. We will refer to $\{N_+, N_-, Y\}$ as an $\mathfrak{sl}_2$-triple.

A representation is called *irreducible* if $V$ has no proper subspaces invariant under $\rho(\mathfrak{sl}(2,\mathbb{C}))$. We say that $\rho$ is real if $V_{\mathbb{C}}$ is the complexification of a real vector space $V$ and $\rho(\mathfrak{sl}(2,\mathbb{R})) \subset \mathfrak{gl}(V)$.

**Example A.14.** For each integer $n$ we may define an irreducible representation $\rho_n$ on $V_{\mathbb{C}} = \mathbb{C}^n$ as follows. Suppose that $n = 2k+1$, then we label the standard basis of $V$ as: $\{e_{-2k}, e_{-2k+2}, \dots, e_0, \dots e_{2k-2}, e_{2k}\}$ and define:

$$Y(e_{2j}) \ := \ 2j \cdot e_{2j} \ ; \quad N_-(e_{2j}) = e_{2j-2} \ ; \quad N_+(e_{2j}) = \mu_{2j} \cdot e_{2j+2}, \tag{A.32}$$

where the integers $\mu_{2j}$ are the unique solution to the recursion equations:

$$\mu_{2j-2} - \mu_{2j} = 2j \ ; \quad \mu_{-2k-2} = 0. \tag{A.33}$$

It is easy to check that the first two commutation relations are satisfied. On the other hand,

$$[N_+, N_-](e_{2j}) = N_+(e_{2j-2}) - N_-(\mu_{2j} \cdot e_{2j-2}) = (\mu_{2j-2} - \mu_{2j}) \cdot e_{2j} = 2j \cdot e_{2j} = Y(e_{2j}).$$

**Exercise 39.** Find the solution to the equations (A.33).

**Exercise 40.** Extend the construction of the representation in Example A.14 to the case $n = 2k$.

We note that for the representation $\rho_n$ we have $N_-^n = 0$ and $N_-^{n-1} \neq 0$; that is, the index of nilpotency of $N_-$ (and of $N_+$) is $n-1$. At the same time, the eigenvalues of $Y$ range from $-n+1$ to $n-1$. We will refer to $n-1$ as the *weight* of the representation $\rho_n$. This notion of weight is consistent with that defined for Jordan blocks above.

The basic structure theorem about representations of $\mathfrak{sl}(2,\mathbb{C})$ is the following

**Theorem A.15.** *Every finite dimensional representation of* $\mathfrak{sl}(2,\mathbb{C})$ *splits as a direct sum of irreducible representations. Moreover, an irreducible representation of dimension* $n$ *is isomorphic to* $\rho_n$.

*Proof.* We refer to [39, Chapter V, Section 3] for a proof. ☐

Suppose now that $\rho\colon \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{gl}(V)$ is a representation and set $N = N_-$. Let $k$ be the unipotency index of $N$ then $\rho$ splits as a direct sum of irreducible representations of weight at most $k$ and we have:

$$W_\ell(N) \ = \ \bigoplus_{j \leq \ell} E_j(Y). \tag{A.34}$$

Indeed, it is enough to check this statement for each irreducible representation and there the statement is clear.

**Exercise 41.** Let $N \in \mathfrak{gl}(V)$ be nilpotent and $Y \in \mathfrak{gl}(V)$ be semisimple. Then the following are equivalent.

i) There exists an $\mathfrak{sl}_2$-triple $\{N_+, N_-, Y\}$ with $N_- = N$.
ii) $[Y, N] = -2N$ and the weight filtration of $N$ is given by (A.34).

The above exercise implies that if $N$ is a nilpotent element, $W_\ell$ its weight filtration and we $\{V_\ell\}$ is a *splitting* of the filtration $W$ in the sense that:

$$W_\ell \;=\; V_\ell \oplus W_{\ell-1}, \tag{A.35}$$

then if we define $Y \in \mathfrak{gl}(V)$ by $Y(v) = \ell v$ if $v \in V_\ell$, the pair $\{N, Y\}$ may be extended to an $\mathfrak{sl}_2$-triple. Moreover if $N$ is defined over $\mathbb{Q}$ (resp. over $\mathbb{R}$) and the splitting is defined over $\mathbb{Q}$ (resp. over $\mathbb{R}$) so is the $\mathfrak{sl}_2$-triple.

**Exercise 42.** Apply Exercise 41 to prove that if $N$ is a nilpotent transformation, then there exists an $\mathfrak{sl}_2$-triple with $N = N_-$. This is a version of the Jacobson-Morosov Theorem.

A.7. **Lefschetz decomposition.** Let $N$ be a nilpotent transformation with nilpotency index $k$. Then, for any $\ell$, with $0 \le \ell \le k$ we have

$$N^\ell \colon Gr^W_\ell \to Gr^W_{-\ell}$$

is an isomorphism. We define

$$\mathcal{P}_\ell \;:=\; \ker\{N^{\ell+1} \colon Gr^W_\ell \to Gr^W_{-\ell-2}\} \tag{A.36}$$

and call it the $\ell$-th *primitive* space.

**Example A.16.** For any $k$ we have $\mathcal{P}_k = Gr^W_k$ since $N^{k+1} = 0$. Suppose $k = 1$ then, $\mathcal{P}_0 = Gr^W_0 = \ker(N)/\mathrm{Im}(N)$.

**Exercise 43.** Let $k = 2$. Prove that $\mathcal{P}_1 = Gr^W_1$ but that

$$\mathcal{P}_0 \;=\; \ker(N)/(\ker(N) \cap \mathrm{Im}(N)) \subset Gr^N_0.$$

**Proposition A.17.** *Let $N \in \mathfrak{gl}(V)$ be a nilpotent transformation with nilpotency index $k$. Then for any $\mathfrak{sl}_2$-triple with $N_- = N$ we have*

$$\mathcal{P}_\ell \;:=\; \ker\{N_+ \colon Gr^W_\ell \to Gr^W_{\ell+2}\}.$$

*Moreover, for every $\ell$, $0 \le \ell \le k$, we have:*

$$Gr^W_\ell \;=\; \mathcal{P}_\ell \oplus N(Gr^W_{\ell+2}). \tag{A.37}$$

*Proof.* Let $\{N_+, N_-, Y\}$ with $N_- = N$ be an $\mathfrak{sl}_2$-triple with $N_- = N$. Then $\mathcal{P}_\ell$ is given by all eigenvectors of $Y$ of eigenvalue $\ell$ living in the sum of irreducible components of the representation of weight $\ell$ and this is are exactly the elements by $N_+$. Similarly, it suffices to verify the decomposition (A.37) in each irreducible component which is easy to do. □

The decomposition (A.37), or more precisely, the decomposition obtained from (A.37) inductively:

$$Gr^W_\ell \;=\; \mathcal{P}_\ell \oplus N(\mathcal{P}_{\ell+2}) \oplus N^2(\mathcal{P}_{\ell+4}) + \cdots \tag{A.38}$$

is called the *Lefschetz decomposition*.

**Example A.18.** The only interesting term in the Lefschetz decomposition for $k = 2$ occurs for $\ell = 0$, where, according to Exercises 38 and 43 we get:

$$Gr^W_0 \;=\; \mathcal{P}_0 \oplus N(\mathcal{P}_2) \;=\; \ker(N)/(\ker(N) \cap \mathrm{Im}(N)) \oplus N(V/\ker(N^2)).$$

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