Dominic Joyce **Differential Geometry** **Nairobi 2019**

Miniprojects

Project 1. Define connections ∇ on a vector bundle $E \to X$, and the curvature of ∇. Explain why a Riemannian manifold (X, g) has a natural connection ∇ on TX, the Levi-Civita connection. Discuss Riemann curvature. Give some idea of reasons why it is important, e.g. General Relativity.

Project 2. Give an introduction to the theory of Lie groups and Lie algebras. One good book (there are several) is R. Carter, G. Segal and I. MacDonald, *Lectures on Lie Groups and Lie algebras*, LMS, 1995.

Project 4. Explain the proof of de Rham's theorem of the isomorphism between de Rham cohomology of a manifold and sheaf cohomology over $\mathbb{R}$, using sheaf cohomology and fine sheaves.

Project 5. Discuss Hodge theory for compact, oriented Riemannian manifolds; the isomorphism between de Rham cohomology and the vector spaces of harmonic forms. Include Poincaré duality.

Project 6. Write about spin geometry for Riemannian manifolds: Clifford algebras and the spin representation, spin structures on a manifold, spin bundles and spinors, the Dirac operator. Could mention quantum theory, relevance of spinors to spin $\frac{1}{2}$ particles like electrons.

Project 7. Give a broad-brush account (not much detail necessary) of some milestones in the theory of 3-manifolds: the Poincaré Conjecture and its proof by Perelman (and hangers-on); perhaps also describe Thurston's Geometrization Programme.

Miniproject on de Rham cohomology

Explain the definition of the de Rham cohomology $H^*_{\mathrm{dR}}(X,\mathbb{R})$ of a manifold X, and review its important properties. Then do either **(a)** or **(b)**:

(a)(i) *De Rham's Theorem* says that $H^*_{\mathrm{dR}}(X,\mathbb{R})\cong H^*_{\mathrm{top}}(X,\mathbb{R})$, for $H^*_{\mathrm{top}}(X,\mathbb{R})$ cohomology as defined in Algebraic Topology (there are several possible definitions). Give a proof of De Rham's Theorem, briefly explaining what background you need.

[*You are advised to define $H^*_{\mathrm{top}}(X,\mathbb{R})$ to be either Čech cohomology, or sheaf cohomology of the constant sheaf $\underline{\mathbb{R}}$ on X. If using Čech cohomology you may assume there exists an open cover $\{U_i : i \in I\}$ of X in which $U_{i_1}\cap\cdots\cap U_{i_k}$ is either empty or diffeomorphic to $\mathbb{R}^{\dim X}$ for all $i_1,\ldots,i_k \in I$.*]

(ii) Suppose we weaken the definition of smooth manifold X by not requiring X to be Hausdorff. Then exterior forms and de Rham cohomology $H^*_{\mathrm{dR}}(X,\mathbb{R})$, and topological cohomology $H^*_{\mathrm{top}}(X,\mathbb{R})$, still make sense. Is your proof still valid, and if not, why not? Discuss the example of X the 'line with two origins' $\mathbb{R}$ with 0 replaced by two points $\{0,0'\}$, with projection $\pi : X \to \mathbb{R}$, such that $U \subset X$ is open if $\pi(U) \subset \mathbb{R}$ is open. In Algebraic Topology terms, X is homotopic to the circle $\mathcal{S}^1$.

(b) Let X be a manifold, and $U,V \subseteq X$ be open subsets with $U\cup V = X$. The *Mayer–Vietoris sequence* is the long exact sequence of real vector spaces

$$\cdots \to H^k_{\mathrm{dR}}(X,\mathbb{R}) \xrightarrow{\rho^U_X\oplus\rho^V_X} \begin{array}{c}H^k_{\mathrm{dR}}(U,\mathbb{R})\\ \oplus H^k_{\mathrm{dR}}(V,\mathbb{R})\end{array} \xrightarrow{\rho^{U\cap V}_U\oplus-\rho^{U\cap V}_V} H^k_{\mathrm{dR}}(U\cap V,\mathbb{R}) \xrightarrow{\partial} H^{k+1}_{\mathrm{dR}}(X,\mathbb{R}) \to \cdots,$$

where the *restriction map* $\rho^U_X : H^k_{\mathrm{dR}}(X,\mathbb{R}) \to H^k_{\mathrm{dR}}(U,\mathbb{R})$ is pullback by the inclusion $U \hookrightarrow X$, and ∂ is the *continuation map* (which needs to be defined).

(i) Working at the level of exterior forms, prove that the Mayer–Vietoris sequence is exact, defining the continuation map ∂ during your proof.

[*You may assume the existence of a partition of unity for the open cover $\{U,V\}$ of X.*]

(ii) Use the Mayer–Vietoris sequence to compute the de Rham cohomology $H^*_{\mathrm{dR}}(X,\mathbb{R})$ when X is (A) the annulus $A = \mathcal{S}^1\times(0,1)$; (B) the 2-sphere $\mathcal{S}^2$; and (C) the genus g surface Σ_g for $g = 1,2,\ldots$.

[*You may assume the de Rham cohomology of $\mathbb{R}^2$.*]