Introduction to Differential Geometry

Lecture 1 of 10: Smooth manifolds

Dominic Joyce, Oxford University
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These slides available at
`http://people.maths.ox.ac.uk/~joyce/`

Plan of talk:

1. Manifolds

Differential Geometry is the study of (*smooth*) *manifolds* — a class of n-dimensional spaces on which one can do calculus (differentiation, integration, partial differential equations, ...). Often one studies a manifold X with an additional geometric structure, such as a Riemannian metric g.
Much of mathematical physics is written in the language of manifolds, e.g. Einstein's General Relativity models the universe as a 4-dimensional manifold with a Lorentzian metric.
Manifolds come in several flavours — topological manifolds (the simplest), smooth manifolds, complex manifolds (on which holomorphic functions and complex analysis make sense), Banach manifolds (modelled on possibly infinite-dimensional Banach spaces), etc. We'll discuss topological and smooth manifolds.

1.1. Topological manifolds

Topological manifolds are a special kind of topological space.

Definition

A topological space X is a *topological manifold* of dimension $n \geqslant 0$ if it is Hausdorff and second countable, and every point $x \in X$ has an open neighbourhood homeomorphic to an open set in $\mathbb{R}^n$.

The natural maps $f : X \to Y$ between topological manifolds X, Y are continuous maps of topological spaces.

Remark

Global topological assumptions vary. Some authors assume paracompact instead of second countable. (Our assumptions imply paracompact.) Some authors assume X connected.

1.2. Smooth manifolds

If X is a topological manifold, and $f : X \to \mathbb{R}$ a continuous function, there is no meaningful notion of when f is differentiable, or smooth, so we cannot do calculus on X.
A *smooth manifold* is a topological manifold X with some extra structure, a *smooth structure*, which tells us which functions $f : X \to \mathbb{R}$ are smooth. Smooth structures are usually defined in terms of an *atlas of charts* $A = \big\{(U_i, \phi_i) : i \in I\big\}$ on X.

Definition

Let X be a topological space, and $n \in \mathbb{N}$. An *n-dimensional chart* (U, ϕ) on X is an open set $U \subseteq \mathbb{R}^n$ and a map $\phi : U \to X$ which is a homeomorphism with an open set $\phi(U) \subseteq X$.

Compatibility of charts

Definition

Let X be a topological space, and $(U_1, \phi_1), (U_2, \phi_2)$ be n-dimensional charts on X. Then we have a diagram of homeomorphisms

$$\begin{tikzcd}
\mathbb{R}^n \supseteq U_1 \supseteq \phi_1^{-1}(\phi_2(U_2)) \arrow[rr, "\phi_2^{-1}\circ\phi_1"] \arrow[dr, "\phi_1|_{\phi_1^{-1}(\phi_2(U_2))}"'] & & \phi_2^{-1}(\phi_1(U_1)) \supseteq U_2 \subseteq \mathbb{R}^n \arrow[dl, shift left, "\phi_2|_{\phi_2^{-1}(\phi_1(U_1))}"] \\
& \phi_1(U_1)\cap\phi_2(U_2) \subseteq X. \arrow[ur, shift left, "\phi_2|^{-1}_{\phi_2^{-1}(\phi_1(U_1))}"] &
\end{tikzcd}$$

We say that $(U_1, \phi_1), (U_2, \phi_2)$ are *compatible* if the *transition map* $\phi_2^{-1} \circ \phi_1 : \phi_1^{-1}(\phi_2(U_2)) \to \phi_2^{-1}(\phi_1(U_1))$ is a diffeomorphism (i.e. a smooth map with smooth inverse) between open subsets of $\mathbb{R}^n$.

Compatibility of charts

Definition of atlases and manifolds

Definition

Let X be a topological space, and $n \in \mathbb{N}$. An *n-dimensional smooth atlas* $A = \{(U_i, \phi_i) : i \in I\}$ on X is a family of n-dimensional charts (U_i, ϕ_i) on X for $i \in I$, where I is an indexing set, such that $X = \bigcup_{i \in I} \phi_i(U_i)$ (that is, the charts cover X), and $(U_i, \phi_i), (U_j, \phi_j)$ are compatible for all $i, j \in I$.

A smooth atlas $\{(U_i, \phi_i) : i \in I\}$ is called *maximal* if no other atlas on X contains it as a proper subset.

Every atlas $\{(U_i, \phi_i) : i \in I\}$ is contained in a unique maximal atlas $\{(U'_{i'}, \phi'_{i'}) : i' \in I'\}$, the set of all charts (U', ϕ') on X compatible with (U_i, ϕ_i).

An *n-dimensional smooth manifold* (X, A) is a Hausdorff, second countable topological space X with a maximal n-dimensional smooth atlas $A = \{(U_i, \phi_i) : i \in I\}$. We write $\dim X = n$.

Remarks

• The terms ‘charts’ and ‘atlas’ come from maps of the world. You can't cover the sphere $\mathcal{S}^2$ with one chart in $\mathbb{R}^2$, without cutting. But you can describe $\mathcal{S}^2$ completely with a finite number of charts covering the whole of $\mathcal{S}^2$ (e.g. an atlas including a map of every country in the world), plus information on how the charts overlap.

• We can use the same idea to define other classes of manifolds:

- To define *manifolds with boundary*, in charts (U, ϕ) we allow $U \subseteq [0,\infty) \times \mathbb{R}^{n-1}$ open as well as $U \subseteq \mathbb{R}^n$ open.
- To define *manifolds with corners*, we allow $U \subseteq [0,\infty)^k \times \mathbb{R}^{n-k}$ open for $k = 0, \ldots, n$.

Manifolds
More basics on manifolds
Topological manifolds
Smooth manifolds
Examples of manifolds defined by atlases
Smooth maps between smooth manifolds
Vector bundles

Remarks

• It's not practical to write down a *maximal* atlas $\{(U_i, \phi_i) : i \in I\}$ in examples – there are too many charts. Instead, you write down an explicit atlas with a small number of charts, and use the fact that it extends uniquely to a maximal atlas.

• The point of taking the atlas $\{(U_i, \phi_i) : i \in I\}$ to be maximal is that it is more canonical, there are fewer arbitrary choices involved.

• Usually we refer to X as the manifold, and don't specify the atlas $A = \{(U_i, \phi_i) : i \in I\}$, taking it to be implicitly given.

• In fact, after the first week in Differential Geometry, it's kind of impolite to refer to atlases at all! We hardly ever use atlases explicitly. Instead of charts (U, ϕ) we usually talk about ‘local coordinates’ $(x_1, \ldots, x_n)$ on X, which is really the function ϕ^{-1} mapping an open set $\phi(U) \subseteq X$ to $\mathbb{R}^n \supseteq U \ni (x_1, \ldots, x_n)$, so that $x_1, \ldots, x_n$ are functions $\phi(U) \to \mathbb{R}$.

1.3. Examples of manifolds defined by atlases

Example (Euclidean space $\mathbb{R}^n$)

$X = \mathbb{R}^n$ is an n-dimensional manifold. It has an atlas $\{(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})\}$ with one chart $(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})$, which extends to a maximal atlas.

Example (The n-sphere $\mathcal{S}^n$)

$X = \mathcal{S}^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : x_0^2 + \cdots + x_n^2 = 1\}$ is an n-dimensional manifold. It has an atlas $\{(U_1, \phi_1), (U_2, \phi_2)\}$ with two charts, where $U_1 = U_2 = \mathbb{R}^n$, $\phi_1(U_1) = \mathcal{S}^n \setminus \{(-1, 0, \ldots, 0)\}$, $\phi_2(U_2) = \mathcal{S}^n \setminus \{(1, 0, \ldots, 0)\}$, and ϕ_1, ϕ_2 are the inverses of

$$\phi_1^{-1} : (x_0, \ldots, x_n) \longmapsto \tfrac{1}{1+x_0}(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$$
$$\phi_2^{-1} : (x_0, \ldots, x_n) \longmapsto \tfrac{1}{1-x_0}(x_1, \ldots, x_n) = (z_1, \ldots, z_n).$$

Note that as $x_0^2 + \cdots + x_n^2 = 1$ we have $(y_1^2 + \cdots + y_n^2)(z_1^2 + \cdots + z_n^2) = \frac{1}{(1+x_0)^2}\frac{1}{(1-x_0)^2}(x_1^2 + \cdots + x_n^2)^2 = 1$. Thus $\phi_2^{-1} \circ \phi_1 : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ maps $(y_1, \ldots, y_n) \longmapsto (z_1, \ldots, z_n) = \frac{1}{y_1^2 + \cdots + y_n^2}(y_1, \ldots, y_n)$. This is smooth, so $(U_1, \phi_1), (U_2, \phi_2)$ are compatible.

Example (The n-torus T^n)

$X = T^n = \mathbb{R}^n/\mathbb{Z}^n$ is an n-dimensional manifold. It has an atlas $\{(U_{\boldsymbol{y}}, \phi_{\boldsymbol{y}}) : \boldsymbol{y} \in I\}$ with $I = \{\boldsymbol{y} = (y_1, \ldots, y_n) : y_i \in \{0, \frac{1}{2}\}\}$ and 2^n charts $(U_{\boldsymbol{y}}, \phi_{\boldsymbol{y}})$ with $U_{\boldsymbol{y}} = (-\frac{1}{3}, \frac{1}{3})^n$ and $\phi_{\boldsymbol{y}} : (x_1, \ldots, x_n) \mapsto (x_1 + y_1 + \mathbb{Z}, \ldots, x_n + y_n + \mathbb{Z})$.
When $n = 1$, so that T^n is the circle $\mathcal{S}^1$, this gives two charts $(U_0, \phi_0), (U_{1/2}, \phi_{1/2})$. The transition map $\phi_{1/2}^{-1} \circ \phi_0$ maps

$$\phi_{1/2}^{-1} \circ \phi_0 : (-\tfrac{1}{3}, -\tfrac{1}{6}) \amalg (\tfrac{1}{6}, \tfrac{1}{3}) \longrightarrow (-\tfrac{1}{3}, -\tfrac{1}{6}) \amalg (\tfrac{1}{6}, \tfrac{1}{3}),$$

$$\phi_{1/2}^{-1} \circ \phi_0 : x \longmapsto \begin{cases} x + \frac{1}{2}, & x \in (-\frac{1}{3}, -\frac{1}{6}), \\ x - \frac{1}{2}, & x \in (\frac{1}{6}, \frac{1}{3}). \end{cases}$$

So $\phi_{1/2}^{-1} \circ \phi_0$ is smooth.

1.4. Smooth maps between smooth manifolds

Definition

Let X, Y be smooth manifolds of dimensions m, n, with maximal atlases $A = \{(U_i, \phi_i) : i \in I\}$ and $B = \{(V_j, \psi_j) : j \in J\}$.
Suppose $f : X \to Y$ is a continuous map of topological spaces. We say that f is *smooth* if for all charts (U_i, ϕ_i) on X and (V_j, ψ_j) on Y, the map $\psi_j^{-1} \circ f \circ \phi_i : (f \circ \phi_i)^{-1}(\psi_j(V_j)) \to V_j$ is a smooth map between open subsets of $\mathbb{R}^m, \mathbb{R}^n$, where

$$\begin{tikzcd}
\mathbb{R}^m \supseteq U_i \supseteq (f \circ \phi_i)^{-1}(\psi_j(V_j)) \arrow[rr, "\psi_j^{-1} \circ f \circ \phi_i"'] \arrow[d, "\phi_i|_{(f\circ\phi_i)^{-1}(\psi_j(V_j))}"] & & V_j \subseteq \mathbb{R}^n \\
X \supseteq f^{-1}(\psi_j(V_j)) \arrow[rr, "f|_{f^{-1}(\psi_j(V_j))}"] & & \psi_j(V_j) \subseteq Y. \arrow[u, "\psi_j^{-1}"']
\end{tikzcd}$$

It is enough to check this for sets of charts $(U_i, \phi_i), (V_j, \psi_j)$ covering X, Y, not for all charts on X, Y.

If $f : X \to Y$ and $g : Y \to Z$ are smooth maps of manifolds then $g \circ f : X \to Z$ is smooth (exercise). Also the identity map $\mathrm{id}_X : X \to X$ is smooth (exercise). Thus manifolds and smooth maps form a category **Man**.

A smooth map $f : X \to Y$ is called a *diffeomorphism* if f is a bijection and $f^{-1} : Y \to X$ is smooth. Diffeomorphisms are the natural notion of when two manifolds are 'the same'.

Example

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$, is smooth, and a homeomorphism at the level of topological spaces. But f is not a diffeomorphism, as $f^{-1} : \mathbb{R} \to \mathbb{R}$, $f^{-1}(x) = x^{1/3}$, is not smooth at 0.

The smooth structure on X is determined by X as a topological space, plus the set (or $\mathbb{R}$-algebra) of smooth functions $f : X \to \mathbb{R}$.

An advanced remark:

Given a topological manifold X, one can ask:

(a) Does X have a smooth structure? (That is, does there exist an atlas A making X into a smooth manifold?)

(b) If X has a smooth structure, is it unique up to diffeomorphism? (That is, if A, B are maximal atlases on X, must there exist a diffeomorphism $f : (X, A) \to (X, B)$, not necessarily the identity?)

In dimensions 4 and above, the answer to both questions can be no. There are examples of compact topological 4-manifolds which do not have a smooth structure. And $\mathbb{R}^4$ itself has many exotic smooth structures. The smooth 4-dimensional Poincaré conjecture (unsolved) says that $\mathcal{S}^4$ has no exotic smooth structure. Dimensions 5 and above are quite well understood, but dimension 4 still has many puzzles. This is an interesting area.

Products of manifolds

Let X, Y be manifolds. Then the product $X \times Y$ as a topological space has a unique manifold structure, with $\dim(X \times Y) = \dim X + \dim Y$, such that if (U, ϕ) and (V, ψ) are charts on X, Y then $(U \times V, \phi \times \psi)$ is a chart on $X \times Y$. We call $X \times Y$ the *product manifold*.

The product has projections $\pi_X : X \times Y \to X$, $\pi_Y : X \times Y \to Y$ which are smooth. Products are associative and commutative, up to canonical diffeomorphism.

1.5. Vector bundles

Most differential geometry studies not the manifolds themselves, but other geometric structures (e.g. Riemannian metrics, lectures 7+8) living on manifolds. So we need to think about geometric structures on manifolds. One of the most basic and important geometric structures on a manifold X is a *vector bundle* $E \to X$. Heuristically, a vector bundle is a family of real vector spaces E_x for $x \in X$, all of the same (finite) dimension $\dim E_x = \operatorname{rank} E$, the *rank* of E, such that E_x 'varies smoothly with $x \in X$'. The next definition makes precise what this 'varies smoothly' means.

The definition of vector bundle

Definition

Let X be a manifold. A *rank k vector bundle $E \to X$ on X* is a manifold E with $\dim E = \dim X + k$, a smooth map $\pi : E \to X$, and a real vector space structure on the fibre $E_x := \pi^{-1}(x)$ for each $x \in X$, such that each $x \in X$ has an open neighbourhood U_x and a diffeomorphism $\Phi_x : \pi^{-1}(U_x) \to U_x \times \mathbb{R}^k$ in a commuting:

$$\begin{tikzcd}
\pi^{-1}(U_x) \arrow[rr, "\Phi_x"'] \arrow[d, "\pi|_{\pi^{-1}(U_x)}"] & & U_x \times \mathbb{R}^k \arrow[d, "\pi_{U_x}"'] \\
U_x \arrow[rr, equal] & & U_x,
\end{tikzcd}$$

where $\Phi_x|_{E_y} : E_y \to \{y\} \times \mathbb{R}^k \cong \mathbb{R}^k$ identifies the vector space structures on E_y and $\mathbb{R}^k$ for all $y \in U_x$. We write $\operatorname{rank} E = k$.
A *section* of E is a smooth map $s : X \to E$ with $\pi \circ s = \mathrm{id}_X$.
We can add sections $(s+t)(x) = s(x) + t(x)$ and multiply them by $\mathbb{R}$, so the set $C^\infty(E)$ of sections of E is a vector space.

Examples of vector bundles

Example (Trivial vector bundles)

Let X be a manifold, and $k \geqslant 0$. Then $\pi_X : X \times \mathbb{R}^k \to X$ is a vector bundle, the *trivial vector bundle* on X with fibre $\mathbb{R}^k$.

Example (The Möbius strip)

Let $X = \mathcal{S}^1 = \mathbb{R}/\mathbb{Z}$ and $E = \mathbb{R}^2/\mathbb{Z}$, where $\mathbb{Z}$ acts on $\mathbb{R}^2$ by $n : (x, y) \mapsto (x + n, (-1)^n y)$ for $n \in \mathbb{Z}$ and $(x, y) \in \mathbb{R}^2$. Define $\pi : E \to X$ by $\pi : (x, y)\mathbb{Z} \mapsto x + \mathbb{Z}$. Then $E \to X$ is a non-trivial vector bundle of rank 1, called the *Möbius strip*.
The fibres E_x for $x \in \mathcal{S}^1$ have isomorphisms $E_x \xrightarrow{\cong} \mathbb{R}$, but these isomorphisms are only canonical up to sign. If you deform x round the circle once, the isomorphism $E_x \xrightarrow{\cong} \mathbb{R}$ changes sign.

Remarks

• Operations on vector spaces also make sense on vector bundles. For example, a vector bundle $E \to X$ has a dual vector bundle $E^* \to X$, with fibre $(E^*)_x = (E_x)^*$ at $x \in X$, for $(E_x)^*$ the dual vector space of E_x. If $E, F \to X$ are vector bundles we can form the vector bundle *direct sum* $E \oplus F \to X$ and *tensor product* $E \otimes F \to X$, with fibres $E_x \oplus F_x$ and $E_x \otimes F_x$ at $x \in X$.
• If $E \to X$ and $F \to X$ are vector bundles over X, a *morphism* $\alpha : E \to F$ is a smooth map $\alpha : E \to F$ with $\pi \circ \alpha = \pi : E \to X$ such that $\alpha|_{E_x} : E_x \to F_x$ is a linear map for each $x \in X$.
• If $f : X \to Y$ is a smooth map of manifolds and $E \to Y$ is a vector bundle, there is a *pullback vector bundle* $f^*(E) \to X$, with $f^*(E)_x = E_{f(x)}$. Here $f^*(E) = \{(x, e) \in X \times E : f(x) = \pi(e)\}$. As a manifold, this is a *transverse fibre product* $f^*(E) = X \times_Y E$, which exists as π is a *submersion* – explained in §2.4.

Introduction to Differential Geometry

Lecture 2 of 10: More basics on manifolds.
Immersions, embeddings, submanifolds,
submersions, and transverse fibre products

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Plan of talk:

2. More basics on manifolds

2.1. Tangent bundles and cotangent bundles

Every smooth manifold X comes with a natural vector bundle $TX \to X$ called the *tangent bundle*, with $\mathrm{rank}\, TX = \dim X$, and a dual vector bundle called the *cotangent bundle* $T^*X \to X$.

Sections $v \in C^\infty(TX)$ of TX are called *vector fields on* X.

Sections $\alpha \in C^\infty(T^*X)$ of T^*X are called 1-*forms on* X.

Definition (Intrinsic definition of tangent vectors)

Let X be a manifold, and write $C^\infty(X)$ for the $\mathbb{R}$-algebra of smooth functions $\alpha : X \to \mathbb{R}$. Let $x \in X$. A *tangent vector v at x* is an $\mathbb{R}$-linear map $v : C^\infty(X) \to \mathbb{R}$ with $v(\alpha\beta) = v(\alpha)\beta(x) + \alpha(x)v(\beta)$ for all $\alpha, \beta \in C^\infty(X)$.
Write T_xX for the $\mathbb{R}$-vector space of tangent vectors v at x.
As a set, $TX = \{(x, v) : x \in X,\ v \in T_xX\}$, and the projection $\pi : TX \to X$ maps $\pi : (x, v) \mapsto x$.

Definition (Tangent vectors in local coordinates)

Let X be a manifold, and $(x_1,\ldots,x_n)$ be local coordinates on an open subset $V \subseteq X$. (That is, (U,ϕ) is a chart on X, $V=\phi(U)$, and $(x_1,\ldots,x_n)=\phi^{-1}: V \to U \subseteq \mathbb{R}^n$.) Let $\alpha : X \to \mathbb{R}$ be smooth. Then we may write $\alpha|_V$ as a smooth function $\alpha(x_1,\ldots,x_n)$ for $(x_1,\ldots,x_n) \in U \subseteq \mathbb{R}^n$.

Let $x \in V \subseteq X$ have coordinates $(x_1,\ldots,x_n)$. One can show $v : C^\infty(X) \to \mathbb{R}$ lies in T_xX if and only if it is of the form

$$v : \alpha \longmapsto v_1 \frac{\partial\alpha}{\partial x_1}(x_1,\ldots,x_n) + \cdots + v_n \frac{\partial\alpha}{\partial x_n}(x_1,\ldots,x_n),$$

for $v_1,\ldots,v_n \in \mathbb{R}$. We write $v = v_1 \frac{\partial}{\partial x_1}|_x + \cdots + v_n \frac{\partial}{\partial x_n}|_x$.

Thus $T_xX \cong \mathbb{R}^n \ni (v_1,\ldots,v_n)$, with basis $\frac{\partial}{\partial x_1}|_x,\ldots,\frac{\partial}{\partial x_n}|_x$. The dual vector space is $T_x^*X := (T_xX)^*$. Write $\mathrm{d}x_1|_x,\ldots,\mathrm{d}x_n|_x$ for the basis of T_x^*X dual to $\frac{\partial}{\partial x_1}|_x,\ldots,\frac{\partial}{\partial x_n}|_x$.

Definition (Manifold structure on the tangent bundle)

With the same notation, define a chart $(TU, T\phi)$ on TX by $TU = U \times \mathbb{R}^n \subseteq \mathbb{R}^{2n}$ and $T\phi : TU \to TX$ given by

$$T\phi : (x_1,\ldots,x_n,v_1,\ldots,v_n) \longmapsto \Bigl(\phi(x_1,\ldots,x_n), v_1 \tfrac{\partial}{\partial x_1}\bigr|_{\phi(x_1,\ldots,x_n)} + \cdots + v_n \tfrac{\partial}{\partial x_n}\bigr|_{\phi(x_1,\ldots,x_n)}\Bigr).$$

There is a unique manifold structure on TX such that $(TU, T\phi)$ is a chart on TX whenever (U,ϕ) is a chart on X, and this makes $TX \to X$ into a vector bundle, the *tangent bundle*.

The dual vector bundle is the *cotangent bundle* T^*X.

If $(x_1,\ldots,x_n)$ are local coordinates on an open $V \subseteq X$, then $\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}$ are a basis of sections of $TX|_V$, and $\mathrm{d}x_1,\ldots,\mathrm{d}x_n$ are a basis of sections of $T^*X|_V$. The notation means what it says: $\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}$ are partial derivatives, and $\mathrm{d}x_1,\ldots,\mathrm{d}x_n$ are derivatives of the functions $x_1,\ldots,x_n$. We have $\frac{\partial}{\partial x_i}\cdot \mathrm{d}x_j = \frac{\partial x_j}{\partial x_i} = \delta_{ij}$.

2.2. Action of smooth maps on vectors

Definition

Let X, Y be manifolds, and $f : X \to Y$ a smooth map. For $x \in X$, recall that a vector v at x ($v \in T_x X$) is an $\mathbb{R}$-linear map $v : C^\infty(X) \to \mathbb{R}$ with $v(\alpha\beta) = v(\alpha)\beta(x) + \alpha(x)v(\beta)$. Let $y = f(x) \in Y$, and define $w : C^\infty(Y) \to \mathbb{R}$ by $w(\gamma) = v(\gamma \circ f)$. Then w is a tangent vector to y at Y, $w \in T_y Y$. Write $\mathrm{d}f(v) = w$. Define $Tf : TX \to TY$ by $Tf : (x, v) \mapsto (f(x), \mathrm{d}f(v))$.

If $g : Y \to Z$ is another smooth map then $\mathrm{d}g \circ \mathrm{d}f(v)$ maps $\gamma \mapsto v((\gamma \circ f) \circ g)$ and $\mathrm{d}(g \circ f)(v)$ maps $\gamma \mapsto v(\gamma \circ (f \circ g))$. So $T(g \circ f) = Tg \circ Tf$, and T is a functor **Man** $\to$ **Man**.

Manifolds
More basics on manifolds
Tangent bundles and cotangent bundles
Action of smooth maps on vectors
Immersions, embeddings, and submanifolds
Submersions

On cotangent bundles, we do *not* have a map $T^*f : T^*X \to T^*Y$ in the same way. But we can also write Tf as a morphism of vector bundles $\mathrm{d}f : TX \to f^*(TY)$ on X mapping $(x, v) \mapsto (x, \mathrm{d}f(v))$. Then we have a dual morphism $(\mathrm{d}f)^* : f^*(T^*Y) \to T^*X$ of vector bundles on X.
Suppose $(x_1, \ldots, x_m)$ are local coordinates on $x \in V \subseteq X$, and $(y_1, \ldots, y_n)$ local coordinates on $y = f(x) \in W \subseteq Y$.
On $V \cap f^{-1}(W) \subseteq X$, we can write f in coordinates as

$$f = \bigl(f_1(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m)\bigr).$$

Then $Tf, \mathrm{d}f$ are given by the matrix of functions $\bigl(\frac{\partial f_j}{\partial x_i}(x_1, \ldots, x_m)\bigr)_{i=1,\ldots,m}^{j=1,\ldots,n}$ w.r.t. the bases of sections $(\frac{\partial}{\partial x_i}), (\frac{\partial}{\partial y_j})$ for TX, TY, and $(\mathrm{d}f)^*$ by the transpose matrix.

2.3. Immersions, embeddings, and submanifolds

Definition

Let $f : X \to Y$ be a smooth map of manifolds. We say that f is an *immersion* if $\mathrm{d}f : T_x X \to T_y Y$ is an injective linear map for all $x \in X$ with $f(x) = y \in Y$.
We say that f is an *embedding* if f is an immersion, and $f : X \to f(X)$ is a homeomorphism of topological spaces, where $f(X) \subseteq Y$ is the image of f.
This implies that $f : X \to Y$ is an injective map. But injective immersions need not be embeddings, as $f(X)$ might not be homeomorphic to X (exercise: find an example).
If $f : X \to Y$ is an immersion/embedding then $\dim X \leqslant \dim Y$, as $\mathrm{d}f : T_x X \to T_y Y$ is injective, $\dim T_x X = \dim X$, $\dim T_y Y = \dim Y$.

Submanifolds

Definition

Let Y be a manifold. An *immersed* or *embedded submanifold* in Y is a manifold X and an immersion or embedding $i : X \hookrightarrow Y$.
Usually we write X for the submanifold, leaving i implicit.

Submanifolds are the natural 'subobjects' in differential geometry, like subgroups in groups, etc.
When people say 'submanifold' without specifying embedded or immersed, they usually mean embedded.
As we will explain shortly, an embedded submanifold $i : X \hookrightarrow Y$ is actually determined (at least up to canonical isomorphism) by the subset $i(X) \subseteq Y$, so people often define (embedded) submanifolds to be special subsets $X \subseteq Y$.
Immersed submanifolds $i : X \hookrightarrow Y$ are not determined by $i(X)$.

Local normal form for immersions and embeddings

We can choose local coordinates to put immersions and embeddings in a simple form:

Proposition 2.1

Let X, Y be manifolds of dimensions m, n with $m \leqslant n$, $f : X \to Y$ be an immersion, and $x \in X$ with $f(x) = y \in Y$. Then we can find local coordinates $(x_1, \ldots, x_m)$ near x in X and $(y_1, \ldots, y_n)$ near y in Y such that f acts in coordinates as

$$f : (x_1, \ldots, x_m) \longmapsto (y_1, \ldots, y_n) = (x_1, \ldots, x_m, 0, \ldots, 0).$$

If such local coordinates exist for all $x \in X$, then f is an immersion.

Notice that this means (if f is an embedding, at least) that $f(X)$ can be written near y as the zeroes $y_{m+1} = \cdots = y_n = 0$ of $n - m$ smooth real functions on Y.

Proposition 2.2

Let $i : X \hookrightarrow Y$ be an embedding of manifolds. Then the image $i(X)$, as a topological subspace of Y, has a unique smooth structure such that the inclusion $i(X) \hookrightarrow Y$ is an embedding, and then $i : X \to i(X)$ is a diffeomorphism.

Proof.

By Proposition 2.1, for each $y \in i(X) \subseteq Y$ we can find a chart (V, ψ) on Y with $y \in \psi(V) \subseteq Y$ and $\psi^{-1}(i(X)) = \{(y_1, \ldots, y_n) \in V : y_{m+1} = \cdots = y_n = 0\}$, where $\dim X = m$, $\dim Y = n$. Define $U = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : (x_1, \ldots, x_m, 0, \ldots, 0) \in V \subseteq \mathbb{R}^n\}$ and $\phi : U \to i(X)$ by $\phi : (x_1, \ldots, x_m) \mapsto \psi(x_1, \ldots, x_m, 0, \ldots, 0)$. Then (U, ϕ) is a chart on $i(X)$. The set of all such charts (U, ϕ) is an atlas on $i(X)$. The rest is easy. $\square$

Proposition 2.2 means that we can often define manifolds as subsets of other manifolds, without going to the trouble of writing down an atlas.

Example

$\mathcal{S}^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : x_0^2 + \cdots + x_n^2 = 1\}$ is an embedded submanifold of $\mathbb{R}^{n+1}$, with the smooth structure defined in §1.3.

But most subsets are not submanifolds.

Example

$C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^2\}$ is not a submanifold of $\mathbb{R}^3$, since it is not even a topological manifold near $(0, 0, 0)$ (exercise).

In Corollary 2.7 in §2.4 we give a criterion for when a subset defined as the zeroes of some smooth equations is a submanifold.

The Whitney Embedding Theorem

Theorem (Whitney)

Let X be a manifold of dimension m, and $n > 2m$. Then there exist embeddings $f : X \hookrightarrow \mathbb{R}^n$ (in fact, a generic smooth $f : X \to \mathbb{R}^n$ is an embedding). We can choose f with $f(X)$ closed in $\mathbb{R}^n$. Hence, every manifold is diffeomorphic to a (closed) submanifold of $\mathbb{R}^n$ for $n \gg 0$.

The proof needs that manifolds are second countable as topological spaces. Whitney also showed that an m-manifold X can be embedded in $\mathbb{R}^{2m}$ (though not just by a generic smooth map), and in fact this minimal dimension can be improved in some cases.

Example 2.3

The real projective plane $\mathbb{RP}^2$ can be embedded in $\mathbb{R}^4$, but not in $\mathbb{R}^3$.

2.4. Submersions

In a similar way to immersions in §2.3, define:

Definition

Let $f : X \to Y$ be a smooth map of manifolds. We say that f is a *submersion* if $\mathrm{d}f : T_xX \to T_yY$ is a surjective linear map for all $x \in X$ with $f(x) = y \in Y$.

If $f : X \to Y$ is a submersion then $\dim X \geqslant \dim Y$, as $\mathrm{d}f : T_xX \to T_yY$ is surjective, $\dim T_xX = \dim X$, $\dim T_yY = \dim Y$.

Example 2.4

Let X, Y be manifolds, and $\pi_X : X \times Y \to X$ the projection, a smooth map. Then $T_{(x,y)}(X \times Y) = T_xX \oplus T_yY$ for $x \in X$, and $\mathrm{d}\pi_X : T_{(x,y)}(X \times Y) \to T_xX$ is the projection $T_xX \oplus T_yY \to T_xX$, and so surjective. Thus π_X is a submersion.

Local normal form for submersions

As for Proposition 2.1, we can choose local coordinates to put submersions in a simple form:

Proposition 2.5

Let X, Y be manifolds of dimensions m, n with $m \geqslant n$, $f : X \to Y$ be a submersion, and $x \in X$ with $f(x) = y \in Y$. Then we can find local coordinates $(x_1, \ldots, x_m)$ near x in X and $(y_1, \ldots, y_n)$ near y in Y such that f acts in coordinates as

$$f : (x_1, \ldots, x_m) \longmapsto (y_1, \ldots, y_n) = (x_1, \ldots, x_n).$$

If such local coordinates exist for all $x \in X$, then f is a submersion.

Notice that this means that f is locally modelled on the projection $\pi_{\mathbb{R}^n} : \mathbb{R}^n \times \mathbb{R}^{m-n} \to \mathbb{R}^n$, as in Example 2.4.

Proposition 2.6

Let $f : X \to Y$ be a submersion of manifolds. Then for each $y \in Y$, $f^{-1}(y) \subseteq X$ is a closed, embedded submanifold of X, of dimension $\dim X - \dim Y$.

Proof.

By Proposition 2.5, for each $x \in f^{-1}(y) \subseteq X$ we can find charts (U, ϕ) on X and (V, ψ) on Y with $x \in \phi(U) \subseteq f^{-1}(\psi(V)) \subseteq X$, so that f acts in coordinates by $(x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_n)$, where $\dim X = m$, $\dim Y = n$. Let $(y_1, \ldots, y_n)$ be the coordinates of y, set $W = \{(z_1, \ldots, z_{m-n}) \in \mathbb{R}^{m-n} : (y_1, \ldots, y_n, z_1, \ldots, z_{m-n}) \in U\}$, and define $\xi : W \to f^{-1}(y)$ by
$\xi : (z_1, \ldots, z_{m-n}) \mapsto \phi(y_1, \ldots, y_n, z_1, \ldots, z_{m-n})$. Then (W, ξ) is a chart on $f^{-1}(y)$. The set of all such charts (W, ξ) is an atlas on $f^{-1}(y)$. The rest is easy. □

For $f^{-1}(y)$ to be an embedded submanifold of X, we only need f to be a submersion near $f^{-1}(y)$ in X. As being a submersion is an open condition, it is enough to check it at each point of $f^{-1}(y)$.

Corollary

Let $f : X \to Y$ be a smooth map, and $y \in Y$. If $\mathrm{d}f : T_xX \to T_yY$ is surjective for all $x \in X$ with $f(x) = y$, then $f^{-1}(y)$ is an embedded submanifold of X, of dimension $\dim X - \dim Y$.

Taking $Y = \mathbb{R}^n$ and $y = (0, \ldots, 0)$ gives:

Corollary 2.7

Let X be a manifold, and $f_1, \ldots, f_n : X \to \mathbb{R}$ smooth functions. Suppose $\mathrm{d}f_1|_x, \ldots, \mathrm{d}f_n|_x$ are linearly independent in T_x^*X for all $x \in X$ with $f_1(x) = \cdots = f_n(x) = 0$ (then we call $f_1, \ldots, f_n$ **independent***). Then $\{x \in X : f_1(x) = \cdots = f_n(x) = 0\}$ is an embedded submanifold of X, with dimension $\dim X - n$.*

Sard's Theorem

Theorem (Sard)

Let $f : X \to Y$ be any smooth map of manifolds. Then for generic $y \in Y$ (actually, in the complement of a null set), f is a submersion in an open neighbourhood of $f^{-1}(y)$ in X. Hence $f^{-1}(y)$ is a closed, embedded submanifold of X, of dimension $\dim X - \dim Y$, by Proposition 2.6.

If $\dim X < \dim Y$ then $f^{-1}(y) = \emptyset$ for generic Y, that is, $f(X)$ has measure zero in Y.
Sard's Theorem means that every positive-dimensional manifold has lots of submanifolds, and submanifolds (and so, examples of manifolds) are easy to find.

Transversality and fibre products

Let $\mathcal{C}$ be a category (for instance, the category of manifolds **Man**). Then in $\mathcal{C}$ we have objects $X, Y, Z, \ldots$, and morphisms $f : X \to Y$, $g : Y \to Z$, $\ldots$, with composition of morphisms $g \circ f : X \to Z$, and identity morphisms $\mathrm{id}_X : X \to X$, with the usual properties, e.g. composition is associative.

Definition 2.8

Let $g : X \to Z$ and $h : Y \to Z$ be morphisms in $\mathcal{C}$. A *fibre product* (W, e, f) for g, h is an object W in $\mathcal{C}$ and morphisms $e : W \to X$, $f : W \to Y$ with $g \circ e = h \circ f : W \to Z$, with the universal property that if $e' : W' \to X$, $f' : W' \to Y$ are morphisms in $\mathcal{C}$ with $g \circ e' = h \circ f'$, then there is a unique morphism $b : W' \to W$ in $\mathcal{C}$ with $e' = e \circ b$ and $f' = f \circ b$. Often we write $X \times_{g,Z,h} Y$ or $X \times_Z Y$ for W, and $\pi_X : X \times_Z Y \to X$, $\pi_Y : X \times_Z Y \to Y$ for e, f.

Fibre products need not exist, but if they do exist they are unique up to canonical isomorphism in $\mathcal{C}$. So, fibre products are a way of constructing a new object W from morphisms $g : X \to Z$, $h : Y \to Z$, fitting into a commutative diagram

$$\begin{tikzcd} W \arrow[r, dotted, "f"] \arrow[d, dotted, "e"] & Y \arrow[d, "h"] \\ X \arrow[r, "g"] & Z. \end{tikzcd}$$

Category theory ideas, such as fibre products, probably seem weird when you first meet them, but later they are very useful.

As an example of how to use the universal property in Definition 2.8, suppose W is a fibre product $X \times_{g,Z,h} Y$ in the category of manifolds **Man**. Take W' to be the point $*$. Then morphisms $b : * \to W$ are in 1-1 correspondence with pairs (e', f') of morphisms $e' : * \to X$, $f' : * \to Y$ with $g \circ e' = h \circ f'$, via $b \mapsto (e', f') = (e \circ b, f \circ b)$. But morphisms $b : * \to W$ correspond to points $w \in W$, and similarly $e' : * \to X$, $f' : * \to Y$ correspond to $x \in X$, $y \in Y$. Hence *as sets* we have a natural identification

$$W \cong \big\{(x, y) \in X \times Y : g(x) = h(y)\big\}. \tag{2.1}$$

So to decide if a fibre product $X \times_{g,Z,h} Y$ exists in **Man**, we must work out whether (2.1) naturally has the structure of a manifold.

Transverse fibre products in **Man**

Definition

Smooth maps of manifolds $g : X \to Z$, $h : Y \to Z$ are called *transverse* if for all $x \in X$ and $y \in Y$ with $g(x) = h(y) = z \in Z$, the map $\mathrm{d}g \oplus \mathrm{d}h : T_x X \oplus T_y Y \to T_z Z$ is surjective.

Theorem 2.9

Suppose $g : X \to Z$, $h : Y \to Z$ are transverse smooth maps of manifolds. Then a fibre product $W = X \times_{g,Z,h} Y$ exists in **Man**, *with* $\dim W = \dim X + \dim Y - \dim Z$.

Idea of proof: working in local coordinates, we show W in (2.1) is an embedded submanifold of $X \times Y$, of the correct dimension. □

If g or h is a submersion, then g, h are automatically transverse. Proposition 2.6 follows from Theorem 2.9 with g a submersion and $Y = *$ a point. Product manifolds $X \times Y$ are transverse fibre products $X \times_* Y$ with $Z = *$ the point.

Examples

Example

The n-sphere is
$\mathcal{S}^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}$. It is an embedded submanifold of $\mathbb{R}^{n+1}$, and thus a manifold.
We can prove this by applying Corollary 2.7 to the function $f : \mathbb{R}^{n+1} \to \mathbb{R}$, $f(x_1, \ldots, x_{n+1}) = x_1^2 + \cdots + x_{n+1}^2 - 1$.

Example

The n-torus T^n is $\mathcal{S}^1 \times \cdots \times \mathcal{S}^1$, the product of n copies of $\mathcal{S}^1$, using products of manifolds.

Example

The n-dimensional *real projective space* is $\mathbb{RP}^n = \mathcal{S}^n/\{\pm 1\}$, with the quotient topology. It has a unique manifold structure such that the projection $\pi : \mathcal{S}^n \to \mathbb{RP}^n$ is a local diffeomorphism.