Introduction to Differential Geometry

Lecture 3 of 10: Tensors

Dominic Joyce, Oxford University
September 2019

2019 Nairobi Workshop in Algebraic Geometry

These slides available at
http://people.maths.ox.ac.uk/∼joyce/

Plan of talk:

3. Tensors and exterior forms

3.1. Some linear algebra

We start with a reminder on some basic operations on vector spaces. For simplicity, all vector spaces will be finite-dimensional over $\mathbb{R}$. If U is a vector space, the *dual vector space* is $U^* = \mathrm{Hom}(U, \mathbb{R})$, with $\dim U^* = \dim U$. If $u^1, \ldots, u^m$ is a basis of U, there is a dual basis $u_1, \ldots, u_m$ of U^*, with $u_j(u^i) = \delta_{ij}$ for $i, j = 1, \ldots, m$. We can identify $U = (U^*)^*$.

If U, V are vector spaces, the *direct sum* is

$$U \oplus V = U \times V = \big\{(u, v) : u \in U,\ v \in V\big\}.$$

It is a vector space of dimension $\dim U + \dim V$. If $u^1, \ldots, u^m$ and $v^1, \ldots, v^n$ are bases of U, V, then $u^1, \ldots, u^m, v^1, \ldots, v^n$ is a basis of $U \oplus V$. So we can *add vector spaces*. Direct sum is associative and commutative, $U \oplus V = V \oplus U$, $U \oplus (V \oplus W) = (U \oplus V) \oplus W$.

We can also *multiply vector spaces*. For U, V vector spaces, the *tensor product* $U \otimes V$ is a natural vector space with $\dim(U \otimes V) = \dim U \cdot \dim V$. There is a bilinear operation

$$\otimes : U \times V \longrightarrow U \otimes V, \quad (u, v) \longmapsto u \otimes v.$$

If $u^1, \ldots, u^m$ and $v^1, \ldots, v^n$ are bases of U, V, then $\{u^i \otimes v^j : i = 1, \ldots, m,\ j = 1, \ldots, n\}$ is a basis of $U \otimes V$.

Formally, we may define

$$U \otimes V = \big\{\text{bilinear maps } \alpha : U^* \times V^* \longrightarrow \mathbb{R}\big\},$$

and for $u \in U$, $v \in V$, define $u \otimes v \in U \otimes V$ to be the bilinear map

$$u \otimes v : U^* \times V^* \longrightarrow \mathbb{R}, \quad u \otimes v : (\alpha, \beta) \longmapsto \alpha(u) \cdot \beta(v).$$

Tensor products are associative and commutative and distributive over direct sum, $U \otimes V = V \otimes U$, $U \otimes (V \otimes W) = (U \otimes V) \otimes W$, $U \otimes (V \oplus W) = (U \otimes V) \oplus (U \otimes W)$, just as you would expect.

Symmetric and exterior (antisymmetric) products

Let V be a vector space. Then we can form the *n-fold tensor product* $\bigotimes^n V = \overbrace{V \otimes \cdots \otimes V}^{n \text{ copies}}$. The symmetric group S_n acts on $\bigotimes^n V$ by permutations on the n factors, so that $\sigma \in S_n$ acts by

$$\sigma : v_1 \otimes \cdots \otimes v_n \longmapsto v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

for $v_1, \ldots, v_n \in V$. The n^{th} *symmetric power* $S^n V$ is the subspace of $\bigotimes^n V$ invariant under S_n, with $\dim S^n V = \binom{\dim V+n-1}{n}$

$$S^n V = \left\{ \boldsymbol{v} \in \bigotimes^n V : \sigma(\boldsymbol{v}) = \boldsymbol{v} \text{ for all } \sigma \in S_n \right\}.$$

The n^{th} *exterior power* $\Lambda^n V$ is the subspace of $\bigotimes^n V$ anti-invariant under S_n, with $\dim \Lambda^n V = \binom{\dim V}{n}$

$$\Lambda^n V = \left\{ \boldsymbol{v} \in \bigotimes^n V : \sigma(\boldsymbol{v}) = \operatorname{sign}(\sigma)\, \boldsymbol{v} \text{ for all } \sigma \in S_n \right\}.$$

For $n = 2$ we have $\bigotimes^2 V = S^2 V \oplus \Lambda^2 V$.
We can identify $\bigotimes^2 \mathbb{R}^n$ with $n \times n$ matrices, $S^2\mathbb{R}^n$ with symmetric matrices, and $\Lambda^2\mathbb{R}^n$ with antisymmetric matrices.

Symmetric and exterior products

There are projections $\Pi^S : \bigotimes^n V \to S^n V$ and $\Pi^\Lambda : \bigotimes^n V \to \Lambda^n V$ by symmetrization and antisymmetrization, given by

$$\Pi^S(\boldsymbol{v}) = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma(\boldsymbol{v}), \quad \Pi^\Lambda(\boldsymbol{v}) = \frac{1}{n!} \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, \sigma(\boldsymbol{v}).$$

The *symmetric product* $\odot$ is tensor product $\otimes$ followed by symmetrization Π^S, so for example
$v_1 \odot \cdots \odot v_n = \Pi^S(v_1 \otimes \cdots \otimes v_n)$ for $v_1, \ldots, v_n \in V$.
The *exterior product* or *wedge product* $\wedge$ is tensor product $\otimes$ followed by antisymmetrization Π^Λ, so for example we have
$\wedge : \Lambda^m V \times \Lambda^n V \to \Lambda^{m+n} V$, $\alpha \wedge \beta = \Pi^\Lambda(\alpha \otimes \beta)$.
Both $\odot, \wedge$ are associative. We have $\beta \odot \alpha = \alpha \odot \beta$ and $\beta \wedge \alpha = (-1)^{\deg \alpha \deg \beta} \alpha \wedge \beta$.

3.2. Operations on vector bundles; tensors

Now let X be a smooth manifold, and as in §1.5 consider vector bundles $E \to X$, $F \to X$, so that for all $x \in X$ the fibres E_x, F_x are vector spaces. The operations on vector spaces in §3.1 all make sense for vector bundles. So we can form the *dual vector bundle* E^* with $\mathrm{rank}\, E^* = \mathrm{rank}\, E$ and fibres $(E^*)_x = (E_x)^*$, the *direct sum vector bundle* $E \oplus F \to X$, with
$\mathrm{rank}(E \oplus F) = \mathrm{rank}\, E + \mathrm{rank}\, F$ and fibres $(E \oplus F)_x = E_x \oplus F_x$,
the *tensor product bundle* $E \otimes F \to X$, with
$\mathrm{rank}(E \otimes F) = \mathrm{rank}\, E \cdot \mathrm{rank}\, F$ and fibres $(E \otimes F)_x = E_x \otimes F_x$.
Given $E \to X$, we can form the *n-fold tensor product* $\bigotimes^n E \to X$, the n^{th} *symmetric power* $S^n E \to X$ and the n^{th} *exterior power* $\Lambda^n E \to X$, with fibres $\bigotimes^n(E_x), S^n(E_x), \Lambda^n(E_x)$.
We can take direct sums and tensor products of sections: if $e \in C^\infty(E)$, $f \in C^\infty(F)$ then $e \oplus f \in C^\infty(E \oplus F)$, and so on.

Tensor bundles and tensors

As in §2.1, any manifold X has two natural vector bundles, the tangent bundle $TX \to X$ and cotangent bundle $T^*X \to X$. So we can make many more bundles by direct sums, tensor products, symmetric products, and exterior products, of TX, T^*X.
The *tensor bundles* on X are $\bigotimes^k TX \otimes \bigotimes^l T^*X$ for $k, l \geqslant 0$ (where if $k = 0$ or $l = 0$ we omit that term). They are vector bundles on X, of rank $(\dim X)^{k+l}$.
A *tensor* T on X is a smooth section of some tensor bundle, $T \in C^\infty(\bigotimes^k TX \otimes \bigotimes^l T^*X)$.
This is very general, and includes many interesting geometric structures.

Examples of interesting classes of tensors

Example

A *vector field* v on X is a section of TX. This is a tensor with $k=1$ and $l=0$.

Example

An *l-form* for $l \geqslant 0$, or *exterior form*, on X, is a section of $\Lambda^l T^*X$. As $\operatorname{rank} \Lambda^l T^*X = \binom{\dim X}{l}$, this is only nonzero for $l=0,\ldots,\dim X$. Since $\Lambda^l T^*X$ is a subbundle of $\bigotimes^l T^*X = \bigotimes^0 TX \otimes \bigotimes^l T^*X$, l-forms are tensors with $k=0$.

Example

A *Riemannian metric* g is a smooth section of S^2T^*X such that $g|_x \in S^2T_x^*X$ is a positive definite quadratic form on T_xX for all $x \in X$. As $S^2T^*X \subset \bigotimes^2 T^*X$, this is a tensor with $k=0$, $l=2$.

3.3. Index notation for tensors

Here is some useful notation for tensors, introduced by physicists. Let X be an n-manifold, and $T \in C^\infty(\bigotimes^k TX \otimes \bigotimes^l T^*X)$ a tensor of type (k,l) on X. Let $(x^1,\ldots,x^n)$ be local coordinates on an open set $U \subseteq X$. (For consistent notation, we use superscripts x^i rather than subscripts x_i; x^i means the $i^{\rm th}$ variable, not a power of x.) Then $\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^n}$ are a basis of sections of TX on U, and $\mathrm{d}x^1,\ldots,\mathrm{d}x^n$ a basis of sections of T^*X on U. Hence we may write

$$T|_U = \sum_{\substack{a_1,\ldots,a_k=1,\ldots,n\\ b_1,\ldots,b_l=1,\ldots,n}} T^{a_1a_2\cdots a_k}_{b_1b_2\cdots b_l} \frac{\partial}{\partial x^{a_1}} \otimes\cdots\otimes \frac{\partial}{\partial x^{a_k}} \otimes \mathrm{d}x^{b_1} \otimes\cdots\otimes \mathrm{d}x^{b_l}. \quad (3.1)$$

Here $T^{a_1a_2\cdots a_k}_{b_1b_2\cdots b_l} : U \to \mathbb{R}$ is a smooth function for all values of $a_1,\ldots,a_k,b_1,\ldots,b_l \in \{1,\ldots,n\}$.

Thus, on U the tensor T is uniquely determined by the real functions $T^{a_1\cdots a_k}_{b_1\cdots b_l}$ for all a_i, b_j, and vice versa. So we can identify T with such n^{k+l}-tuples of functions $\bigl(T^{a_1\cdots a_k}_{b_1\cdots b_l}\bigr)^{a_1,\ldots,a_k=1,\ldots,n}_{b_1,\ldots,b_l=1,\ldots,n}$, which we can think of as a kind of generalized matrix.

If $(\tilde x^1,\ldots,\tilde x^n)$ is another coordinate system on $\tilde U\subseteq X$, and $\tilde T^{a_1\cdots a_k}_{b_1\cdots b_l}$ the corresponding functions from $T\vert_{\tilde U}$, then using $\frac{\partial}{\partial \tilde x^i}=\sum_{j=1}^n\frac{\partial x^j}{\partial\tilde x^i}\cdot\frac{\partial}{\partial x^j}$, $\mathrm{d}\tilde x^i=\sum_{j=1}^n\frac{\partial\tilde x^i}{\partial x^j}\cdot\mathrm{d}x^j$, on $U\cap\tilde U$ we have

$$\tilde T^{a_1\cdots a_k}_{b_1\cdots b_l}=\sum_{\substack{c_1,\ldots,c_k=1,\ldots,n\\ d_1,\ldots,d_l=1,\ldots,n}}\frac{\partial\tilde x^{a_1}}{\partial x^{c_1}}\cdots\frac{\partial\tilde x^{a_k}}{\partial x^{c_k}}\cdot\frac{\partial x^{d_1}}{\partial\tilde x^{b_1}}\cdots\frac{\partial x^{d_l}}{\partial\tilde x^{b_l}}\cdot T^{c_1\cdots c_k}_{d_1\cdots d_l}. \quad (3.2)$$

This tells you how the tuples $\bigl(T^{a_1\cdots a_k}_{b_1\cdots b_l}\bigr)^{a_1,\ldots,a_k=1,\ldots,n}_{b_1,\ldots,b_l=1,\ldots,n}$ transform under change of coordinates.

Upper indices T^a are called *contravariant* (*vector*) *indices*. Lower indices T_b are called *covariant* (*1-form*) *indices*.

Tensors
Exterior forms
Some linear algebra
Operations on vector bundles; tensors
Index notation for tensors
The Lie bracket of vector fields
Exponentiating vector fields

In the index notation, we write the tensor T (on all of X, not just on one coordinate chart $U\subseteq X$) as $T^{a_1\cdots a_k}_{b_1\cdots b_l}$. We could interpret this in several ways. We could view it just as a formal symbol, telling us that T is a section of $\bigotimes^k TX\otimes\bigotimes^l T^*X$. Or, we could understand it to mean 'every time we have coordinates $(x^1,\ldots,x^n)$ on $U\subseteq X$, then we get an n^{k+l}-tuple $\bigl(T^{a_1\cdots a_k}_{b_1\cdots b_l}\bigr)^{a_1,\ldots,a_k=1,\ldots,n}_{b_1,\ldots,b_l=1,\ldots,n}$ of smooth functions $T^{a_1\cdots a_k}_{b_1\cdots b_l}:U\to\mathbb{R}$ as in (3.1), and under change of coordinates, these n^{k+l}-tuples transform as in (3.2)'.

Examples of tensor notation

Example

A *vector field* v on X is written v^a. In coordinates $(x^1, \ldots, x^n)$ this means functions $(v^1, \ldots, v^n)$ with $v = v^1 \frac{\partial}{\partial x^1} + \cdots + v^n \frac{\partial}{\partial x^n}$.

Example

An *l-form* on X is a tensor $\alpha_{b_1 \cdots b_l}$ with $\alpha_{b_1 \cdots b_{i-1} b_j b_{i+1} \cdots b_{j-1} b_i b_{j+1} \cdots b_l} = -\alpha_{b_1 \cdots b_l}$ for all $1 \leqslant i < j \leqslant l$. So a 2-form is α_{ab} with $\alpha_{ba} = -\alpha_{ab}$.

Example

A *Riemannian metric* is a tensor g_{ab} with $g_{ab} = g_{ba}$, with $(g_{ab})_{a,b=1,\ldots,n}$ a positive definite $n \times n$ matrix of functions.

Index notation makes it easy to describe (anti)symmetries of tensors, by permuting indices.

The Einstein summation convention

As TX, T^*X are dual, there is a dual pairing $TX \times T^*X \to \mathbb{R}$. This induces vector bundle morphisms $\bigotimes^{k+1} TX \otimes \bigotimes^{l+1} T^*X \to \bigotimes^k TX \otimes \bigotimes^l T^*X$ by contracting together a TX and a T^*X factor (need to specify which factors). In index notation, this is done by the *Einstein summation convention*: if an index c occurs twice in a tensor in a formula, once as an upper and once as a lower index, then (thinking in terms of tuples of functions) we are to sum the index c from $1, \ldots, n = \dim X$, even though the sum $\sum_{c=1}^n$ is not written.

Example

Let $v \in C^\infty(TX)$ be a vector field, and $\alpha \in C^\infty(T^*X)$ a 1-form. In index notation we write $v = v^a$, $\alpha = \alpha_b$.
Then $v^a\alpha_b$ in index notation means $v \otimes \alpha \in C^\infty(TX \otimes T^*X)$. But $v^a\alpha_a$ means the smooth function $\alpha(v) : X \to \mathbb{R}$. In coordinates, $v^a\alpha_a$ means $v^1\alpha_1 + \cdots + v^n\alpha_n$.

Example

Let $v, w \in C^\infty(TX)$ be vector fields, and $g \in C^\infty(S^2T^*X)$ a Riemannian metric. Then $v = v^a$, $w = w^b$, $g = g_{ab}$ in index notation, and $g_{ab}v^aw^b$ means the function $g(v, w)$, the inner product of v, w using g, and $g_{ab}v^av^b$ means the function $|v|^2$.

3.4. The Lie bracket of vector fields

In the next sections we will discuss various ways in which we can *differentiate* tensors, or more general sections of vector bundles. One of the simplest of these is the Lie bracket of vector fields.

Definition

Let X be a manifold, and $v, w \in C^\infty(TX)$ be vector fields on X. We will define a vector field $[v, w] \in C^\infty(TX)$ called the Lie bracket of v and w. In local coordinates $(x^1, \ldots, x^n)$ on $U \subseteq X$, this is given in index notation by the formula

$$[v, w]^a = v^b \frac{\partial w^a}{\partial x^b} - w^b \frac{\partial v^a}{\partial x^b}. \tag{3.3}$$

That is, if $v = v^1 \frac{\partial}{\partial x^1} + \cdots + v^n \frac{\partial}{\partial x^n}$ and $w = w^1 \frac{\partial}{\partial x^1} + \cdots + w^n \frac{\partial}{\partial x^n}$, then $[v, w] = u^1 \frac{\partial}{\partial x^1} + \cdots + u^n \frac{\partial}{\partial x^n}$, where

$$u^a = \sum_{b=1}^n v^b \frac{\partial w^a}{\partial x^b} - w^b \frac{\partial v^a}{\partial x^b}. \tag{3.4}$$

Exercise 3.1

Show that the Lie bracket $[v, w]$ *in* (3.3) *is well-defined. That is, as a vector field it is independent of the choice of local coordinates* $(x^1, \ldots, x^n)$ *used to define it.*

Proposition 3.2

The Lie bracket of vector fields satisfies $[u, v] = -[v, u]$ *and*

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0 \tag{3.5}$$

for all vector fields $u, v, w \in C^\infty(TX)$.

Equation (3.5) is called the *Jacobi identity*. It means that vector fields $C^\infty(TX)$ are an (infinite-dimensional) Lie algebra.

Lie derivatives of tensors

Definition

Let X be a manifold, $v \in C^\infty(TX)$ be a vector field, and $T \in C^\infty(\bigotimes^k TX \otimes \bigotimes^l T^*X)$ a tensor. We will define a tensor $\mathcal{L}_v T \in C^\infty(\bigotimes^k TX \otimes \bigotimes^l T^*X)$ called the *Lie derivative of* T *along* v. In local coordinates $(x^1, \ldots, x^n)$ on $U \subseteq X$, this is given in index notation by the formula

$$\begin{aligned}(\mathcal{L}_v T)^{a_1\cdots a_k}_{b_1\cdots b_l} = v^c \tfrac{\partial}{\partial x^c} T^{a_1\cdots a_k}_{b_1\cdots b_l} &- \textstyle\sum_{i=1}^k T^{a_1\cdots a_{i-1} c a_{i+1}\cdots a_k}_{b_1\cdots b_l} \frac{\partial v^{a_i}}{\partial x^c} \\ &+ \textstyle\sum_{j=1}^l T^{a_1\cdots a_k}_{b_1\cdots b_{j-1} c b_{j+1}\cdots b_l} \frac{\partial v^c}{\partial x^{b_j}}.\end{aligned} \tag{3.6}$$

This is well-defined, i.e. independent of the choice of coordinates $(x^1, \ldots, x^n)$. If $T = w$ is a vector field then $\mathcal{L}_v w = [v, w]$.

We can think of $\mathcal{L}_v T$ as 'the derivative of T in the direction v'. But note that (3.6) involves derivatives of v as well as T, so $\mathcal{L}_v T$ is not pointwise linear in v. That is, in general $\mathcal{L}_{fv+gw}T \neq f\mathcal{L}_v T + g\mathcal{L}_w T$ for vector fields v, w and functions $f, g : X \to \mathbb{R}$.

Example

In coordinates $(x^1, \ldots, x^n)$, take $v = \frac{\partial}{\partial x^i}$, so that $v^1, \ldots, v^n$ are $v^a = 1$ for $a = i$ and $v^a = 0$ otherwise. Then (3.6) becomes

$$(\mathcal{L}_v T)^{a_1\cdots a_k}_{b_1\cdots b_l} = \frac{\partial}{\partial x^i} T^{a_1\cdots a_k}_{b_1\cdots b_l},$$

as you would expect.

3.5. Exponentiating vector fields

Let X be a compact manifold (for simplicity), and $v \in C^\infty(TX)$ a vector field. A *flow-line* of v is a smooth map $\gamma : \mathbb{R} \to X$ satisfying the differential equation $\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) = v|_{\gamma(t)} \in T_{\gamma(t)}V$ for all $t \in \mathbb{R}$. Results on o.d.e.s imply that for each $x \in X$, there is a unique flow-line γ_x with $\gamma_x(0) = x$. Here we need X compact so that flow-lines cannot 'fall off the edge of X', so that γ could only be defined on an open interval, not all of $\mathbb{R}$. (Consider $X = (0, 1)$, noncompact and $v = \frac{\partial}{\partial x}$. Then γ is only defined on $(-x, 1 - x)$.) Define $\exp(tv) : X \to X$ for $t \in \mathbb{R}$ by $\exp(tv) : x \mapsto \gamma_x(t)$, for γ_x the flow-line of v with $\gamma_x(0) = x$ as above. Then $\exp(tv)$ is a diffeomorphism of X depending smoothly on t, with $\exp(0) = \mathrm{id}_X$ and $\exp(sv) \circ \exp(tv) = \exp((s + t)v)$ for $s, t \in \mathbb{R}$.

If $T \in C^\infty(\bigotimes^k TX \otimes \bigotimes^l T^*X)$ is a tensor on X, then $\exp(tv)^*(T)$ is a tensor depending smoothly on $t \in \mathbb{R}$. One can show that

$$\mathcal{L}_v T = \frac{\mathrm{d}}{\mathrm{d}t}\left[\exp(tv)^*(T)\right]\Big|_{t=0}.$$

That is, $\mathcal{L}_v T$ measures the infinitesimal change of T under the flow of v.

Introduction to Differential Geometry

Lecture 4 of 10: Exterior forms

Dominic Joyce, Oxford University
September 2019

2019 Nairobi Workshop in Algebraic Geometry

These slides available at
http://people.maths.ox.ac.uk/∼joyce/

Plan of talk:

4. Exterior forms

4.1. Exterior forms and the de Rham differential

Let X be a manifold, of dimension n. Then we have vector bundles $\Lambda^k T^*X$ for $k = 0, 1, \ldots, n$ (note that $\Lambda^k T^*X = 0$ for $k > n$). Sections α of $\Lambda^k T^*X$ are called *k-forms*, and form a (generally infinite-dimensional) vector space $C^\infty(\Lambda^k T^*X)$. In index notation $\alpha = \alpha_{a_1 \cdots a_k}$, and is antisymmetric in the indices $a_1, \ldots, a_k$ (i.e. if you exchange any two a_i, a_j, you change the sign).

As in §3.1–§3.2 we have the *exterior product* (*wedge product*)

$$\wedge : C^\infty(\Lambda^k T^*X) \times C^\infty(\Lambda^l T^*X) \longrightarrow C^\infty(\Lambda^{k+l} T^*X),$$

acting in index notation by

$$(\alpha \wedge \beta)_{a_1 \cdots a_{k+l}} = \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \mathrm{sign}(\sigma) \alpha_{a_{\sigma(1)} \cdots a_{\sigma(k)}} \beta_{a_{\sigma(k+1)} \cdots a_{\sigma(k+l)}}. \quad (4.1)$$

Pullback of forms by smooth maps

Let $f : X \to Y$ be a smooth map of manifolds. As in §2.2 we have $Tf : TX \to TY$, which can be interpreted as a vector bundle morphism $\mathrm{d}f : TX \to f^*(TY)$ on X, with a dual morphism $(\mathrm{d}f)^* : f^*(T^*Y) \to T^*X$. Taking exterior powers gives vector bundle morphisms on X

$$\Lambda^k(\mathrm{d}f)^* : f^*(\Lambda^k T^*Y) \longrightarrow \Lambda^k T^*X.$$

Let $\alpha \in C^\infty(\Lambda^k T^*Y)$ be a k-form on Y. Then we have a pullback $f^{-1}(\alpha) \in C^\infty(f^*(\Lambda^k T^*Y))$ on X. Define the *pullback k-form* to be

$$f^*(\alpha) = \Lambda^k(\mathrm{d}f)^*\bigl[f^{-1}(\alpha)\bigr] \in C^\infty(\Lambda^k T^*X).$$

Pullback is (contravariantly) functorial, $(g \circ f)^*(\beta) = f^* \circ g^*(\beta)$ for smooth $g : Y \to Z$ and $\beta \in C^\infty(\Lambda^k T^*Z)$.

If $X \subseteq Y$ is a submanifold, we write $\alpha|_X$ for $i^*(\alpha)$, with $i : X \hookrightarrow Y$ the inclusion.

Definition

The *de Rham differential* $\mathrm{d} : C^\infty(\Lambda^k T^*X) \longrightarrow C^\infty(\Lambda^{k+1} T^*X)$ for $k \geqslant 0$ is defined in local coordinates $(x^1, \dots, x^n)$ on $U \subseteq X$, using index notation, by the formula

$$(\mathrm{d}\alpha)_{a_1 \cdots a_{k+1}} = \sum_{i=1}^{k+1} (-1)^{i-1} \frac{\partial}{\partial x^{a_i}} \alpha_{a_1 \cdots a_{i-1} a_{i+1} \cdots a_{k+1}}. \tag{4.2}$$

Exercise 4.1

Show that the de Rham differential is well-defined. That is, as a $k+1$-form, $\mathrm{d}\alpha$ is independent of the choice of local coordinates $(x^1, \dots, x^n)$ used to define it.

Properties of the de Rham differential

From equations (4.1) and (4.2) we can prove:

Proposition 4.2

For all forms α, β, γ on X, the de Rham differential satisfies

$$\mathrm{d} \circ \mathrm{d}\alpha = 0, \quad \mathrm{d}(\beta \wedge \gamma) = (\mathrm{d}\beta) \wedge \gamma + (-1)^{\deg \beta} \beta \wedge (\mathrm{d}\gamma). \quad (4.3)$$

Proposition 4.3

Let $f : X \to Y$ be smooth map of manifolds and $\alpha \in C^\infty(\Lambda^k T^* Y)$. Then

$$\mathrm{d}(f^*(\alpha)) = f^*(\mathrm{d}\alpha). \quad (4.4)$$

4.2. Homology and cohomology

A reminder of some algebraic topology: let X be a topological space, and $\mathbb{F}$ a field (for simplicity). Then we can define the *homology groups* $H_k(X, \mathbb{F})$ and *cohomology groups* $H^k(X, \mathbb{F})$ for $k \in \mathbb{N}$, which are vector spaces over $\mathbb{F}$, with $H^k(X, \mathbb{F}) \cong H_k(X, \mathbb{F})^*$
If $f : X \to Y$ is continuous there are functorial *pushforward maps* $f_* : H_k(X, \mathbb{F}) \to H_k(Y, \mathbb{F})$ on homology, and *pullback maps* $f^* : H^k(Y, \mathbb{F}) \to H^k(X, \mathbb{F})$ on cohomology. There are *cup products* $\cup : H^k(X, \mathbb{F}) \times H^l(X, \mathbb{F}) \to H^{k+l}(X, \mathbb{F})$ making $H^*(X, \mathbb{F})$ into a supercommutative graded algebra.
If X is a compact, oriented manifold of dimension n, then *Poincaré duality* says that $H^k(X, \mathbb{F}) \cong H_{n-k}(X, \mathbb{F})$.
The *Betti numbers* of X are $b^k(X) = \dim H^k(X, \mathbb{R})$.
Homology and cohomology are important topological invariants of a space, one of the most basic things you can compute.

De Rham cohomology

Definition

Let X be a smooth manifold. The *de Rham cohomology group* $H^k_{\rm dR}(X,\mathbb{R})$ *of* X, for $k=0,\ldots,\dim X$, is

$$H^k_{\rm dR}(X,\mathbb{R})=\frac{\mathrm{Ker}\bigl(\mathrm{d}:C^\infty(\Lambda^k T^*X)\longrightarrow C^\infty(\Lambda^{k+1}T^*X)\bigr)}{\mathrm{Im}\bigl(\mathrm{d}:C^\infty(\Lambda^{k-1}T^*X)\longrightarrow C^\infty(\Lambda^k T^*X)\bigr)}.$$

This makes sense as $\mathrm{d}\circ\mathrm{d}=0$, by Proposition 4.2. The second equation of (4.3) implies that we can define a *cup product*

$$\begin{aligned}\cup: H^k_{\rm dR}(X,\mathbb{R})\times H^l_{\rm dR}(X,\mathbb{R})&\longrightarrow H^{k+l}_{\rm dR}(X,\mathbb{R}),\\ (\beta+\mathrm{Im}\,\mathrm{d})\cup(\gamma+\mathrm{Im}\,\mathrm{d})&\longmapsto\beta\wedge\gamma+\mathrm{Im}\,\mathrm{d},\end{aligned}$$

which is associative and supercommutative as $\wedge$ is.
If X is compact then $H^k_{\rm dR}(X,\mathbb{R})$ is finite-dimensional.

If $f:X\to Y$ is a smooth map of manifolds then Proposition 4.3 implies that we can define *pullback maps*

$$f^*:H^k_{\rm dR}(Y,\mathbb{R})\longrightarrow H^k_{\rm dR}(X,\mathbb{R}),\quad f^*(\alpha+\mathrm{Im}\,\mathrm{d})=f^*(\alpha)+\mathrm{Im}\,\mathrm{d}.$$

These pullback maps are *independent of* $f:X\to Y$ *up to smooth (or continuous) deformation*. That is, if $g:X\times[0,1]\to Y$ is smooth and $f_0,f_1:X\to Y$ are $f_0(x)=g(x,0)$, $f_1(x)=g(x,1)$ then $f_0^*=f_1^*:H^k_{\rm dR}(Y,\mathbb{R})\to H^k_{\rm dR}(X,\mathbb{R})$.

Theorem (The de Rham Theorem)

There are natural isomorphisms $H^k_{\rm dR}(X,\mathbb{R})\cong H^k(X,\mathbb{R})$, *where* $H^k(X,\mathbb{R})$ *is the* $k^{\rm th}$ *real cohomology group of the underlying topological space* X. *These isomorphisms are compatible with cup products and pullbacks on* $H^*_{\rm dR}(-,\mathbb{R})$ *and* $H^*(-,\mathbb{R})$.

Cohomology of products, the Künneth Theorem

Let X, Y be topological spaces, and $\mathbb{F}$ a field. We have a product topological space $X \times Y$ with projections $\pi_X : X \times Y \to X$, $\pi_Y : X \times Y \to Y$.

Theorem (The Künneth Theorem)

For each $k \geqslant 0$ there is an isomorphism

$$\bigoplus_{i,j\geqslant 0:i+j=k} H^i(X,\mathbb{F}) \otimes_{\mathbb{F}} H^j(Y,\mathbb{F}) \longrightarrow H^k(X \times Y,\mathbb{F})$$

acting by $\bigoplus_{i+j=k} \alpha^i \otimes \beta^j \longmapsto \sum_{i+j=k} \pi_X^*(\alpha^i) \cup \pi_Y^*(\beta^j)$, for $\alpha^i \in H^i(X,\mathbb{F})$ and $\beta^j \in H^j(Y,\mathbb{F})$.

In particular, this applies to de Rham cohomology of products of manifolds.

Betti numbers and the Euler characteristic

Let X be a manifold (usually compact). The *Betti numbers* of X are $b^k(X) = \dim H^k_{\mathrm{dR}}(X,\mathbb{R})$. The *Euler characteristic* is $\chi(X) = \sum_{k=0}^{\dim X}(-1)^k b^k(X)$. They are topological invariants of X. If X is compact then $H^k_{\mathrm{dR}}(X,\mathbb{R})$ is finite-dimensional, so these are well defined. If X is compact and odd-dimensional then $\chi(X) = 0$.
The Künneth Theorem implies that $\chi(X \times Y) = \chi(X)\chi(Y)$.
The Euler characteristic is very important, and crops up in many different places. For example, if X is a compact manifold then the number of zeroes of a generic vector field v on X, counted with multiplicity, is $\chi(X)$.
The Gauss–Bonnet Theorem says that if (X, g) is a compact Riemannian 2-manifold with Gaussian curvature κ then $\int_X \kappa \,\mathrm{d}V_g = 2\pi\chi(X)$.

4.3. Examples

Example

The de Rham cohomology of $\mathbb{R}^n$ for $n \geqslant 0$ is

$$H^k_{\rm dR}(\mathbb{R}^n,\mathbb{R}) = \begin{cases} \mathbb{R}, & k=0, \\ 0, & k>0. \end{cases}$$

When $n=0$, so that $\mathbb{R}^0 = *$ is a point, this is immediate from the definitions. To prove it when $n>0$, consider the smooth maps $i: * \to \mathbb{R}^n$, $i: * \mapsto (0,\ldots,0)$, and $\pi: \mathbb{R}^n \to *$, $\pi: (x_1,\ldots,x_n) \mapsto *$. These induce maps $i^*: H^k_{\rm dR}(\mathbb{R}^n,\mathbb{R}) \to H^k_{\rm dR}(*,\mathbb{R})$ and $\pi^*: H^k_{\rm dR}(*,\mathbb{R}) \to H^k_{\rm dR}(\mathbb{R}^n,\mathbb{R})$. Since $\pi \circ i = \mathrm{id}: * \to *$ we see that $i^* \circ \pi^*$ is the identity on $H^k_{\rm dR}(*,\mathbb{R})$. Conversely, although $i \circ \pi \neq \mathrm{id}: \mathbb{R}^n \to \mathbb{R}^n$, we can smoothly deform $i \circ \pi$ to id, so $\pi^* \circ i^*$ is the identity on $H^k_{\rm dR}(\mathbb{R}^n,\mathbb{R})$. Hence i^*, π^* are inverse, and $H^k_{\rm dR}(\mathbb{R}^n,\mathbb{R}) \cong H^k_{\rm dR}(*,\mathbb{R})$.

Example

The de Rham cohomology of $\mathcal{S}^n$ for $n>0$ is

$$H^k_{\rm dR}(\mathcal{S}^n,\mathbb{R}) \cong \begin{cases} \mathbb{R}, & k=0 \text{ or } k=n, \\ 0, & \text{otherwise.} \end{cases} \tag{4.5}$$

Example

The de Rham cohomology of T^n for $n \geqslant 0$ is

$$H^k_{\rm dR}(T^n,\mathbb{R}) \cong \mathbb{R}^{\binom{n}{k}}.$$

This follows from (4.5) for $H^*_{\rm dR}(\mathcal{S}^1,\mathbb{R})$ and the Künneth Theorem.

Considering $H^1_{\rm dR}(-,\mathbb{R})$ we see that:

Corollary

There is no diffeomorphism $\mathcal{S}^n \cong T^n$ for $n \geqslant 2$.

De Rham cohomology is useful for distinguishing manifolds.