Introduction to Differential Geometry

Lecture 7 of 10: Riemannian manifolds

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These slides available at
http://people.maths.ox.ac.uk/∼joyce/

Plan of talk:

7. Riemannian manifolds
7.1. Riemannian metrics

In Euclidean geometry on $\mathbb{R}^n$, by Pythagoras' Theorem the distance between two points $\boldsymbol{x} = (x^1, \ldots, x^n)$ and $\boldsymbol{y} = (y^1, \ldots, y^n)$ is

$$d_{\mathbb{R}^n}(\boldsymbol{x}, \boldsymbol{y}) = \left[(x^1 - y^1)^2 + \cdots + (x^n - y^n)^2\right]^{1/2}.$$

Note that *squares of distances*, rather than distances, behave nicely, algebraically. If $\gamma = (\gamma^1, \ldots, \gamma^n) : [0,1] \to \mathbb{R}^n$ is a smooth path in $\mathbb{R}^n$, then the *length* of γ is

$$l(\gamma) = \int_0^1 \left[\left(\frac{\mathrm{d}\gamma^1}{\mathrm{d}t}\right)^2 + \cdots + \left(\frac{\mathrm{d}\gamma^n}{\mathrm{d}t}\right)^2\right]^{1/2} \mathrm{d}t.$$

Note that this is unchanged under reparametrizations of $[0,1]$: replacing t by an alternative coordinate $\tilde{t}$ multiplies $\frac{\mathrm{d}\gamma^i}{\mathrm{d}t}$ by $\frac{\mathrm{d}t}{\mathrm{d}\tilde{t}}$ and $\mathrm{d}t$ by $\frac{\mathrm{d}\tilde{t}}{\mathrm{d}t}$, which cancel.

Regarding $\gamma : [0,1] \to \mathbb{R}^n$ as a smooth map of manifolds, we have

$$l(\gamma) = \int_0^1 g_{\mathbb{R}^n}|_{\gamma(t)}\left(\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t), \frac{\mathrm{d}\gamma}{\mathrm{d}t}(t)\right)^{1/2} \mathrm{d}t,$$

where $\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) \in T_{\gamma(t)}\mathbb{R}^n \cong \mathbb{R}^n$, and $g_{\mathbb{R}^n}|_{\boldsymbol{x}} = (\mathrm{d}x^1)^2 + \cdots + (\mathrm{d}x^n)^2$ in $S^2 T^*_{\boldsymbol{x}}\mathbb{R}^n \cong S^2(\mathbb{R}^n)^*$ for $\boldsymbol{x} \in \mathbb{R}^n$, so that $g_{\mathbb{R}^n} \in C^\infty(S^2 T^*\mathbb{R}^n)$. This is a simple example of a Riemannian metric on a manifold, being used to define lengths of curves.

Definition

Let X be a manifold. A *Riemannian metric* g (or just *metric*) is a smooth section of S^2T^*X such that $g|_x \in S^2T^*_xX$ is a positive definite quadratic form on T_xX for all $x \in X$. In index notation we write $g = g_{ab}$, with $g_{ab} = g_{ba}$. We call (X, g) a *Riemannian manifold*. Let $\gamma : [0,1] \to X$ be a smooth map, considered as a curve in X. The *length* of γ is

$$l(\gamma) = \int_0^1 g|_{\gamma(t)}\Bigl(\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t), \frac{\mathrm{d}\gamma}{\mathrm{d}t}(t)\Bigr)^{1/2}\mathrm{d}t.$$

If X is (path-)connected, we can define a metric d_g on X, in the sense of metric spaces, by

$$d_g(x,y) = \inf_{\substack{\gamma:[0,1]\to X\ C^\infty:\\ \gamma(0)=x,\ \gamma(1)=y}} l(\gamma).$$

Roughly, $d_g(x,y)$ is the length of the shortest curve γ from x to y.

Restricting metrics to submanifolds

Let $i : X \to Y$ be an immersion or an embedding, so that X is a submanifold of Y, and $g \in C^\infty(S^2T^*Y) \subseteq C^\infty(\bigotimes^2 T^*Y)$ be a Riemannian metric on Y. Pulling back gives
$i^\sharp(g) \in C^\infty(S^2 i^*(T^*Y))$. We have a vector bundle morphism $\mathrm{d}i : TX \to i^*(TY)$ on X, which is injective as i is an immersion, and a dual surjective morphism $(\mathrm{d}i)^* : i^*(T^*Y) \to T^*X$.
Symmetrizing gives $S^2(\mathrm{d}i)^* : S^2 i^*(T^*Y) \to S^2T^*X$.
Define $i^*(g) = (S^2(\mathrm{d}i)^*)(i^\sharp(g)) \in C^\infty(S^2T^*X)$. This is defined for any smooth map $i : X \to Y$. But if i is an immersion, so that $\mathrm{d}i : TX \to i^*(TY)$ is injective, then g positive definite implies $i^*(g)$ positive definite, so $i^*(g)$ is a Riemannian metric on X. We call it the *pullback* or *restriction* of g to X, and write it as $g|_X$.

Submanifolds of Euclidean space

Example

Define $g_{\mathbb{R}^n} = (\mathrm{d}x^1)^2 + \cdots + (\mathrm{d}x^n)^2$ in $C^\infty(S^2(\mathbb{R}^n)^*)$. This is a Riemannian metric on $\mathbb{R}^n$, which induces the usual notions of lengths of curves and distance in Euclidean geometry. We call $g_{\mathbb{R}^n}$ the *Euclidean metric* on $\mathbb{R}^n$.

Example

Let X be any submanifold of $\mathbb{R}^n$. Then $g_{\mathbb{R}^n}|_X$ is a Riemannian metric on X.

Since any manifold X can be embedded in $\mathbb{R}^n$ for $n \gg 0$ (Whitney Embedding Theorem), this implies

Corollary

Any manifold X admits a Riemannian metric.

These ideas are important even in really basic applied mathematics, physics, geography, etc:

Example

Model the surface of the earth as a sphere $\mathcal{S}^2_R$ of radius $R = 6,371$km about 0 in $\mathbb{R}^3$. Then the Riemannian metric $g_E = g_{\mathbb{R}^3}|_{\mathcal{S}^2_R}$ determines lengths of paths on the earth. Define spherical polar coordinates (θ,φ) (latitude and longitude) on $\mathcal{S}^2_R \setminus \{N,S\}$ by $\boldsymbol{x}(\theta,\varphi) = (R\sin\theta\cos\varphi, R\sin\theta\sin\varphi, R\cos\theta)$. Then

$$
\begin{aligned}
g_E &= ((\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2)|_{\mathcal{S}^2} \\
&= (\mathrm{d}(R\sin\theta\cos\varphi))^2 + (\mathrm{d}(R\sin\theta\sin\varphi))^2 + (\mathrm{d}(R\cos\theta))^2 \\
&= R^2\big[(\cos\theta\cos\varphi\,\mathrm{d}\theta - \sin\theta\sin\varphi\,\mathrm{d}\varphi)^2 \\
&+ (\cos\theta\sin\varphi\,\mathrm{d}\theta + \sin\theta\cos\varphi\,\mathrm{d}\varphi)^2 + (-\sin\theta\,\mathrm{d}\theta)^2\big] \\
&= R^2[\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2].
\end{aligned}
$$

7.2. The Levi-Civita connection

Any Riemannian manifold (X, g) has a natural connection ∇ on TX, called the *Levi-Civita connection*. This is known as the 'Fundamental Theorem of Riemannian Geometry'.

Theorem (The Fundamental Theorem of Riemannian Geometry)

Let (X, g) be a Riemannian manifold. Then there exists a unique connection ∇ on TX, such that ∇ is torsion-free, and the induced connection ∇' on $\bigotimes^2 T^*X$ satisfies $\nabla' g = 0$. We call ∇ the **Levi-Civita connection** *of g.*

Usually we write ∇ for all the induced connections on $\bigotimes^k TX \otimes \bigotimes^l T^*X$, without comment. So ∇ allows us to differentiate all tensors T on a Riemannian manifold (X, g), without making any arbitrary choices.

Proof of the Fundamental Theorem

The theorem is local in X, so it is enough to prove it in coordinates $(x^1, \ldots, x^n)$ defined on open $U \subseteq X$. Let ∇ be a connection on TX, with Christoffel symbols $\Gamma^a_{bc} : U \to \mathbb{R}$, as in §6.5, so $g = \sum_{a,b=1}^n g_{ab}\, \mathrm{d}x^a \otimes \mathrm{d}x^b$ with $g_{ab} = g_{ba}$. Then $\bigl(g_{ab}\bigr)_{a,b=1}^n$ is a symmetric, positive-definite, invertible matrix of functions on U. Write $\bigl(g^{ab}\bigr)_{a,b=1}^n$ for the inverse matrix of functions.
Then ∇ torsion-free is equivalent to

$$\Gamma^a_{bc} = \Gamma^a_{cb}, \tag{7.1}$$

and $\nabla'_c g_{ab} = 0$ is equivalent to

$$\tfrac{\partial}{\partial x_c} g_{ab} - \Gamma^d_{ac} g_{db} - \Gamma^d_{bc} g_{ad} = 0. \tag{7.2}$$

Calculation shows that (7.1)–(7.2) have a unique solution for Γ^a_{bc}:

$$\Gamma^a_{bc} = \tfrac{1}{2} g^{ad} \bigl(\tfrac{\partial}{\partial x_c} g_{db} + \tfrac{\partial}{\partial x_b} g_{dc} - \tfrac{\partial}{\partial x_d} g_{bc} \bigr). \tag{7.3}$$

This gives the unique connection ∇ we want. □

7.3. The Riemann curvature tensor

Let (X,g) be a Riemannian manifold. Then by the FTRG we have a natural connection ∇ on TX. As in §6.3, the curvature of ∇ is $R\in C^\infty(TX\otimes T^*X\otimes\Lambda^2T^*X)$, which is called the *Riemann curvature tensor* of g. In index notation $R=R^a_{bcd}$, and it is characterized by the formula for all vector fields $u,v,w\in C^\infty(TX)$

$$\begin{aligned}&R^a_{bcd}u^bv^cw^d\\ &=v^c\nabla_c(w^d\nabla_du^a)-w^c\nabla_c(v^d\nabla_cu^a)-(v^c\nabla_cw^d-w^c\nabla_cv^d)\nabla_du^a\\ &=v^cw^d(\nabla_c\nabla_du^a-\nabla_d\nabla_cu^a),\end{aligned}\tag{7.4}$$

using $[v,w]^d=v^c\nabla_cw^d-w^c\nabla_cv^d$ as ∇ is torsion-free. Thus

$$R^a_{bcd}u^b=(\nabla_c\nabla_d-\nabla_d\nabla_c)u^a.\tag{7.5}$$

Riemann curvature in coordinates

Let (X,g) be a Riemannian manifold with Riemann curvature R, and $(x^1,\ldots,x^n)$ be coordinates on $U\subseteq X$. From (6.7) we have

$$R^a_{bcd}=\frac{\partial}{\partial x^c}\Gamma^a_{bd}-\frac{\partial}{\partial x^d}\Gamma^a_{bc}+\Gamma^a_{ec}\Gamma^e_{bd}-\Gamma^a_{ed}\Gamma^e_{bc}.$$

Substituting in (7.3) gives an expression for R in coordinates. This is rather long, but we expand the first part:

$$R^a_{bcd}=\tfrac12g^{ae}\Big(\frac{\partial^2g_{ed}}{\partial x^b\partial x^c}+\frac{\partial^2g_{bc}}{\partial x^e\partial x^d}-\frac{\partial^2g_{ec}}{\partial x^b\partial x^d}-\frac{\partial^2g_{bd}}{\partial x^e\partial x^c}\Big)+\Gamma^a_{ec}\Gamma^e_{bd}-\Gamma^a_{ed}\Gamma^e_{bc}.$$

As Γ^a_{bc} involves g_{ab},g^{ab} and $\frac{\partial}{\partial x_c}g_{ab}$, we see that R^a_{bcd} depends on g_{ab}, its first and second derivatives, and its inverse g^{ab}.

Flat and locally Euclidean metrics

A Riemannian metric g is called *flat* if it has Riemann curvature $R^a_{bcd} = 0$. In a similar way to Theorem 6.2, one can prove:

Theorem

Let (X, g) be a flat Riemannian manifold. Then for each $x \in X$, there exist coordinates $(x^1, \ldots, x^n)$ on an open neighbourhood U of x in X with $g|_U = (\mathrm{d}x^1)^2 + \cdots + (\mathrm{d}x^n)^2$.

That is, a flat Riemannian manifold (X, g) is locally isometric to Euclidean space $(\mathbb{R}^n, g_{\mathbb{R}^n})$. Here an *isometry* of Riemannian manifolds (X, g), (Y, h) is a diffeomorphism $f : X \to Y$ with $f^*(h) = g$. ('Iso-metry' from Greek 'same distance'.)

Symmetries of Riemann curvature

It is often more convenient to work with $R_{abcd} = g_{ae} R^e_{bcd}$ (also called the Riemann curvature) rather than R^a_{bcd}. Since $R^a_{bcd} = g^{ae} R_{abcd}$, the two are equivalent.

Theorem

Let (X, g) be a Riemannian manifold, with Riemann curvature R_{abcd}. Then R_{abcd} and $\nabla_e R_{abcd}$ satisfy the equations

$$R_{abcd} = -R_{abdc} = -R_{bacd} = R_{cdab}, \tag{7.6}$$

$$R_{abcd} + R_{adbc} + R_{acdb} = 0, \tag{7.7}$$

and

$$\nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec} = 0. \tag{7.8}$$

This can be proved using the coordinate expressions for $R_{abcd}, \nabla_e R_{abcd}$. Here (7.7) and (7.8) are the *first* and *second Bianchi identities*, as in §6.5 with torsion $T^a_{bc} = 0$.

Ricci curvature and scalar curvature

Let (X, g) be a Riemannian manifold. The Riemann curvature tensor R^a_{bcd} of g is a complicated object. Often it is helpful to work with components of R^a_{bcd} which are simpler.

Definition

The *Ricci curvature* of g is $R_{ab} = R^c_{acb} = g^{cd}R_{cadb}$. By (7.6) it satisfies $R_{ab} = R_{ba}$, so $R_{ab} \in C^\infty(S^2T^*X)$. The *scalar curvature* of g is $s = g^{ab}R_{ab} = g^{ab}R^c_{acb}$, so that $s : X \to \mathbb{R}$ is smooth.

Here R_{ab} is the trace of R^a_{bcd}, and s the trace of R_{ab}.
We say that g is *Einstein* if $R_{ab} = \lambda g_{ab}$ for $\lambda \in \mathbb{R}$, and *Ricci-flat* if $R_{ab} = 0$. Einstein and Ricci-flat metrics are important for many reasons; they arise in Einstein's General Relativity.

7.4. Volume forms and integrating functions

Let (X, g) be a Riemannian manifold, of dimension n. Then g induces norms $|\,.\,|_g$ on the bundles of k-forms Λ^kT^*X, and in particular on Λ^nT^*X.
If X is also oriented (§5.2) then $\Lambda^nT^*X \setminus 0$ is divided into positive forms and negative forms, where positive forms are all proportional by positive constants.
Therefore there is a unique positive n-form $\mathrm{d}V_g \in C^\infty(\Lambda^nT^*X)$ with $|\mathrm{d}V_g|_g = 1$. We call $\mathrm{d}V_g$ the *volume form* of g.
It can be characterized as follows: if $x \in X$ and $(v_1, \ldots, v_n)$ is an oriented basis of T_xX which is orthonormal w.r.t. $g|_x$, then $\mathrm{d}V_g \cdot (v_1 \wedge \cdots \wedge v_n) = 1$. In local coordinates $(x^1, \ldots, x^n)$ we have

$$\mathrm{d}V_g = \pm\big[\det(g_{ab})^n_{a,b=1}\big]^{1/2}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n,$$

where the sign depends on whether $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ is an oriented or anti-oriented basis of T_xX.

Let (X, g) be an oriented Riemannian manifold (say compact, for simplicity), and $f : X \to \mathbb{R}$ a smooth function. Then $f \, \mathrm{d}V_g$ is an n-form on X, with X oriented, so as in §5.3 we have the integral

$$\int_X f \, \mathrm{d}V_g.$$

Remark

Changing the orientation of X changes the sign of both the operator $\int_X$, and of the n-form $\mathrm{d}V_g$, so $\int_X f \, \mathrm{d}V_g$ is unchanged. Thus the orientation on X is not really important. We ignore the orientation issue from now on.

Thus, *we can integrate functions on Riemannian manifolds*.

We can use these ideas to define *Lebesgue spaces* and *Sobolev spaces*, Banach spaces of functions (or tensors, etc.) on Riemannian manifolds, which are important in many p.d.e. problems. Let (X, g) be a Riemannian manifold, not necessarily compact. We say a smooth function $f : X \to \mathbb{R}$ is *compactly-supported* if $\operatorname{supp} f = \overline{\{x \in X : f(x) \neq 0\}}$ is contained in a compact subset of X. Write $C^\infty_{\mathrm{cs}}(X)$ for the vector space of compactly supported functions $f : X \to \mathbb{R}$.

For real $p \geqslant 1$ and integer $k \geqslant 0$, define the *Lebesgue norm* $\|\,.\,\|_{L^p}$ and *Sobolev norm* $\|\,.\,\|_{L^p_k}$ on $C^\infty_{\mathrm{cs}}(X)$ by

$$\|f\|_{L^p} = \Bigl(\int_X |f|^p \mathrm{d}V_g\Bigr)^{1/p}, \quad \|f\|_{L^p_k} = \Bigl(\sum_{j=0}^k \int_X \bigl|\nabla^j f\bigr|^p \mathrm{d}V_g\Bigr)^{1/p}.$$

Then define the Banach spaces $L^p(X)$ and $L^p_k(X)$ to be the completions of $C^\infty_{\mathrm{cs}}(X)$ w.r.t. the norms $\|\,.\,\|_{L^p}$ and $\|\,.\,\|_{L^p_k}$. Note that $L^p(X) = L^p_0(X)$. $L^2(X)$ and $L^2_k(X)$ are Hilbert spaces.

We define Banach spaces of tensors such as $L^p_k(\bigotimes^l TX\otimes\bigotimes^m T^*X)$ by completion of $C^\infty_{\rm cs}(\bigotimes^l TX\otimes\bigotimes^m T^*X)$ in the same way.

Example

Let (X,g) be a compact, connected Riemannian manifold. The *Laplacian* $\Delta:C^\infty(X)\rightarrow C^\infty(X)$ may be defined by $\Delta f=-g^{ab}\nabla_a\nabla_b f$. Then Δ extends uniquely to a bounded linear operator on Banach spaces $\Delta:L^p_{k+2}(X)\rightarrow L^p_k(X)$.
It is known that if $p>1$ and $k\geqslant 0$ then

$$\Delta:\bigl\{f\in L^p_{k+2}(X):\textstyle\int_X f\,\mathrm{d}V_g=0\bigr\}\rightarrow\bigl\{h\in L^p_k(X):\textstyle\int_X h\,\mathrm{d}V_g=0\bigr\}$$

is an isomorphism of topological vector spaces. That is, if $h\in L^p_k(X)$ then the linear elliptic p.d.e. $\Delta f=h$ has a solution f in $L^p_{k+2}(X)$ iff $\int_X h\,\mathrm{d}V_g=0$, and if $\int_X f\,\mathrm{d}V_g=0$ then the solution f is unique.

Introduction to Differential Geometry

Lecture 8 of 10: More about Riemannian manifolds

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Plan of talk:

8. More about Riemannian manifolds
8.1. Examples: spheres and hyperbolic spaces

Let g, h be Riemannian metrics on a manifold X. We call g, h *conformally equivalent* if $g = f \cdot h$ for smooth $f : X \to (0, \infty)$. Then g, h define the same notion of angles between vectors in T_xX, since angles depend only on ratios between distances. We will show that the complement of a point in the sphere $(\mathcal{S}^n_R, g_R)$ of radius R in $\mathbb{R}^{n+1}$ is conformally equivalent to Euclidean space $(\mathbb{R}^n, h_{\mathrm{Euc}})$. Define a bijection between points $(y_0, y_1, \ldots, y_n)$ in $\mathcal{S}^n_R \setminus (R, 0, \ldots, 0)$ and $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ such that $(R, 0, \ldots, 0)$, $(y_0, y_1, \ldots, y_n)$ and $(0, x_1, \ldots, x_n)$ are collinear in $\mathbb{R}^{n+1}$. An easy calculation shows that

$$(y_0, y_1, \ldots, y_n) = \Big(\frac{R^3}{R^2 + r^2}, \frac{R^2 x_1}{R^2 + r^2}, \ldots, \frac{R^2 x_n}{R^2 + r^2}\Big),$$

where $r^2 = x_1^2 + \cdots + x_n^2$.

Regarding $(x_1,\ldots,x_n)$ as coordinates on $\mathcal{S}^n_R\setminus(R,0,\ldots,0)$, this enables us to compute g_R in the coordinates $(x_1,\ldots,x_n)$: we have

$$\begin{aligned} g_R &= \mathrm{d}y_0^2+\mathrm{d}y_1^2+\cdots+\mathrm{d}y_n^2 \\ &= \Big(\frac{R^3\cdot 2r\mathrm{d}r}{(R^2+r^2)^2}\Big)^2+\sum_{i=1}^n\Big(\frac{R^2\mathrm{d}x_i}{R^2+r^2}-\frac{R^2x_i\cdot 2r\mathrm{d}r}{(R^2+r^2)^2}\Big)^2 \qquad (8.1)\\ &= \frac{R^4}{(R^2+r^2)^2}(\mathrm{d}x_1^2+\cdots+\mathrm{d}x_n^2)=\frac{R^4}{(R^2+x_1^2+\cdots+x_n^2)^2}\cdot h_{\mathrm{Euc}}. \end{aligned}$$

Hence $(\mathcal{S}^n_R, g_R)$ (take away a point) is conformally equivalent to $(\mathbb{R}^n, h_{\mathrm{Euc}})$. Observe that translations in $\mathbb{R}^n$ preserve h_{Euc}, and so preserve the conformal structure of $\mathcal{S}^n_R$, but are not isometries of $\mathcal{S}^n_R$. The group of isometries of $(\mathcal{S}^n_R, g_R)$ is $O(n+1)$, a compact Lie group of dimension $\frac{1}{2}n(n+1)$ (next time). But the group of conformal transformations (angle-preserving maps) of $(\mathcal{S}^n_R, g_R)$ is larger, it is $O_+(n+1,1)$, a noncompact Lie group of dimension $\frac{1}{2}(n+1)(n+2)$.

Hyperbolic space $\mathcal{H}^n$

Equation (8.1) has an interesting feature: we can replace R by an imaginary number iR, and get a new Euclidean metric on $\mathbb{R}^n$:

$$\frac{R^4}{(R^2-x_1^2-\cdots-x_n^2)^2}\cdot h_{\mathrm{Euc}}$$

which is defined except on the sphere of radius R in $\mathbb{R}^n$. Taking $R=1$, define *n-dimensional hyperbolic space* $(\mathcal{H}^n, g_{\mathcal{H}^n})$ by

$$\mathcal{H}^n=\big\{(x_1,\ldots,x_n)\in\mathbb{R}^n : x_1^2+\cdots+x_n^2<1\big\},$$
$$g_{\mathcal{H}^n}=\frac{1}{(1-x_1^2-\cdots-x_n^2)^2}\cdot(\mathrm{d}x_1^2+\cdots+\mathrm{d}x_n^2).$$

Morally this is a 'sphere of radius $\sqrt{-1}$'. It has a large isometry group $O_+(n,1)$, of dimension $\frac{1}{2}n(n+1)$. Whereas spheres $\mathcal{S}^n_R$ are Einstein with positive scalar curvature, hyperbolic spaces are Einstein with negative scalar curvature. Hyperbolic spaces were historically important in the development of non-Euclidean geometry.

8.2. Riemannian 2-manifolds and surfaces in $\mathbb{R}^3$

For a Riemannian 2-manifold (X, g), the Ricci curvature R_{ab} and Riemann curvature R^a_{bcd} are determined by the scalar curvature s and g by $R_{ab} = \frac{1}{2} s g_{ab}$ and $R^a_{bcd} = \frac{1}{2} s(\delta^a_c g_{bd} - \delta^a_d g_{bc})$. The scalar curvature s is often called the *Gaussian curvature*, and written κ.

Suppose X is a 2-submanifold of $\mathbb{R}^3$, (s, t) are coordinates on X, and the embedding $X \hookrightarrow \mathbb{R}^3$ is $\boldsymbol{r}(s, t) = \big(x(s, t), y(s, t), z(s, t)\big)$. Then the Riemann metric $g = g_{\mathbb{R}^3}|_X$ on X (often called the *first fundamental form*) is

$$g = E\mathrm{d}s^2 + F(\mathrm{d}s\mathrm{d}t + \mathrm{d}t\mathrm{d}s) + G\mathrm{d}t^2, \quad \text{with}$$
$$E = \left|\frac{\partial \boldsymbol{r}}{\partial s}\right|^2, \quad F = \left\langle \frac{\partial \boldsymbol{r}}{\partial s}, \frac{\partial \boldsymbol{r}}{\partial t} \right\rangle, \quad G = \left|\frac{\partial \boldsymbol{r}}{\partial t}\right|^2.$$

Define $\boldsymbol{n}$ to be the unit normal vector to X in $\mathbb{R}^3$, that is,

$$\boldsymbol{n} = \frac{\frac{\partial \boldsymbol{r}}{\partial s} \times \frac{\partial \boldsymbol{r}}{\partial t}}{\left|\frac{\partial \boldsymbol{r}}{\partial s} \times \frac{\partial \boldsymbol{r}}{\partial t}\right|}.$$

Then the *second fundamental form* is

$$\mathbb{I} = L\mathrm{d}s^2 + M(\mathrm{d}s\mathrm{d}t + \mathrm{d}t\mathrm{d}s) + N\mathrm{d}t^2, \quad \text{with}$$
$$L = \boldsymbol{n} \cdot \frac{\partial^2 \boldsymbol{r}}{\partial s^2}, \quad M = \boldsymbol{n} \cdot \frac{\partial^2 \boldsymbol{r}}{\partial s \partial t}, \quad N = \boldsymbol{n} \cdot \frac{\partial^2 \boldsymbol{r}}{\partial t^2}.$$

The *principal curvatures* κ_1, κ_2 are the solutions λ of

$$\det\left[\lambda \begin{pmatrix} E & F \\ F & G \end{pmatrix} - \begin{pmatrix} L & M \\ M & N \end{pmatrix}\right] = 0.$$

The Gaussian curvature (= scalar curvature) is

$$\kappa = \kappa_1 \kappa_2 = (LN - M^2)/(EG - F^2).$$

Although $L, M, N, \kappa_1, \kappa_2$ depend on the embedding of X in $\mathbb{R}^3$, the Gaussian curvature $\kappa = \kappa_1\kappa_2$ depends only on (X, g).

A sphere $\mathcal{S}^2_R$ of radius R in $\mathbb{R}^3$ has principal curvatures $\kappa_1 = \kappa_2 = R^{-1}$ everywhere, so $\kappa = R^{-2}$.

The Gauss–Bonnet Theorem

Recall that if X is a compact n-manifold it has finite-dimensional de Rham cohomology groups $H^k_{\mathrm{dR}}(X,\mathbb{R})$ for $k=0,\ldots,n$. The *Betti numbers* are $b^k(X)=\dim H^k_{\mathrm{dR}}(X,\mathbb{R})$, and the *Euler characteristic* is $\chi(X)=\sum_{k=0}^n(-1)^k b^k(X)$. If $n=2$ and X is a surface of genus g then $\chi(X)=2-2g$.

Theorem (Gauss–Bonnet)

Let (X,g) be a compact Riemannian 2-manifold, with Gauss curvature κ. Then
$$\int_X \kappa \,\mathrm{d}V_g = 2\pi\chi(X).$$

This is an avatar of a lot of important geometry in higher dimensions – index theorems, characteristic classes.
For a simpler analogy, let $\gamma:\mathcal{S}^1\to\mathbb{R}^2$ be an immersed curve, and $\kappa:\mathcal{S}^1\to\mathbb{R}$ be the curvature (rate of change of angle of tangent direction). Then $\int_{\mathcal{S}^1}\kappa\,\mathrm{d}s=2\pi W(\gamma)$, where $\int\cdots\mathrm{d}s$ is integration w.r.t arc-length, and $W(\gamma)$ is the winding number of γ.

Minimal surfaces in $\mathbb{R}^3$

Let $X\hookrightarrow\mathbb{R}^3$ be an (oriented) embedded surface in $\mathbb{R}^3$. The *mean curvature* $H:X\to\mathbb{R}^3$ is $H=\frac{1}{2}(\kappa_1+\kappa_2)$, the average of the principal curvatures of X. The *mean curvature vector* is $H\boldsymbol{n}$. (The sign of H depends on the orientation of X, but $H\boldsymbol{n}$ is independent of orientation.) We call X a *minimal surface* if $H=0$. It turns out that X is minimal if and only if X is locally volume-minimizing in $\mathbb{R}^3$ (the equation $H=0$ is the Euler–Lagrange equation for the volume functional on surfaces in $\mathbb{R}^3$).
Minimal surfaces are important in physical problems – if you dip a twisted loop of wire in the washing up and it is spanned by a bubble, this will be a minimal surface (to first approximation), as the surface tension in the bubble tries to minimize its area. Finding a minimal surface with given boundary is called *Plateau's problem*.
More generally, a bubble separating two regions in $\mathbb{R}^3$ with different air pressures should satisfy the p.d.e. $H=$ constant (e.g. a sphere).

Isothermal coordinates

Let (X, g) be a Riemannian 2-manifold. Then near each point $x \in X$ there exists a local coordinate system (x_1, x_2) such that

$$g = f(x_1, x_2) \cdot (\mathrm{d}x_1^2 + \mathrm{d}x_2^2).$$

for $f(x_1, x_2)$ a smooth positive function. That is, in 2 dimensions any Riemannian metric is locally conformally equivalent to the Euclidean plane $(\mathbb{R}^2, g_{\mathrm{Euc}})$. Such coordinates (x_1, x_2) are called *isothermal coordinates*. This is false in dimension > 2.

If also X is oriented, and we take (x_1, x_2) to be oriented coordinates, we can take $x_1 + ix_2$ to be a complex local coordinate on X. Such complex coordinates have holomorphic transition functions, and make X into a Riemann surface.

Basically, a conformal structure (Riemannian metric modulo conformal equivalence) on an oriented 2-manifold is equivalent to the data of how to rotate vectors 90° in each tangent space T_xX (i.e. multiply by i in $\mathbb{C}$), and this is equivalent to a complex structure.

8.3. Geodesics

Let (X, g) be a Riemannian manifold. Consider a smooth immersed curve $\gamma : [a, b] \to X$. The length of γ is

$$l(\gamma) = \int_a^b g(\dot\gamma(t), \dot\gamma(t))^{1/2}\,\mathrm{d}t.$$

To a first approximation, a geodesic is a *locally length-minimizing curve* γ, that is, it satisfies the Euler–Lagrange equations for the length functional l on curves γ. Actually, this turns out not to be well behaved. If $F : [a', b'] \to [a, b]$ is any diffeomorphism then γ is locally length-minimizing iff $\gamma \circ F$ is locally length-minimizing, as length is independent of parametrization. Thus, geodesics defined this way would come in infinite-dimensional families.

Instead, we define the *energy* of a curve γ in (X, g) by

$$E(\gamma) = \int_a^b g(\dot\gamma(t), \dot\gamma(t))\,\mathrm{d}t.$$

We define a *geodesic* $\gamma : [a, b] \to X$ or $\gamma : \mathbb{R} \to X$ to satisfy the Euler–Lagrange equations for the energy functional E on curves γ. Then γ is a geodesic iff:

- γ is *locally length-minimizing*, i.e. γ satisfies the Euler–Lagrange equation for the length functional l; and
- γ is *parametrized with constant speed*, that is, $g(\dot\gamma(t), \dot\gamma(t))$ is (locally) constant along γ.

Example

Take (X, g) to be Euclidean n-space $(\mathbb{R}^n, h_{\mathrm{Euc}})$. Then $\gamma = (\gamma_1, \ldots, \gamma_n) : [a, b] \to \mathbb{R}^n$ satisfies the geodesic equations iff $\frac{\mathrm{d}^2\gamma_i}{\mathrm{d}t^2} = 0$ for $i = 1, \ldots, n$. Hence geodesics are of the form $\gamma(t) = \boldsymbol{a}t + \boldsymbol{b}$ for $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$. That is, they are straight lines in $\mathbb{R}^n$ traversed with constant speed.

The geodesic equations in local coordinates

Let $(x_1, \ldots, x_n)$ be local coordinates on X. Write $g = g_{ij}(x_1, \ldots, x_n)$, and let g^{ij} be the inverse matrix of functions. Write a smooth map $\gamma : [a, b] \to X$ as $\gamma = \bigl(x_1(t), \ldots, x_n(t)\bigr)$ in coordinates. Then γ satisfies the geodesic equations iff we can extend γ to a $2n$-tuple $\bigl(x_1(t), \ldots, x_n(t), y_1(t), \ldots, y_n(t)\bigr)$ satisfying the o.d.e.s

$$\begin{aligned} \frac{\mathrm{d}x_j}{\mathrm{d}t} &= \sum_{i=1}^n g^{ij}(x_1(t), \ldots, x_n(t)) \cdot y_i(t), \\ \frac{\mathrm{d}y_k}{\mathrm{d}t} &= -\tfrac{1}{2} \sum_{i,j=1}^n \frac{\partial g^{ij}}{\partial x_k}(x_1(t), \ldots, x_n(t)) \cdot y_i(t) y_j(t). \end{aligned} \tag{8.2}$$

Here $D\gamma = \bigl(x_1(t), \ldots, x_n(t), y_1(t), \ldots, y_n(t)\bigr)$ is naturally a curve in the cotangent bundle T^*X, and $\gamma = \bigl(x_1(t), \ldots, x_n(t)\bigr)$ is its projection to X. We can think of $D\gamma$ as a flowline of a fixed vector field v on T^*X depending on g, called the *geodesic flow*.

From the geodesic equations (8.2) and standard results about o.d.e.s we see that for any $x \in X$ and any vector $v \in T_xX$, there exists a unique solution $\gamma : I \to X$ to the geodesic equations with $\gamma(0) = x$ and $\dot{\gamma}(0) = v$, where $0 \in I \subseteq \mathbb{R}$ is an open interval, which we can take to be maximal.
A Riemannian manifold (X, g) is called *complete* if we can take $I = \mathbb{R}$ for all such x, v. If X is compact then any g is complete, but many noncompact Riemannian manifolds such as $(\mathbb{R}^n, g_{\rm Euc})$ and $(\mathcal{H}^n, h_{\mathcal{H}^n})$ are complete. Roughly, to be complete means that the boundary/edge of (X, g) is at infinite distance from the interior of X.

Riemannian manifolds
More about Riemannian manifolds
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Riemannian 2-manifolds and surfaces in $\mathbb{R}^3$
Geodesics

Example

In the 2-sphere $\mathcal{S}^2_R$ of radius R in $\mathbb{R}^3$, the geodesics are the *great circles*, that is, intersections of $\mathcal{S}^2_R$ with a plane $\mathbb{R}^2$ in $\mathbb{R}^3$ passing through the centre $(0, 0, 0)$ of $\mathcal{S}^2_R$. So for example on the earth, the equator is a closed geodesic.
Note that geodesics need not globally be a shortest path: you can make the equator shorter by deforming it through lines of latitude. But geodesics have stationary length, and a geodesic γ gives the shortest path between points x, y on γ if x, y are sufficiently close.

Example

Take $(\mathcal{H}^2, g_{\mathcal{H}^2})$ to be the hyperbolic plane $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ with $g_{\mathcal{H}^2} = (1 - x^2 - y^2)^{-1}({\rm d}x^2 + {\rm d}y^2)$. Then geodesics in $\mathcal{H}^2$ are the intersection of $\mathcal{H}^2$ with circles and straight lines in $\mathbb{R}^2$ which intersect the unit circle $x^2 + y^2 = 1$ at right angles.

Geodesic triangles in a Riemannian 2-manifold

Let (X, g) be a Riemannian 2-manifold. Suppose we are given points $A, B, C \in X$, and geodesic segments AB, BC, CA in X with endpoints A, B, C, which enclose a triangle ABC homeomorphic to a disc D^2. Let α, β, γ be the internal angles of the triangle at A, B, C computed using g. (That is, α is the angle in $(T_A X, g|_A)$ between the tangent vectors to AB, AC at A, etc.)
Then one can show that

$$\alpha + \beta + \gamma - \pi = \int_{ABC} \kappa \, \mathrm{d}V_g, \tag{8.3}$$

where $\kappa : X \to \mathbb{R}$ is the Gaussian curvature of g.
If (X, g) is $(\mathbb{R}^2, g_{\mathrm{Euc}})$ then $\kappa = 0$ and (8.3) becomes $\alpha + \beta + \gamma = \pi$, that is, the angles in a triangle in $\mathbb{R}^2$ add up to π.

If (X, g) is the unit sphere $\mathcal{S}^2$ then $\kappa = 1$, so (8.3) becomes $\alpha + \beta + \gamma = \pi + \mathrm{area}(ABC)$. Thus, the angles in a triangle on $\mathcal{S}^2$ add up to more than π.
If (X, g) is the hyperbolic plane $(\mathcal{H}^2, g_{\mathcal{H}^2})$ then $\kappa = -1$, so (8.3) becomes $\alpha + \beta + \gamma = \pi - \mathrm{area}(ABC)$. Thus, the angles in a triangle on $\mathcal{S}^2$ add up to less than π. Also, all triangles have area less than π, however long their sides.
We can use (8.3) to prove the Gauss–Bonnet Theorem. Suppose (X, g) is a compact Riemannian 2-manifold. Choose a division of X into small triangles $\Delta_1, \ldots, \Delta_N$ with geodesic sides, and sum (8.3) over $1, \ldots, N$. We get

$$2\pi(\#\text{vertices}) - \pi(\#\text{triangles}) = \int_X \kappa \, \mathrm{d}V_g.$$

Since $2\#\text{edges} = 3\#\text{triangles}$ we have

$$\#\text{triangles} = 2(\#\text{edges} - \#\text{triangles}).$$

Then using $\chi(X) = \#\text{vertices} - \#\text{edges} + \#\text{triangles}$ proves G–B.