Introduction to Differential Geometry

Lecture 9 of 10: Lie groups and Lie algebras

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These slides available at
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Plan of talk:

9. Lie groups and Lie algebras
9.1. Lie groups

A Lie group is a group which is also a manifold:

Definition

A *Lie group* is a smooth manifold G equipped with a *multiplication map* $\mu : G \times G \to G$, an *inverse map* $i : G \to G$, and an *identity element* $1 \in G$, such that μ, i are smooth maps of manifolds, and $\mu, i, 1$ satisfy the usual group axioms, i.e. for all $a, b, c \in G$ we have

$$\mu(a, \mu(b, c)) = \mu(\mu(a, b), c), \quad \mu(a, 1) = \mu(1, a) = a,$$
$$\mu(a, i(a)) = \mu(i(a), a) = 1.$$

We usually write $\mu(a, b) = ab$ and $i(a) = a^{-1}$.

Lie groups are very well understood. The theory of Lie groups is a beautiful area of mathematics.

Here are some obvious definitions:

Definition

Let G, H be Lie groups. A *morphism* $\Phi : G \to H$ is a smooth map of manifolds $\Phi : G \to H$ which is also a morphism of groups, that is, $\Phi(ab^{-1}) = \Phi(a)\Phi(b)^{-1}$ for all $a, b \in G$.

Definition

A *Lie subgroup* G of a Lie group H is a subgroup $G \subseteq H$ which is also an (embedded) submanifold. Then G is a Lie group.

Definition

Let G be a Lie group and X a manifold. An *action* of G on X is a smooth map $\rho : G \times X \to X$ that is a group action of G on X, that is, $\rho(a, \rho(b, x)) = \rho(\mu(a, b), x)$ and $\rho(1, x) = x$ for all $a, b \in G$ and $x \in X$.

In general, to turn some definition in group theory into differential geometry, you replace groups by Lie groups, sets by manifolds, and require all maps in the definition to be smooth maps of manifolds.

We can talk about Lie groups using ideas from group theory (e.g. kernel, abelian), or topology (e.g. compact, connected, simply-connected, universal cover), or smooth manifolds (e.g. dimension $\dim G$, submanifold, immersion, submersion). It is a rich subject. Products of Lie groups are Lie groups.

We can also define *complex Lie groups* G, with G a complex manifold, and μ, i holomorphic maps. Lie groups are sometimes called *real Lie groups*, in contrast with complex Lie groups.

Symmetry groups of objects in differential geometry are often Lie groups. For example, the isometry group of the sphere $\mathcal{S}^{n-1} \subset \mathbb{R}^n$ with the round metric $g_{\mathbb{R}^n}|_{\mathcal{S}^{n-1}}$ is the orthogonal group $\mathrm{O}(n)$.

9.2. Examples of Lie groups

Example

$\mathbb{R}^n$ is a noncompact abelian Lie group, with

$$\mu\big((x^1,\ldots,x^n),(y^1,\ldots,y^n)\big)=(x^1+y^1,\ldots,x^n+y^n),$$
$$i\big((x^1,\ldots,x^n)\big)=(-x^1,\ldots,-x^n) \quad \text{and} \quad 1=(0,\ldots,0).$$

Example

$T^n=\mathbb{R}^n/\mathbb{Z}^n$ is a compact, abelian Lie group.

Example

Let G be any finite or countable group. Regard G as a 0-dimensional manifold, with the discrete topology (i.e. G is a disjoint union of points). Then G is a Lie group.

Example

Write $\mathrm{GL}(n,\mathbb{R})$ for the group of invertible $n \times n$ matrices $(A_{ij})_{i,j=1}^n$ over $\mathbb{R}$, under matrix multiplication. Then $\mathrm{GL}(n,\mathbb{R})$ is an open subset of $\mathbb{R}^{n^2}$ (all $n \times n$ matrices), defined by the open condition $\det(A_{ij}) \neq 0$, so $\mathrm{GL}(n,\mathbb{R})$ is a manifold. Matrix multiplication and inverses are smooth maps (consider the explicit formulae), so $\mathrm{GL}(n,\mathbb{R})$ is a noncompact Lie group, of dimension n^2.

Example

Let V be a finite-dimensional real vector space, and write $\mathrm{GL}(V)$ for the group of linear isomorphisms $\alpha : V \rightarrow V$. Then $\mathrm{GL}(V)$ is a Lie group isomorphic to $\mathrm{GL}(n,\mathbb{R})$ for $n = \dim V$.

Definition

Let G be a Lie group. A *representation* of G is a finite-dimensional real vector space V and a Lie group morphism $\Phi : G \rightarrow \mathrm{GL}(V)$.

Example

Define $\mathrm{O}(n)$ to be the subgroup of *orthogonal matrices* in $\mathrm{GL}(n,\mathbb{R})$, that is, the subgroup of $A = (A_{ij})_{i,j=1}^n$ satisfying $A^t A = I_n$, where A^t is the transpose of A, and I_n the $n \times n$ identity matrix. Equivalently, $\mathrm{O}(n)$ is the automorphisms of $\mathbb{R}^n$ preserving the Euclidean metric $(\mathrm{d}x^1)^2 + \cdots + (\mathrm{d}x^n)^2$.
Consider the map $\Phi : \mathrm{GL}(n,\mathbb{R}) \rightarrow S^2(\mathbb{R}^n)$, where $S^2(\mathbb{R}^n) = \mathbb{R}^{n(n+1)/2}$ is the space of symmetric $n \times n$ matrices, mapping $\Phi : A \mapsto A^t A$. Then $O(n) = \Phi^{-1}(I_n)$. The derivative of Φ at $A \in \mathrm{GL}(n,\mathbb{R})$ is $\mathrm{d}\Phi|_A : B \mapsto A^t B + B^t A$. This is surjective, as any $C \in S^2(\mathbb{R}^n)$ has $C = \mathrm{d}\Phi|_A(B)$ for $B = \frac{1}{2}(A^t)^{-1}C$. So Φ is a submersion, and thus $\mathrm{O}(n) = \Phi^{-1}(I_n)$ is a submanifold of $\mathrm{GL}(n,\mathbb{R})$, of dimension $n(n-1)/2$, by Proposition 2.5, §2.4. Thus $\mathrm{O}(n)$ is a Lie group. Taking trace of $A^t A = I_n$ gives $\sum_{i,j} A_{ij}^2 = n$ for $A \in \mathrm{O}(n)$, so $\mathrm{O}(n)$ is closed and bounded in $\mathbb{R}^{n^2}$, and so compact.

Example

Write $\mathrm{SL}(n,\mathbb{R})$ for the subgroup of $A=(A_{ij})_{i,j=1}^n$ in $\mathrm{GL}(n,\mathbb{R})$ with $\det A=1$. The map $\det:\mathrm{GL}(n,\mathbb{R})\to\mathbb{R}\setminus\{0\}$ is a submersion, so $\mathrm{SL}(n,\mathbb{R})$ is a Lie subgroup of $\mathrm{GL}(n,\mathbb{R})$, of dimension n^2-1.

Example

If $A\in\mathrm{O}(n)$ then $A^tA=I_n$, so $\det(A^t)\det A=(\det A)^2$ $=\det I_n=1$, and $\det A\in\{\pm1\}$. Thus $\det:\mathrm{O}(n)\to\{\pm1\}$ is a morphism of Lie groups. Write $\mathrm{SO}(n)=\{A\in\mathrm{O}(n):\det A=1\}$. Then $\mathrm{SO}(n)$ is a compact Lie subgroup of $\mathrm{O}(n)$, the *special orthogonal group*, of dimension $n(n-1)/2$. It is connected.

Example

Write $\mathrm{GL}(n,\mathbb{C})$ for the group of invertible $n\times n$ matrices $(A_{ij})_{i,j=1}^n$ over $\mathbb{C}$. It is a real Lie group of dimension $2n^2$, and also a complex Lie group. We can view $\mathrm{GL}(n,\mathbb{C})$ as a Lie subgroup of $\mathrm{GL}(2n,\mathbb{R})$.

Example

Define the *unitary group* $\mathrm{U}(n)$ to be the Lie subgroup of $A\in\mathrm{GL}(n,\mathbb{C})$ with $\bar{A}^tA=I_n$, where $\bar{A}^t$ is the complex conjugate transpose matrix. Equivalently, A is the $\mathbb{C}$-linear automorphisms of $\mathbb{C}^n$ preserving the Hermitian metric $|\mathrm{d}z^1|^2+\cdots+|\mathrm{d}z^n|^2$ on $\mathbb{C}^n$. Then $\mathrm{U}(n)$ is a compact, connected real Lie group (but not a complex Lie group) of dimension n^2. We can write $\mathrm{U}(n)$ as the intersection of $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{O}(2n)$ in $\mathrm{GL}(2n,\mathbb{R})$.

Example

Write $\mathrm{SL}(n,\mathbb{C})$ for the subgroup of $A\in\mathrm{GL}(n,\mathbb{C})$ with $\det_{\mathbb{C}}A=1$. It is a Lie subgroup of $\mathrm{SL}(n,\mathbb{C})$, of real dimension $2n^2-2$.

Example

The *special unitary group* $\mathrm{SU}(n)$ is the subgroup of $A\in\mathrm{SL}(n,\mathbb{C})$ with $\bar{A}^tA=I_n$. It is a Lie group of dimension n^2-1.

9.3. Lie algebras of Lie groups

Let G be a Lie group. Then for each $\gamma \in G$ we have the *left translation map* $L_\gamma : G \to G$ mapping $L_\gamma : \delta \mapsto \gamma\delta$, which is a diffeomorphism of G, and satisfies $L_\gamma \circ L_{\gamma'} = L_{\gamma\gamma'}$.

Let $v \in C^\infty(TG)$ be a vector field on G, and $\gamma \in G$. As L_γ is a diffeomorphism, we have the pullback $L_\gamma^*(v) \in C^\infty(TG)$. We say that v is *left-invariant* if $L_\gamma^*(v) = v$ for all $\gamma \in G$.

Write $\mathfrak{g}$ for the vector space of left-invariant vector fields. Then we have a map $\mathfrak{g} \to T_1 G$ mapping $v \mapsto v|_1$. This is an isomorphism, as every $w \in T_1 G$ determines a unique left-invariant vector field v with $v|_1 = w$, by $v|_\gamma = (\mathrm{d}L_\gamma|_1)_*(w)$, where $\mathrm{d}L_\gamma|_1 : T_1 G \to T_\gamma G$.

Recall that in §3.4 we define the *Lie bracket* $[v, w]$ of vector fields v, w on G, with $[v, w] = -[w, v]$, which satisfies the *Jacobi identity* for all $u, v, w \in C^\infty(TG)$:

$$[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0. \tag{9.1}$$

As the Lie bracket depends only on the manifold structure of G, it is preserved by diffeomorphisms. Thus if $v, w \in \mathfrak{g}$ then

$$L_\gamma^*([v,w]) = \bigl[L_\gamma^*(v), L_\gamma^*(w)\bigr] = [v,w]$$

for all $\gamma \in G$ so $[v, w]$ is also left-invariant, and $[\,,\,] : C^\infty(TG) \times C^\infty(TG) \to C^\infty(TG)$ maps $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.

We have defined an interesting structure in linear algebra: a finite-dimensional vector space $\mathfrak{g}$ with an antisymmetric, bilinear bracket $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying (9.1).

Definition

A *Lie algebra* is a finite-dimensional vector space $\mathfrak{g}$ (over $\mathbb{R}$ or $\mathbb{C}$ in these lectures) with a bilinear *Lie bracket* $[\,,\,]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ satisfying $[v,w]=-[w,v]$ and the Jacobi identity

$$[u,[v,w]]+[v,[w,u]]+[w,[u,v]]=0 \quad \text{for all } u,v,w\in\mathfrak{g}.$$

The discussion above shows that every Lie group G has a natural Lie algebra $\mathfrak{g}$ over $\mathbb{R}$, where we can take $\mathfrak{g}$ to be the vector space of left-invariant vector fields, or $\mathfrak{g}=T_1G$. So $\dim\mathfrak{g}=\dim G$.
If G is a complex Lie group, it has a natural Lie algebra over $\mathbb{C}$.
One can study Lie algebras by themselves, using algebraic methods, without considering Lie groups. Lie algebras make sense over general fields $\mathbb{K}$, for which there is no concept of manifold.

Example

Let A be any finite-dimensional $\mathbb{R}$-algebra. Define a Lie bracket $[\,,\,]$ on A by $[a,b]=ab-ba$ for $a,b\in A$. Then $[a,b]=-[b,a]$, and

$$\begin{aligned}[a,[b,c]]+[b,[c,a]]+[c,[a,b]]&=a(bc-cb)-(bc-cb)a\\&\quad+b(ca-ac)-(ca-ac)b+c(ab-ba)-(ab-ba)c=0,\end{aligned}$$

as multiplication in A is associative. So $A,[\,,\,]$ is a Lie algebra.

Example

Real $n\times n$ matrices $(A_{ij})_{i,j=1}^n$ are an $\mathbb{R}$-algebra, and hence a Lie algebra. Write $\mathfrak{gl}(n,\mathbb{R})$ for the Lie algebra of $n\times n$ matrices with Lie bracket $[A,B]=AB-BA$. It is the Lie algebra of $\mathrm{GL}(n,\mathbb{R})$.

Example

Write $\mathfrak{so}(n)$ for the Lie algebra of $n\times n$ antisymmetric matrices $(A_{ij})_{i,j=1}^n$ with $A_{ij}=-A_{ji}$, and Lie bracket $[A,B]=AB-BA$. It is a Lie subalgebra of $\mathfrak{gl}(n,\mathbb{R})$, the Lie algebra of $\mathrm{SO}(n)$ and $\mathrm{O}(n)$.

The adjoint representation

Let G be a Lie group, with Lie algebra $\mathfrak{g}$. We defined $\mathfrak{g}$ using left translation maps $L_\gamma : G \to G$, $L_\gamma : \delta \mapsto \gamma\delta$. We also have *right translation maps* $R_\gamma : G \to G$, $R_\gamma : \delta \mapsto \delta\gamma$, which commute with left translation. If $v \in \mathfrak{g}$ and $\gamma, \delta \in G$ then

$$L_\gamma^*\big(R_\delta^*(v)\big) = R_\delta^*\big(L_\gamma^*(v)\big) = R_\delta^*(v),$$

as L_γ, R_δ commute. So $R_\delta^*(v) \in \mathfrak{g}$. This defines a linear isomorphism $R_\delta^* : \mathfrak{g} \to \mathfrak{g}$, that is, $R_\delta^* \in \mathrm{GL}(\mathfrak{g})$.
Define the *adjoint representation* $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ by $\mathrm{Ad} : \delta \mapsto R_\delta^*$. This is a morphism of Lie groups, and so a representation of G on $\mathfrak{g}$. It preserves the Lie bracket, that is,

$$\mathrm{Ad}(\delta)\big([v, w]\big) = \big[\mathrm{Ad}(\delta)(v), \mathrm{Ad}(\delta)w\big].$$

9.4. Fundamental groups

We will be interested in the question: to what extent is a Lie group G determined by its Lie algebra $\mathfrak{g}$? To answer this, we need to recall some ideas from algebraic topology. In particular, we need to understand the ideas of *fundamental group* $\pi_1(X)$, *simply connected space*, and *universal cover*. Let X be a topological space (usually assumed path-connected), and fix a basepoint x_0 in X.
A *loop* γ *in* X *based at* x_0 is a continuous map $\gamma : [0,1] \to X$ with $\gamma(0) = \gamma(1) = x_0$. If γ, γ' are loops in X based at x_0, a *homotopy* from γ to γ' is a continuous map $H : [0,1]^2 \to X$ with $H(0,t) = \gamma(t)$, $H(1,t) = \gamma'(t)$ and $H(s,0) = H(s,1) = x_0$ for all $s, t \in [0,1]$. Write $\gamma \sim \gamma'$ if there exists a homotopy from γ to γ'.
Then $\sim$ is an equivalence relation on loops.
Write $[\gamma]$ for the $\sim$-equivalence class of γ.

Define the *fundamental group* $\pi_1(X)$ of X to be the set of $\sim$-equivalence classes $[\gamma]$ of loops γ in X based at x_0.
We make $\pi_1(X)$ into a group with operation $[\gamma]\cdot[\delta]=[\epsilon]$, where

$$\epsilon:[0,1]\longrightarrow X,\quad \epsilon(t)=\begin{cases}\gamma(2t), & t\in[0,\frac{1}{2}],\\ \delta(2t-1), & t\in[1/2,1],\end{cases}$$

and identity $1=[\iota]$ where $\iota:[0,1]\to X$, $\iota:t\mapsto x_0$, and inverses $[\gamma]^{-1}=[\gamma^{-1}]$, where $\gamma^{-1}(t)=\gamma(1-t)$.
If X is path-connected then $\pi_1(X)$ is independent of the choice of basepoint x_0 up to isomorphism.
We say that X is *simply-connected* if $\pi_1(X)=\{1\}$.

Example

- $\pi_1(\mathbb{R}^n) = \{1\}$
- $\pi_1(T^n) \cong \mathbb{Z}^n$
- $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}_2$
- $\pi_1(S^1) \cong \mathbb{Z}$
- $\pi_1(\mathcal{S}^n) = \{1\}$ for $n \geqslant 2$
- $\pi_1(\mathrm{SO}(n)) \cong \mathbb{Z}_2$ for $n \geqslant 3$.

Let X be path-connected, with basepoint x_0. Then we can define a path-connected, simply-connected topological space $\tilde{X}$ called the *universal cover* of X, and a free, continuous action of $\pi_1(X)$ on $\tilde{X}$, such that $\tilde{X}/\pi_1(X) \cong X$. That is, we have a continuous projection $\pi : \tilde{X} \to X$, called the *covering map*, which is a local homeomorphism, whose fibres $\pi^{-1}(x)$ for $x \in X$ are the orbits of $\pi_1(X)$ in $\tilde{X}$.

Here is how to define the universal cover $\tilde{X}$. Consider continuous paths $\delta : [0,1] \to X$ with $\delta(0) = x_0$, but do not require $\delta(1) = x_0$. Let δ, δ' be such paths, with $\delta(1) = \delta'(1) = x$ say. A homotopy from δ to δ' is a continuous map $H : [0,1]^2 \to X$ with $H(0,t) = \delta(t)$, $H(1,t) = \delta'(t)$ and $H(s,0) = x_0$, $H(s,1) = x$ for all $s,t \in [0,1]$. Write $\delta \sim \delta'$ if there exists a homotopy from δ to δ'. Write $[\delta]$ for the $\sim$-equivalence class of δ.
Define $\tilde{X}$ to be the set of such equivalence classes $[\delta]$, and $\pi : \tilde{X} \to X$ by $\pi : [\delta] \mapsto \delta(1)$. Note that $\pi^{-1}(x_0) = \pi_1(X)$.
Then $\tilde{X}$ is naturally a topological space, and π a continuous covering map. As for multiplication in $\pi_1(X)$, define an action of $\pi_1(X)$ on $\tilde{X}$ by $[\gamma] \cdot [\delta] = [\epsilon]$ for $[\gamma] \in \pi_1(X)$ and $[\delta] \in \tilde{X}$, where

$$\epsilon : [0,1] \longrightarrow X, \quad \epsilon(t) = \begin{cases} \gamma(2t), & t \in [0, \frac{1}{2}], \\ \delta(2t-1), & t \in [1/2, 1]. \end{cases}$$

Introduction to Differential Geometry

Lecture 10 of 10: More about Lie groups and Lie algebras

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Plan of talk:

10. More about Lie groups and Lie algebras
10.1. Relating Lie algebras and Lie groups

Let G, H be Lie groups, with Lie algebras $\mathfrak{g}, \mathfrak{h}$. If $G \cong H$ as Lie groups, then $\mathfrak{g} \cong \mathfrak{h}$ as Lie algebras. When can it happen that $\mathfrak{g} \cong \mathfrak{h}$, but $G \not\cong H$? Consider two examples:

Example

The Lie groups $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ both have Lie algebra $\{A \in \mathfrak{gl}(n, \mathbb{R}) : A^t + A = 0\}$. But $\mathrm{O}(n) \not\cong \mathrm{SO}(n)$. Note that $\mathrm{SO}(n)$ is connected, but $\mathrm{O}(n)$ has two connected components, $\{\det A = 1\}$ and $\{\det A = -1\}$.

Example

The Lie groups $\mathbb{R}^n$ and $T^n = \mathbb{R}^n/\mathbb{Z}^n$ both have Lie algebra $\mathbb{R}^n$ with Lie bracket $[\,,\,] = 0$ (as they are abelian), but $\mathbb{R}^n \not\cong T^n$. Here $\mathbb{R}^n$ is simply-connected, but $\pi_1(T^n) = \mathbb{Z}^n$.

These examples show that we can have $\mathfrak{g} \cong \mathfrak{h}$ but $G \not\cong H$ if the conected components of G, H are different, or the fundamental groups $\pi_1(G), \pi_1(H)$ are different. If we restrict to connected, simply-connected Lie groups, one can prove:

Theorem 10.1 (From Lie's second theorem. Also true over $\mathbb{C}$.)

Suppose G, H are connected, simply-connected Lie groups with Lie algebras $\mathfrak{g}, \mathfrak{h}$. If $\mathfrak{g} \cong \mathfrak{h}$ as Lie algebras, then $G \cong H$ as Lie groups.

Broadly, if $\mathfrak{g} \cong \mathfrak{h}$ then G, H are locally isomorphic near $1 \in G$ and $1 \in H$, and being connected and simply-connected forces $G \cong H$. If G is any Lie group, write G_1 for the connected component of G containing 1, a Lie subgroup of G, and $\tilde{G}_1$ for the universal cover of G_1, with basepoint $x_0 = 1$. Then $\tilde{G}_1$ is a connected, simply-connected Lie group with Lie algebra $\mathfrak{g}$. So Theorem 10.1 implies that if G, H are Lie groups with $\mathfrak{g} \cong \mathfrak{h}$ then $\tilde{G}_1 \cong \tilde{H}_1$.

One can go from Lie algebras back to Lie groups:

Theorem 10.2 (Lie's third theorem. Also true over $\mathbb{C}$.)

Let $\mathfrak{g}$ be a real Lie algebra. Then there exists a connected, simply-connected Lie group G with Lie algebra $\mathfrak{g}$.

This G is unique up to canonical isomorphism by Lie's second theorem. Thus, we have a 1-1 correspondence between (isomorphism classes of) Lie algebras and (isomorphism classes of) connected, simply-connected Lie groups.
Now Lie groups G are complicated objects, but Lie algebras $\mathfrak{g}$ are much simpler. Choosing a basis $v_1, \ldots, v_n$ for $\mathfrak{g}$, write $[v_a, v_b] = \sum_{c=1}^n C_{ab}^c v_c$ for $C_{ab}^c \in \mathbb{R}$. Then $\mathfrak{g}$ is completely described by the *structure constants* $\bigl(C_{ab}^c\bigr)_{a,b,c=1}^n$, which satisfy $C_{ab}^c = -C_{ba}^c$ and $\sum_{e=1}^n \bigl(C_{ab}^e C_{ec}^d + C_{bc}^e C_{ea}^d + C_{ca}^e C_{eb}^d\bigr) = 0$.
We can study and classify Lie algebras using linear algebra, and then deduce results about Lie groups.

10.2. The classification of complex Lie algebras

There is a classification theory for Lie algebras, similar in some ways to the classification of finite groups. By Lie's theorems, this is equivalent to classifying Lie groups. The theory is easiest for complex Lie algebras, as $\mathbb{C}$ is algebraically closed.

Definition

An *ideal* $\mathfrak{h}$ in a Lie algebra $\mathfrak{g}$ is a vector subspace $\mathfrak{h} \subseteq \mathfrak{g}$ such that $[g, h] \in \mathfrak{h}$ for all $g \in \mathfrak{g}$ and $h \in \mathfrak{h}$. Then $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are Lie algebras. This is the analogue of normal subgroups of groups.

Definition

A Lie algebra $\mathfrak{g} \neq 0$ is *simple* if it has no ideals $\mathfrak{h} \subseteq \mathfrak{g}$ with $\mathfrak{h} \neq 0, \mathfrak{g}$.

(Compare definition of simple finite group.)

Decomposing Lie algebras into simple Lie algebras

It is easy to show:

Lemma 10.3

Let $\mathfrak{g}$ be a Lie algebra. Then there exists a maximal chain of ideals $0 = \mathfrak{g}_0 \subsetneqq \mathfrak{g}_1 \subsetneqq \mathfrak{g}_2 \subsetneqq \cdots \subsetneqq \mathfrak{g}_n = \mathfrak{g}$ in $\mathfrak{g}$ such that $\mathfrak{g}_i/\mathfrak{g}_{i-1}$ is a simple Lie algebra for $i = 1, \ldots, n$.

Thus, every Lie algebra is 'built' from finitely many simple Lie algebras. We will give a *complete classification* of simple Lie algebras. So we know the building blocks for all Lie algebras. In general how $\mathfrak{g}$ is built from its simple factors $\mathfrak{g}_i/\mathfrak{g}_{i-1}$ is complicated, but for Lie algebras of compact Lie groups things are nice:

Theorem 10.4

Let G be a compact real Lie group with Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ is **semisimple***, that is, a direct sum of simple Lie algebras.*

Outline of the classification of simple Lie algebras over $\mathbb{C}$

There is a long, complicated story which gives a complete classification of simple Lie algebras $\mathfrak{g}$ over $\mathbb{C}$ up to isomorphism. Some of the important ideas are:

- Every Lie algebra $\mathfrak{g}$ has a *Cartan subalgebra* $\mathfrak{h}$, a maximal abelian subalgebra, the Lie algebra of a maximal torus in the associated Lie group G. The *rank* of $\mathfrak{g}$ is $\dim \mathfrak{h}$.
- Every $\mathfrak{g}$ with CSA $\mathfrak{h}$ has a *Cartan decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathbb{C} \cdot e_\alpha,$$

  where $\Phi \subset \mathfrak{h}^*$ is a finite set of *roots*.
- The Killing form $\langle\, , \,\rangle$ of $\mathfrak{g}$ gives an inner product on $\mathfrak{h}^*$.
- To each such triple $(\mathfrak{h}^*, \Phi, \langle\, , \,\rangle)$ we associate a finite graph, a *Dynkin diagram*. The possible Dynkin diagrams are then classified using graph theory methods.

In the end, the complete list of simple Lie algebras over $\mathbb{C}$ is:

- A_n: $\mathfrak{sl}(n+1,\mathbb{C}) = \mathrm{Lie}\,\mathrm{SL}(n+1,\mathbb{C})$, dimension $n(n+2)$.
- B_n: $\mathfrak{so}(2n+1,\mathbb{C}) = \mathrm{Lie}\,\mathrm{SO}(2n+1,\mathbb{C})$, dimension $n(2n+1)$.
- C_n: $\mathfrak{sp}(2n,\mathbb{C})$, dimension $n(2n+1)$. The Lie algebra of the *symplectic group* $\mathrm{Sp}(2n,\mathbb{C})$, the group of automorphisms of $\mathbb{C}^{2n}$ preserving the symplectic 2-form $\sum_{j=1}^{n} \mathrm{d}z_j \wedge \mathrm{d}z_{j+n}$.
- D_n: $\mathfrak{so}(2n,\mathbb{C}) = \mathrm{Lie}\,\mathrm{SO}(2n,\mathbb{C})$, dimension $n(2n-1)$.
- E_6: the exceptional Lie algebra $\mathfrak{e}_6$, dimension 78.
- E_7: the exceptional Lie algebra $\mathfrak{e}_7$, dimension 133.
- E_8: the exceptional Lie algebra $\mathfrak{e}_8$, dimension 248.
- F_4: the exceptional Lie algebra $\mathfrak{f}_4$, dimension 52.
- G_2: the exceptional Lie algebra $\mathfrak{g}_2$, dimension 14.

The number n in $A_n, B_n, \ldots$ is the rank of $\mathfrak{g}$.

10.3. Real forms of Lie algebras

Having classified simple Lie algebras $\mathfrak{g}^{\mathbb{C}}$ over $\mathbb{C}$, we can ask what are the real simple Lie algebras $\mathfrak{g}^{\mathbb{R}}$ with $\mathfrak{g}^{\mathbb{C}} \cong \mathfrak{g}^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, and so classify simple Lie algebras over $\mathbb{R}$.

Example

$\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{U}(n)$ are both real Lie subgroups of $\mathrm{GL}(n,\mathbb{C})$. Their Lie algebras are $\mathfrak{gl}(n,\mathbb{R}) = \{A \in \mathfrak{gl}(n,\mathbb{C}) : A = \bar{A}\}$ and $\mathfrak{u}(n) = \{A \in \mathfrak{gl}(n,\mathbb{C}) : A + \bar{A}^t = 0\}$. So $\mathfrak{gl}(n,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{gl}(n,\mathbb{C}) \cong \mathfrak{u}(n) \otimes_{\mathbb{R}} \mathbb{C}$.
Thus $\mathfrak{gl}(n,\mathbb{R})$ and $\mathfrak{u}(n)$ are nonisomorphic real forms of $\mathfrak{gl}(n,\mathbb{C})$. Of the corresponding Lie groups, $\mathrm{U}(n)$ is compact, but $\mathrm{GL}(n,\mathbb{R})$ is not. Also the Killing form $\langle\,,\,\rangle$ of $\mathfrak{gl}(n,\mathbb{C})$ is positive definite on $\mathfrak{gl}(n,\mathbb{R})$, but negative definite on $\mathfrak{u}(n)$.

Real forms $\mathfrak{g}^{\mathbb{R}}$ of simple Lie algebras $\mathfrak{g}^{\mathbb{C}}$ over $\mathbb{C}$ are generally distinguished by the signature (number of positive and negative eigenvalues) of the Killing form $\langle\,,\,\rangle$ on the CSA $\mathfrak{h}^{\mathbb{R}}$ of $\mathfrak{g}^{\mathbb{R}}$.

It turns out that if G is a compact real Lie group, then the Killing form $\langle\,,\,\rangle$ on its (real) Lie algebra $\mathfrak{g}$ is negative semidefinite. Using this one can prove:

Theorem

Every nontrivial simple Lie algebra $\mathfrak{g}^{\mathbb{C}}$ over $\mathbb{C}$ has exactly one real form $\mathfrak{g}^{\mathbb{R}}$ (up to isomorphism) which is the Lie algebra of a compact Lie group G. We may take G connected and simply-connected.

Thus, the classification of simple Lie algebras over $\mathbb{C}$ gives us the classification of compact simple Lie groups.

Example

$\mathrm{SU}(n)$ is the unique compact real form of $\mathrm{SL}(n,\mathbb{C})$.

10.5. Principal bundles

We have talked about vector bundles on manifolds. Using Lie groups we can define another kind of bundle on a manifold:

Definition

Let X be a manifold, and G a Lie group. A *principal G-bundle* $P \to X$ over X is a manifold P, a smooth map $\pi : P \to X$, and an action $\rho : G \times P \to P$ of G on P, such that each $x \in X$ has an open neighbourhood U_x in X with a commutative diagram

$$\begin{tikzcd}
\pi^{-1}(U_x) \arrow[rr, "\cong"'] \arrow[d, "\pi|_{\pi^{-1}(U_x)}"] & & U_x \times G \arrow[d, "\pi_{U_x}"'] \\
U_x \arrow[rr, equal] & & U_x,
\end{tikzcd}$$

compatible with the G-action $\gamma : (u,\delta) \mapsto (u,\gamma\delta)$ on $U_x \times G$.

This is like the definition of vector bundles in §1.5, but with G in place of $\mathbb{R}^k$.

The frame bundle

Example

Let X be a manifold of dimension n. The *frame bundle* $F \to X$ of X is a principal $\mathrm{GL}(n,\mathbb{R})$-bundle on X defined by

$$F = \big\{(x, f_1, \ldots, f_n) : x \in X,\ (f_1, \ldots, f_n) \text{ is a basis for } T_xX\big\},$$

with projection $\pi : (x, f_1, \ldots, f_n) \mapsto x$, and $\mathrm{GL}(n,\mathbb{R})$-action

$$\begin{aligned}(A_{ij})_{i,j=1}^n : (x, f_1, \ldots, f_n) \mapsto (x, A_{11}f_1 + \cdots + A_{1n}f_n, \ldots,& \\ A_{n1}f_1 + \cdots + A_{nn}f_n).&\end{aligned}$$

G-structures on manifolds

Definition

Let G be a Lie subgroup of $\mathrm{GL}(n,\mathbb{R})$. A *$G$-structure* on an n-manifold X is a submanifold $P \subseteq F$ of the frame bundle $F \to X$, such that P is invariant under the G-action on F induced by the $\mathrm{GL}(n,\mathbb{R})$-action, and $\pi|_P : P \to X$ is a principal G-bundle. That is, P is a principal G-subbundle of $F \to X$.

Example

Let g be a Riemannian metric on an n-manifold X. Define

$$P = \big\{(x, f_1, \ldots, f_n) \in F : (f_1, \ldots, f_n) \text{ is orthonormal w.r.t. } g|_x\big\}.$$

Then P is an $\mathrm{O}(n)$-structure on X, for $G = \mathrm{O}(n) \subset \mathrm{GL}(n,\mathbb{R})$ the orthogonal group. This gives a 1-1 correspondence between Riemannian metrics and $\mathrm{O}(n)$-structures.

Example

Write $\mathrm{GL}_+(n,\mathbb{R})$ for the Lie subgroup of $A \in \mathrm{GL}(n,\mathbb{R})$ with $\det A > 0$. Let X be an oriented n-manifold. Define

$$P = \bigl\{(x, f_1, \ldots, f_n) \in F : (f_1, \ldots, f_n) \text{ is an oriented basis}\bigr\}.$$

Then P is a $\mathrm{GL}_+(n,\mathbb{R})$-structure on X, and this gives a 1-1 correspondence between orientations and $\mathrm{GL}_+(n,\mathbb{R})$-structures.

Example

$\mathrm{SO}(n)$-structures on X correspond to choices of a Riemannian metric g and an orientation on X.

Example

For $G = \mathrm{GL}(n,\mathbb{C}) \subset \mathrm{GL}(2n,\mathbb{R})$, a $\mathrm{GL}(n,\mathbb{C})$-structure on a $2n$-manifold X is an 'almost complex structure' on X.

In this way we can define many interesting geometric structures.

Vector bundles associated to principal bundles

Definition

Let G be a Lie group, X a manifold, $\pi : P \to X$ a principal G-bundle, and V a representation of G. Write $E = (P \times V)/G$, and define a map $\pi_X : E \to X$ by $\pi_X(G \cdot (p, v)) = \pi(p)$. Since each $x \in X$ has an open neighbourhood U_x with $\pi^{-1}(U_x) \cong U_x \times G$, we see that

$$\pi_X^{-1}(U_x) \cong (U_x \times G \times V)/G \cong U_x \times V,$$

identifying $G \cdot (u, \gamma, v) \cong (u, \gamma^{-1} \cdot v)$. Thus we can make E into a manifold, and $\pi_X : E \to X$ into a vector bundle, with fibre V.

Example

Take $G = \mathrm{GL}(n,\mathbb{R})$ and P the frame bundle $F \to X$. The vector bundles associated to the representations of $\mathrm{GL}(n,\mathbb{R})$ on $V = \mathbb{R}^n, (\mathbb{R}^n)^*, \bigotimes^k \mathbb{R}^n \otimes \bigotimes^l (\mathbb{R}^n)^*, \Lambda^l(\mathbb{R}^n)^*$ are $TX, T^*X, \bigotimes^k TX \otimes \bigotimes^l T^*X, \Lambda^l T^*X$.

Example

Suppose $G \subseteq \mathrm{GL}(n,\mathbb{R})$ is a Lie subgroup, and $P \to X$ is a G-structure on X. By the previous example

$$\Lambda^l T^*X \cong (F \times \Lambda^l(\mathbb{R}^n)^*)/\mathrm{GL}(n,\mathbb{R}) \cong (P \times \Lambda^l(\mathbb{R}^n)^*)/G.$$

So the l-forms $\Lambda^l T^*X$ are the vector bundle associated to P and the representation $\Lambda^l(\mathbb{R}^n)^*$ of G. Suppose we have a decomposition of G-representations $\Lambda^l(\mathbb{R}^n)^* = \bigoplus_{i=1}^k V_i$. Then we have a decomposition of vector bundles $\Lambda^l T^*X = \bigoplus_{i\in I} E_i$.

Example

The representation of $\mathrm{SO}(4)$ on $\Lambda^2(\mathbb{R}^4)^*$ splits as $\Lambda^2(\mathbb{R}^4)^* = \Lambda^2_+ \oplus \Lambda^2_-$. So on an oriented Riemannian 4-manifold (X,g) we have a splitting $\Lambda^2 T^*X = \Lambda^2_+ T^*X \oplus \Lambda^2_- T^*X$.

Lie groups and principal bundles are a powerful and flexible language for talking about geometric structures.

Connections on principal bundles

In §6 we discussed connections on vector bundles. The third definition of connection in §6.2 involved a splitting $TE = V \oplus H$.

Definition

Let $\pi : P \to X$ be a principal G-bundle on a manifold X. Then we have a surjective vector bundle morphism $\mathrm{d}\pi : TP \to \pi^*(TX)$ on P. Write $V = \mathrm{Ker}(\mathrm{d}\pi)$. It is a G-invariant vector subbundle of TP. Using the G-action, we see that $V \cong P \times \mathfrak{g} \to P$.
A *connection* on P is a vector subbundle H of TP such that $TP = V \oplus H$, and H is invariant under the G-action on P.
Then $\mathrm{d}\pi$ induces an isomorphism $H \cong \pi^*(TX)$.

A connection on P induces (vector bundle) connections on the vector bundles associated to P by G-representations. One can define curvature of principal bundle connections – the whole story for connections on vector bundles extends to principal bundles.