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Initial problem sheet

1. The n-sphere is $\mathcal{S}^n = \{(x_0,\ldots,x_n) \in \mathbb{R}^{n+1} : x_0^2 + \cdots + x_n^2 = 1\}$. It has an atlas $\{(U_1,\phi_1),(U_2,\phi_2)\}$ with two charts, where $U_1 = U_2 = \mathbb{R}^n$, $\phi_1(U_1) = \mathcal{S}^n \setminus \{(-1,0,\ldots,0)\}$, $\phi_2(U_2) = \mathcal{S}^n \setminus \{(1,0,\ldots,0)\}$, and ϕ_1,ϕ_2 are the inverses of

$$\begin{aligned} \phi_1^{-1} &: (x_0,\ldots,x_n) \longmapsto \tfrac{1}{1+x_0}(x_1,\ldots,x_n) = (y_1,\ldots,y_n) \\ \phi_2^{-1} &: (x_0,\ldots,x_n) \longmapsto \tfrac{1}{1-x_0}(x_1,\ldots,x_n) = (z_1,\ldots,z_n). \end{aligned}$$

Show that $\mathcal{S}^n$ is a Hausdorff, second countable topological space. Compute the transition function $\phi_2^{-1} \circ \phi_1$ between (U_1,ϕ_1) and (U_2,ϕ_2), and show that it is smooth with smooth inverse.

Thus $\{(U_1,\phi_1),(U_2,\phi_2)\}$ is an atlas on $\mathcal{S}^n$, which extends to a unique maximal atlas, making $\mathcal{S}^n$ into a smooth n-dimensional manifold.

2. The n-dimensional projective space $\mathbb{RP}^n$ is the set of 1-dimensional vector subspaces of $\mathbb{R}^{n+1}$. Points in $\mathbb{RP}^n$ are written $[x_0,x_1,\ldots,x_n]$ for $(x_0,\ldots,x_n)$ in $\mathbb{R}^{n+1}\setminus\{0\}$, where $[x_0,\ldots,x_n] = \mathbb{R}\cdot(x_0,\ldots,x_n) \subseteq \mathbb{R}^{n+1}$, and $[\lambda x_0,\ldots,\lambda x_n] = [x_0,\ldots,x_n]$ for $\lambda \in \mathbb{R}\setminus\{0\}$. It has the quotient topology induced from the surjective projection $\pi : \mathbb{R}^{n+1}\setminus\{0\} \to \mathbb{RP}^n$, $\pi : (x_0,\ldots,x_n) \mapsto [x_0,\ldots,x_n]$.

Define a chart (V_i,ψ_i) on $\mathbb{RP}^n$ for $i = 0,\ldots,n+1$ by $V_i = \mathbb{R}^n$ and

$$\psi_i(y_1,\ldots,y_n) = [y_1,\ldots,y_{i-1},1,y_i,\ldots,y_n].$$

Compute the transition functions $\psi_j^{-1} \circ \psi_i$ between (V_i,ψ_i) and (V_j,ψ_j), for $0 \le i < j \le n+1$, and that they are smooth with smooth inverses.

Thus $\{(V_i,\psi_i) : i = 0,\ldots,n\}$ is an atlas on $\mathbb{RP}^n$, which extends to a unique maximal atlas, making $\mathbb{RP}^n$ into a smooth n-dimensional manifold.

3. Define $f : \mathcal{S}^n \to \mathbb{RP}^n$ by $f(x_0,\ldots,x_n) = [x_0,\ldots,x_n]$. Show that f is a smooth surjective map of differentiable manifolds, and that for each $y \in \mathbb{RP}^n$, the inverse image $f^{-1}(y)$ consists of two points.