Problem Sheet 1

1. The real projective line $\mathbb{RP}^1$ (see Sheet 0, Question 2) may be written as $\bigl(\mathbb{R}^2 \setminus \{0\}\bigr)/\sim$, with the quotient topology induced from the projection $\pi : \mathbb{R}^2 \setminus \{0\} \to \mathbb{RP}^1$, where $\sim$ is the equivalence relation on $\mathbb{R}^2 \setminus \{0\}$ given by

$$(x, y) \sim (\lambda x, \lambda y) \qquad \text{for some} \quad \lambda \in \mathbb{R}\setminus\{0\}.$$

In a similar way, define $X = \bigl(\mathbb{R}^2 \setminus \{0\}\bigr)/\approx$, with the quotient topology induced from the projection $\pi : \mathbb{R}^2 \setminus \{0\} \to X$, where $\approx$ is the equivalence relation on $\mathbb{R}^2 \setminus \{0\}$ given by

$$(x, y) \approx (\lambda x, \lambda^{-1} y) \qquad \text{for some} \quad \lambda \in \mathbb{R}\setminus\{0\}.$$

Write $[x, y] \in X$ for the $\approx$-equivalence class of $(x, y) \in \mathbb{R}^2 \setminus \{0\}$.

Define $U = V = \mathbb{R}$ and $\phi : U \to X$, $\psi : V \to X$ by $\phi(u) = [u, 1]$ and $\psi(v) = [1, v]$. Show that (U, ϕ) and (V, ψ) are charts on X, with $\phi(U) = X \setminus \{[1, 0]\}$ and $\psi(V) = X \setminus \{[0, 1]\}$. Compute the transition function $\psi^{-1} \circ \phi$ between (U, ϕ) and (V, ψ). Deduce that $\bigl\{(U, \phi), (V, \psi)\bigr\}$ is an atlas on X.

However, X is not a manifold. Why not?

Draw a picture of X, as a topological space.

2. Let X, Y be manifolds. Show carefully that $X \times Y$ has a unique manifold structure such that if $(U, \phi), (V, \psi)$ are charts on X, Y then $(U \times V, \phi \times \psi)$ is a chart on $X \times Y$, and $\dim(X \times Y) = \dim X + \dim Y$.

P.T.O.

3. (Partitions of unity.) Let X be a manifold, and $\{U_i : i \in I\}$ an open cover of X. A *partition of unity on X subordinate to* $\{U_i : i \in I\}$ is a family $\{\eta_i : i \in I\}$ with $\eta_i : X \to \mathbb{R}$ smooth for $i \in I$, satisfying the conditions:

(i) $\eta_i(x) \in [0, 1]$ for all $i \in I$ and $x \in X$.

(ii) The *support* of η_i is $\operatorname{supp} \eta_i = \overline{\{x \in X : \eta(x) \neq 0\}}$, where '$\overline{\cdots}$' means the closure in X. Then $\operatorname{supp} \eta_i \subseteq U_i$ for each $i \in I$.

(iii) Each $x \in X$ has an open neighbourhood V such that $\eta_i|_V$ is nonzero for only finitely many $i \in I$ (that is, $\{\eta_i : i \in I\}$ is *locally finite*).

(iv) $\sum_{i \in I} \eta_i = 1$, where the sum makes sense by (iii) as near any $x \in X$ there are only finitely many nonzero terms.

It is a theorem that for any manifold X and any open cover $\{U_i : i \in I\}$, there exists a partition of unity $\{\eta_i : i \in I\}$ subordinate to $\{U_i : i \in I\}$. We will prove this when X is *compact.* So suppose X is a compact manifold and $\{U_i : i \in I\}$ is an open cover of X.

(a) Show that for each $x \in X$, we can choose $i_x \in I$ and a smooth function $\zeta_x : X \to \mathbb{R}$ with $\zeta_x \geq 0$, $\zeta_x(x) > 0$, and $\operatorname{supp} \zeta_x \subseteq U_{i_x}$.

[Hint: choose local coordinates $(y_1, \ldots, y_n)$ on an open neighbourhood V of x in X, such that x has coordinates $(0, \ldots, 0)$. Choose $i_x \in I$ with $x \in U_{i_x}$, and small $\epsilon > 0$ such that $V \cap U_{i_x}$ contains the ball $\overline{B}_\epsilon(0)$ of all points y with coordinates $(y_1, \ldots, y_n) \in \mathbb{R}^n$ with $y_1^2 + \cdots + y_n^2 \leq \epsilon^2$. Then consider the function $\zeta_x : X \to \mathbb{R}$ given by $\zeta_x(y) = e^{-1/(\epsilon^2 - y_1^2 - \cdots - y_n^2)}$ if $y = (y_1, \ldots, y_n) \in B_\epsilon(0) \subset V \cap U_{i_x}$, and $\zeta_x(y) = 0$ otherwise.]

(b) Set $V_x = \{y \in X : \zeta_x(y) > 0\}$. Show that there is a finite subset $S \subseteq X$ with $X = \bigcup_{x \in S} V_x$.

(c) Show that $\sum_{x \in S} \zeta_x$ is a positive function on X. For each $i \in I$, define

$$\eta_i = \left(\textstyle\sum_{x \in S} \zeta_x\right)^{-1} \cdot \textstyle\sum_{x \in S : i_x = i} \zeta_x.$$

Show that $\{\eta_i : i \in I\}$ is a partition of unity subordinate to $\{U_i : i \in I\}$.