Problem Sheet 4

1. Let $f : X \to Z$ be a smooth map of a compact oriented manifold X of dimension k to a manifold Z, and $\alpha \in \Omega^k(Z)$ be a closed k-form on Z. Show by integrating $f^*(\alpha)$ on X that f defines a linear map $L_f : H^k(Z) \to \mathbb{R}$.

Let $g : Y \to Z$ be a smooth map from a compact oriented $(k+1)$-manifold with boundary Y, such that $\partial Y = X$ and $g|_{\partial Y} = f$. Show using Stokes' Theorem that $L_f = 0$.

2(a) On the circle $\mathcal{S}^1 \subset \mathbb{R}^2$ denote by $\mathrm{d}\theta$ the 1-form

$$\mathrm{d}\theta = \frac{\mathrm{d}x_2}{x_1} = -\frac{\mathrm{d}x_1}{x_2}.$$

Now consider the product manifold $T^n = \mathcal{S}^1 \times \cdots \times \mathcal{S}^1$, and let $\pi_i : T^n \to \mathcal{S}^1$ be the projection onto the i^{th} factor. By considering the exterior product of all the forms $\pi_i^*(\mathrm{d}\theta)$, deduce that the de Rham cohomology classes $\pi_i^*([\mathrm{d}\theta])$ for $i = 1, \ldots, n$ are linearly independent in $H^1(T^n)$.

(b) Let $n > 1$ and let $f : \mathcal{S}^n \to T^n$ be a smooth map. Noting that $H^1(\mathcal{S}^n) = 0$, prove that the degree of f is zero.

3. What is the degree of the map $\mathbf{x} \mapsto -\mathbf{x}$ on the sphere $\mathcal{S}^n$?

4. The *quaternions* consist of the four-dimensional associative algebra $\mathbb{H}$ of expressions $q = x_0 + ix_1 + jx_2 + kx_3$ where $x_i \in \mathbb{R}$ and i, j, k satisfy the relations

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

Show that if $\bar{q} = x_0 - ix_1 - jx_2 - kx_3$ then $q\bar{q} = \|q\|^2$ and $\|ab\|^2 = \|a\|^2\|b\|^2$.

Show that $f(q) = q^2$ defines a smooth map from $\mathbb{R}^4 \cup \{\infty\} \cong \mathcal{S}^4$ to itself. How many solutions are there to the equation $q^2 = 1$?

What is the degree of f?

How many solutions are there to the equation $q^2 = -1$?

P.T.O.

5. Write down in coordinates $x_2, \ldots, x_n$ where $x_1 \neq 0$, the induced Riemannian metric on the sphere $\mathcal{S}^{n-1} \subset \mathbb{R}^n$. Show that its volume form is $\omega = x_1^{-1} \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n$.

6. Let
$$v = a(x, y)\frac{\partial}{\partial x} + b(x, y)\frac{\partial}{\partial y}$$
be a vector field in $\mathbb{R}^2$ such that
$$\mathcal{L}_v(\mathrm{d}x^2 + \mathrm{d}y^2) = 0$$
Solve this equation for a and b. Show that each vector field integrates to a one parameter group of diffeomorphisms, each of which is of the form
$$\varphi(\mathbf{x}) = A\mathbf{x} + \mathbf{c}$$
where A is a rotation and $\mathbf{c}$ a constant vector.