Complex manifolds and Kähler Geometry

Lecture 3 of 16: Exterior forms on complex manifolds

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These slides available at
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Plan of talk:

3.1. Exterior forms on real and complex manifolds

Let X be a real manifold of dimension n. It has cotangent bundle T^*X and vector bundles of k-forms $\Lambda^k T^*X$ for $k = 0, 1, \ldots, n$, with vector space of sections $C^\infty(\Lambda^k T^*X)$. A k-form α in $C^\infty(\Lambda^k T^*X)$ may be written $\alpha_{a_1 \cdots a_k}$ in index notation. The exterior derivative is $\mathrm{d} : C^\infty(\Lambda^k T^*X) \to C^\infty(\Lambda^{k+1} T^*X)$, given by

$$(\mathrm{d}\alpha)_{a_1 \cdots a_{k+1}} = \sum_{j=1}^{k+1} \frac{(-1)^{j+1}}{k+1} \cdot \frac{\partial \alpha_{a_1 \cdots a_{j-1} a_{j+1} \cdots a_{k+1}}}{\partial x_{a_j}}$$

in index notation, where $\mathrm{d}^2 = 0$.

The *de Rham cohomology* of X is

$$H^k_{\mathrm{dR}}(X;\mathbb{R}) = \frac{\mathrm{Ker}\bigl(\mathrm{d} : C^\infty(\Lambda^k T^*X) \to C^\infty(\Lambda^{k+1} T^*X)\bigr)}{\mathrm{Im}\bigl(\mathrm{d} : C^\infty(\Lambda^{k-1} T^*X) \to C^\infty(\Lambda^k T^*X)\bigr)}.$$

It is isomorphic to the usual cohomology $H^k(X;\mathbb{R})$ of the underlying topological space X.
It will be helpful to consider *complex forms* on X, that is, sections of the complex vector bundles of complexified k-forms $\Lambda^k T^*X \otimes_{\mathbb{R}} \mathbb{C}$.
A complex k-form α in $C^\infty(\Lambda^k T^*X \otimes_{\mathbb{R}} \mathbb{C})$ is of the form $\beta + i\gamma$ for β, γ real k-forms. The *complex conjugate* is $\bar{\alpha} = \beta - i\gamma$. The exterior derivative extends to $\mathrm{d} : C^\infty(\Lambda^k T^*X \otimes_{\mathbb{R}} \mathbb{C}) \to C^\infty(\Lambda^{k+1} T^*X \otimes_{\mathbb{R}} \mathbb{C})$ by $\mathrm{d}(\beta + i\gamma) = \mathrm{d}\beta + i\mathrm{d}\gamma$. *Complex de Rham cohomology* $H^k_{\mathrm{dR}}(X;\mathbb{C})$ is defined using complex forms.

Exterior forms on complex manifolds

Now let (X,J) be a complex manifold in the sense of §2, so that X is a real $2n$-manifold, and $J=J_a^b$ is a complex structure. We will use J to decompose k-forms into components.

Define an action of J on complex 1-forms α on X by $(J\alpha)_a=J_a^b\alpha_b$, in index notation. Then $J^2=-1$, so J has eigenvalues $\pm i$, and $T^*X\otimes_{\mathbb R}\mathbb C$ splits as a direct sum of complex subbundles

$$T^*X\otimes_{\mathbb R}\mathbb C=T^*X^{1,0}\oplus T^*X^{0,1},$$

where $T^*X^{1,0}$, $T^*X^{0,1}$ are the eigenspaces of J with eigenvalues $i,-i$. Both have complex rank n. Sections of $T^*X^{1,0}$, $T^*X^{0,1}$ are called (1,0)-*forms* and (0,1)-*forms*. If $\alpha=\beta+i\gamma$ for β,γ real 1-forms, then α is a (1,0)-form if $\gamma_a=-J_a^b\beta_b$, and a (0,1)-form if $\gamma_a=J_a^b\beta_b$. If $\alpha=\beta+i\gamma$ is a (1,0)-form then $\bar\alpha=\beta-i\gamma$ is a (0,1)-form, and vice versa.

For complex k-forms we have $(\Lambda^kT^*X)\otimes_{\mathbb R}\mathbb C=\Lambda^k_{\mathbb C}(T^*X\otimes_{\mathbb R}\mathbb C)$, that is, we can take exterior powers of the real vector bundle and then complexify, or complexify and then take complex exterior powers. Therefore $(\Lambda^kT^*X)\otimes_{\mathbb R}\mathbb C=\Lambda^k_{\mathbb C}(T^*X^{1,0}\oplus T^*X^{0,1})$. Now $\Lambda^k(U\oplus V)\cong\bigoplus_{p+q=k}\Lambda^pU\otimes\Lambda^qV$. Thus we have a natural splitting $(\Lambda^kT^*X)\otimes_{\mathbb R}\mathbb C=\bigoplus_{p+q=k}\Lambda^{p,q}X$, with

$$\Lambda^{p,q}X\cong\Lambda^p_{\mathbb C}(T^*X^{1,0})\otimes\Lambda^q_{\mathbb C}(T^*X^{0,1}).$$

Sections of $\Lambda^{p,q}X$ are called (p,q)-*forms*. We also have

$$C^\infty(\Lambda^kT^*X)\otimes_{\mathbb R}\mathbb C=\bigoplus_{p+q=k}C^\infty(\Lambda^{p,q}X).$$

If α is a (p,q)-form then $\bar\alpha$ is a (q,p)-form. Thus a (p,q)-form α can only be *real* ($\alpha=\bar\alpha$) if $p=q$.
We can also express $\Lambda^{p,q}X$ as an eigenspace. Let J act on complex k-forms $\alpha=\alpha_{a_1\cdots\alpha_k}$ by

$$J(\alpha)_{a_1\cdots a_k}=\sum_{j=1}^{k}J_{a_j}^{b}\alpha_{a_1\cdots a_{j-1}ba_{j+1}\cdots a_k}.$$

As $\alpha_{a_1\cdots a_k}$ is antisymmetric in $a_1,\ldots,a_k$, so is $J(\alpha)_{a_1\cdots a_k}$, so $J(\alpha)$ is a k-form. It turns out that if α is a (p,q)-form then $J(\alpha)=i(p-q)\alpha$, so $\Lambda^{p,q}X$ is the eigensubbundle of J in $(\Lambda^{p+q}T^*X)\otimes_{\mathbb{R}}\mathbb{C}$ with eigenvalue $i(p-q)$.

(p,q)-forms in complex coordinates

A good way to write (p,q)-forms is in complex coordinates on X. This also fits in with the definition of complex manifolds in §1. Let $(z_1,\ldots,z_n)$ be holomorphic coordinates on some open $U\subseteq X$. Then $\mathrm{d}z_1,\ldots,\mathrm{d}z_n$ and their complex conjugates $\mathrm{d}\bar z_1,\ldots,\mathrm{d}\bar z_n$ are complex 1-forms on U. It turns out that $\mathrm{d}z_1,\ldots,\mathrm{d}z_n$ are (1,0)-forms, and $\mathrm{d}\bar z_1,\ldots,\mathrm{d}\bar z_n$ are (0,1)-forms.
More generally, if $1\leqslant a_1<\cdots<a_p\leqslant n$ and $1\leqslant b_1<\cdots<b_q\leqslant n$ then $\mathrm{d}z_{a_1}\wedge\cdots\wedge\mathrm{d}z_{a_p}\wedge\mathrm{d}\bar z_{b_1}\wedge\cdots\wedge\mathrm{d}\bar z_{b_q}$ is a (p,q)-form on U, and these form a basis of sections of $\Lambda^{p,q}X|_U$, so every (p,q)-form α on U may be written uniquely as

$$\alpha=\sum_{\substack{1\leqslant a_1<\cdots<a_p\leqslant n\\ 1\leqslant b_1<\cdots<b_q\leqslant n}}\alpha_{\underline{a}\underline{b}}\mathrm{d}z_{a_1}\wedge\cdots\wedge\mathrm{d}z_{a_p}\wedge\mathrm{d}\bar z_{b_1}\wedge\cdots\wedge\mathrm{d}\bar z_{b_q}$$

for functions $\alpha_{\underline{a}\underline{b}}:U\to\mathbb{C}$, with $\underline{a}=(a_1,\ldots,a_p)$, $\underline{b}=(b_1,\ldots,b_q)$. The rank of $\Lambda^{p,q}X$, as a complex vector bundle, is $\binom{n}{p}\cdot\binom{n}{q}$.

3.2. The ∂ and $\bar\partial$ operators

Let (X, J) be a complex manifold and $f : X \to \mathbb{C}$ be a smooth function. Then $\mathrm{d}f$ is a complex 1-form on X. Since $C^\infty(T^*X \otimes_{\mathbb{R}} \mathbb{C}) = C^\infty(T^*X^{1,0}) \oplus C^\infty(T^*X^{0,1})$, we may write $\mathrm{d}f$ uniquely as $\mathrm{d}f = \partial f + \bar\partial f$, where $\partial f = \frac{1}{2}(\mathrm{d}f - iJ(\mathrm{d}f))$ is a $(1,0)$-form, and $\bar\partial f = \frac{1}{2}(\mathrm{d}f + iJ(\mathrm{d}f))$ is a $(0,1)$-form. Note that $\bar\partial f = 0$ iff $J(\mathrm{d}f) = i\mathrm{d}f$, that is, iff f is *holomorphic*.

Now let α be a (p,q)-form. As in §3.1, let $(z_1, \ldots, z_n)$ be holomorphic coordinates on $U \subseteq X$, with

$$\alpha|_U = \textstyle\sum_{\underline{a},\underline{b}} \alpha_{\underline{ab}} \mathrm{d}z_{a_1} \wedge \cdots \wedge \mathrm{d}z_{a_p} \wedge \mathrm{d}\bar z_{b_1} \wedge \cdots \wedge \mathrm{d}\bar z_{b_q}.$$

As $\mathrm{d}z_{a_j}, \mathrm{d}\bar z_{b_j}$ are closed, we see that

$$\begin{aligned} \mathrm{d}\alpha|_U &= \textstyle\sum_{\underline{a},\underline{b}} \mathrm{d}\alpha_{\underline{ab}} \wedge \mathrm{d}z_{a_1} \wedge \cdots \wedge \mathrm{d}z_{a_p} \wedge \mathrm{d}\bar z_{b_1} \wedge \cdots \wedge \mathrm{d}\bar z_{b_q} \\ &= \textstyle\sum_{\underline{a},\underline{b}} \partial\alpha_{\underline{ab}} \wedge \mathrm{d}z_{a_1} \wedge \cdots \wedge \mathrm{d}z_{a_p} \wedge \mathrm{d}\bar z_{b_1} \wedge \cdots \wedge \mathrm{d}\bar z_{b_q} \\ &+ \textstyle\sum_{\underline{a},\underline{b}} \bar\partial\alpha_{\underline{ab}} \wedge \mathrm{d}z_{a_1} \wedge \cdots \wedge \mathrm{d}z_{a_p} \wedge \mathrm{d}\bar z_{b_1} \wedge \cdots \wedge \mathrm{d}\bar z_{b_q}. \end{aligned} \tag{3.1}$$

The second sum in (3.1) is a $(p+1,q)$-form, and the third a $(p,q+1)$-form. Hence, if α is a (p,q)-form then $\mathrm{d}\alpha$ is the sum of a $(p+1,q)$-form and a $(p,q+1)$-form. Write $\partial\alpha$ for the $(p+1,q)$-form, and $\bar\partial\alpha$ for the $(p,q+1)$-form. Then $\mathrm{d}\alpha = \partial\alpha + \bar\partial\alpha$, so that $\mathrm{d} = \partial + \bar\partial$. In coordinates we have

$$\begin{aligned} \partial\alpha|_U &= \textstyle\sum_{\underline{a},\underline{b}} \partial\alpha_{\underline{ab}} \wedge \mathrm{d}z_{a_1} \wedge \cdots \wedge \mathrm{d}z_{a_p} \wedge \mathrm{d}\bar z_{b_1} \wedge \cdots \wedge \mathrm{d}\bar z_{b_q}, \\ \bar\partial\alpha|_U &= \textstyle\sum_{\underline{a},\underline{b}} \bar\partial\alpha_{\underline{ab}} \wedge \mathrm{d}z_{a_1} \wedge \cdots \wedge \mathrm{d}z_{a_p} \wedge \mathrm{d}\bar z_{b_1} \wedge \cdots \wedge \mathrm{d}\bar z_{b_q}. \end{aligned}$$

We have $\mathrm{d}^2 = 0$, and $\mathrm{d} = \partial + \bar\partial$. Hence $\partial^2 + (\partial\circ\bar\partial + \bar\partial\circ\partial) + \bar\partial^2 = 0$. But ∂^2 and $\partial\circ\bar\partial + \bar\partial\circ\partial$ and $\bar\partial^2$ map (p,q)-forms to $(p+2,q)$-forms, and to $(p+1,q+1)$-forms, and to $(p,q+2)$-forms, respectively. Thus each of them must vanish, and we have

$$\partial^2 = \partial\circ\bar\partial + \bar\partial\circ\partial = \bar\partial^2 = 0.$$

We now have a *double complex:*

$$
\begin{array}{ccccccc}
C^\infty(\Lambda^{p,q}X) & \xrightarrow[\partial]{} & C^\infty(\Lambda^{p+1,q}X) & \xrightarrow[\partial]{} & C^\infty(\Lambda^{p+2,q}X) & \xrightarrow[\partial]{} & \cdots \\
\downarrow{\scriptstyle (-1)^p\bar\partial} & & \downarrow{\scriptstyle (-1)^{p+1}\bar\partial} & & \downarrow{\scriptstyle (-1)^{p+2}\bar\partial} & & \\
C^\infty(\Lambda^{p,q+1}X) & \xrightarrow[\partial]{} & C^\infty(\Lambda^{p+1,q+1}X) & \xrightarrow[\partial]{} & C^\infty(\Lambda^{p+2,q+1}X) & \xrightarrow[\partial]{} & \cdots \\
\downarrow{\scriptstyle (-1)^p\bar\partial} & & \downarrow{\scriptstyle (-1)^{p+1}\bar\partial} & & \downarrow{\scriptstyle (-1)^{p+2}\bar\partial} & & \\
C^\infty(\Lambda^{p,q+2}X) & \xrightarrow[\partial]{} & C^\infty(\Lambda^{p+1,q+2}X) & \xrightarrow[\partial]{} & C^\infty(\Lambda^{p+2,q+1}X) & \xrightarrow[\partial]{} & \cdots \\
\downarrow{\scriptstyle (-1)^p\bar\partial} & & \downarrow{\scriptstyle (-1)^{p+1}\bar\partial} & & \downarrow{\scriptstyle (-1)^{p+2}\bar\partial} & & \\
\vdots & & \vdots & & \vdots & &
\end{array}
$$

The rows and columns are complexes, as $\partial^2 = \bar\partial^2 = 0$. Using $\partial \circ \bar\partial + \bar\partial \circ \partial = 0$ and the sign changes $(-1)^p$ above, we see that the diagram commutes.

Define the *Dolbeault cohomology*

$$H^{p,q}_{\bar\partial}(X) = \frac{\mathrm{Ker}\big(\bar\partial : C^\infty(\Lambda^{p,q}X) \to C^\infty(\Lambda^{p,q+1}X)\big)}{\mathrm{Im}\big(\bar\partial : C^\infty(\Lambda^{p,q-1}X) \to C^\infty(\Lambda^{p,q}X)\big)}.$$

This is related to de Rham cohomology: there is a spectral sequence going from Dolbeault cohomology to de Rham cohomology. We will see later that if X is a compact Kähler manifold then

$$H^k(X;\mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}_{\bar\partial}(X),$$

but this is false for general complex manifolds.
For compact complex manifolds, $H^{p,q}_{\bar\partial}(X)$ is finite-dimensional.

The operators ∂ and $\bar\partial$ are complex conjugate, in the sense that

$$\overline{(\partial\alpha)} = \bar\partial(\bar\alpha).$$

An operator that is often useful is $\mathrm{d}^c = i(\bar\partial - \partial)$. We have

$$\begin{aligned}\overline{(\mathrm{d}^c\alpha)} &= \overline{(i\bar\partial\alpha - i\partial\alpha)} \\ &= -i\overline{(\bar\partial\alpha)} + i\overline{(\partial\alpha)} \\ &= -i\partial\bar\alpha + i\bar\partial\bar\alpha = \mathrm{d}^c\bar\alpha.\end{aligned}$$

Thus d^c is a *real operator*, that is, it takes real forms to real forms. From $\partial^2 = \partial\circ\bar\partial + \bar\partial\circ\partial = \bar\partial^2 = 0$ we find that $(\mathrm{d}^c)^2 = 0$ and $\mathrm{d}\mathrm{d}^c + \mathrm{d}^c\mathrm{d} = 0$.

Holomorphic forms

If $f : X \to \mathbb{C}$ is a smooth function then f is *holomorphic* if $\bar\partial f = 0$. So it seems natural to call a (p, q)-form α holomorphic if $\bar\partial\alpha = 0$. However, it turns out this is a good idea only if $q = 0$, as if $q > 0$ the condition $\bar\partial\alpha = 0$ is too weak to be called holomorphic. For example, if $\dim_{\mathbb{C}} X = n$ then any (p, n)-form α satisfies $\bar\partial\alpha = 0$, as all $(p, n + 1)$-forms are zero.

Suppose α is a $(p, 0)$-form. Then in holomorphic coordinates $(z_1, \ldots, z_n)$ we may write

$$\alpha|_U = \sum_{a_1<\cdots<a_p} \alpha_{a_1\cdots a_p} \mathrm{d}z_{a_1} \wedge \cdots \wedge \mathrm{d}z_{a_p},$$

so that

$$\bar\partial\alpha|_U = \sum_{a_1<\cdots<a_p} \bar\partial\alpha_{a_1\cdots a_p} \wedge \mathrm{d}z_{a_1} \wedge \cdots \wedge \mathrm{d}z_{a_p}.$$

Therefore $\bar\partial\alpha|_U = 0$ iff $\bar\partial\alpha_{a_1\cdots a_p} = 0$ for all $a_1, \ldots, a_p$, so each $\alpha_{a_1\cdots a_p}$ is a holomorphic function.
We call a $(p, 0)$-form α *holomorphic* if $\bar\partial\alpha = 0$. The Dolbeault cohomology group $H^{p,0}_{\bar\partial}(X)$ is just the vector space of holomorphic $(p, 0)$-forms.

The canonical bundle

Let X be a complex manifold of complex dimension n. Then $\Lambda^{n,0}X$ is a complex vector bundle with rank $\binom{n}{n} \cdot \binom{n}{0} = 1$, that is, a *complex line bundle*, and we have a good notion of holomorphic section of $\Lambda^{n,0}X$, so $\Lambda^{n,0}X$ is a *holomorphic line bundle*.
We call $\Lambda^{n,0}X$ the *canonical bundle* of X, usually written K_X. It will be important in understanding the Ricci curvature of Kähler manifolds.

3.3. Exterior forms on almost complex manifolds

Now let X be a $2n$-manifold, and J an almost complex structure on X. How much of §3.1–§3.2 extends to the almost complex case? The definition of (p,q)-forms and

$$(\Lambda^k T^*X)\otimes_{\mathbb{R}}\mathbb{C}=\bigoplus_{p+q=k}\Lambda^{p,q}X$$

all work as in the complex case. However, we cannot choose holomorphic coordinates $(z_1,\ldots,z_n)$ on X. These were used in §3.2 to show that if α is a (p,q)-form then $\mathrm{d}\alpha$ is the sum of a $(p+1,q)$-form $\partial\alpha$ and a $(p,q+1)$-form $\bar\partial\alpha$.

In fact in the almost complex case, if α is a (p,q)-form then $\mathrm{d}\alpha$ is the sum of four components of types $(p+2,q-1)$, $(p+1,q)$, $(p,q+1)$ and $(p-1,q+2)$, where those of type $(p+2,q-1)$ and $(p-1,q+2)$ are bilinear in α and $N=N^a_{bc}$, the Nijenhuis tensor of J. We can still define $\partial\alpha$ and $\bar\partial\alpha$ as the components of $\mathrm{d}\alpha$ of types $(p+1,q)$ and $(p,q+1)$. Then formally we have

$$\mathrm{d}\alpha=\bar N\cdot\alpha+\partial\alpha+\bar\partial\alpha+N\cdot\alpha,$$

where $\bar N\cdot\alpha$, $N\cdot\alpha$ are of types $(p+2,q-1)$ and $(p-1,q+2)$.

In particular, if α is a (1,0)-form then

$$\pi_{\Lambda^{0,2}}(\mathrm{d}\alpha)_{bc} = N\cdot\alpha = N^a_{bc}\alpha_a,$$

and we can *identify* the Nijenhuis tensor with the component of d mapping

$$C^\infty(\Lambda^{1,0}X) \longrightarrow C^\infty(\Lambda^{0,2}X).$$

The extra terms $\bar N\cdot\alpha, N\cdot\alpha$ mean that we no longer have $\bar\partial^2=0$, etc., instead

$$\bar\partial^2\alpha + \partial(N\cdot\alpha) + N\cdot(\partial\alpha) = 0.$$

So the definition of Dolbeault cohomology does not work in the almost complex case.

3.4. (p,q)-forms in terms of representation theory

Here is a more abstract way of explaining the decomposition of k-forms into (p,q)-forms. Let (X,J) be an almost complex $2n$-manifold. The *frame bundle* F of X is a principal $\mathrm{GL}(2n,\mathbb{R})$-bundle, whose fibre over $x\in X$ is the family of bases $(e_1,\ldots,e_{2n})$ for T_xX. Let $P\subset F$ be the subset of $(e_1,\ldots,e_{2n})$ with $Je_{2i-1}=e_{2i}$ for $i=1,\ldots,n$. Then P is a principal subbundle with structure group $\mathrm{GL}(n,\mathbb{C})\subset\mathrm{GL}(2n,\mathbb{R})$, i.e. a $\mathrm{GL}(n,\mathbb{C})$-*structure*.

Now we may write

$$\Lambda^k T^*X \otimes_{\mathbb{R}} \mathbb{C} \cong F \times_{\mathrm{GL}(2n,\mathbb{R})} (\Lambda^k(\mathbb{R}^{2n})^* \otimes_{\mathbb{R}} \mathbb{C}),$$

that is, $\Lambda^k T^*X \otimes_{\mathbb{R}} \mathbb{C}$ is the vector bundle coming from the principal $\mathrm{GL}(2n,\mathbb{R})$-bundle F and the (irreducible) representation $\Lambda^k(\mathbb{R}^{2n})^* \otimes_{\mathbb{R}} \mathbb{C}$ of $\mathrm{GL}(2n,\mathbb{R})$. Given the principal subbundle P, we have

$$\Lambda^k T^*X \otimes_{\mathbb{R}} \mathbb{C} \cong P \times_{\mathrm{GL}(n,\mathbb{C})} (\Lambda^k(\mathbb{R}^{2n})^* \otimes_{\mathbb{R}} \mathbb{C}).$$

But now $\Lambda^k(\mathbb{R}^{2n})^* \otimes_{\mathbb{R}} \mathbb{C}$ is a *reducible* $\mathrm{GL}(n,\mathbb{C})$-representation: the decomposition into irreducibles is

$$\Lambda^k(\mathbb{R}^{2n})^* \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=k} \Lambda^p_{\mathbb{C}}(\mathbb{C}^n)^* \otimes_{\mathbb{C}} \Lambda^q_{\mathbb{C}}\overline{(\mathbb{C}^n)}^*.$$

(Almost) complex $2n$-manifolds have structure group $\mathrm{GL}(n,\mathbb{C}) \subset \mathrm{GL}(2n,\mathbb{R})$, and decomposition of forms into (p,q)-forms corresponds to decomposition of $\Lambda^k(\mathbb{R}^{2n})^* \otimes_{\mathbb{R}} \mathbb{C}$ into irreducible representations of $\mathrm{GL}(n,\mathbb{C})$.
The same works for other groups. For instance, Kähler manifolds have structure group $\mathrm{U}(n)$, so forms and tensors decompose into pieces corresponding to irreducible representations of $\mathrm{U}(n)$.

Complex manifolds and Kähler Geometry

Lecture 4 of 16: Kähler metrics

September 2019

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Plan of talk:

4.1. Hermitian and Kähler metrics

Let (X, J) be a complex manifold, and g be a Riemannian metric on X. As tensors we have $J = J_a^b$, $g = g_{ab}$. We call g *Hermitian* if $g_{ab} = J_a^c J_b^d g_{cd}$. That is, for all vector fields v, w we have $g(v, w) = g(Jv, Jw)$, so J is an isometry with respect to g. To be Hermitian is a natural pointwise compatibility condition between g and J. If g is arbitrary then

$$h_{ab} = \tfrac{1}{2}(g_{ab} + J_a^c J_b^d g_{cd})$$

is Hermitian. Thus, any (X, J) admits many Hermitian metrics.

Let g be a Hermitian metric on (X, J). Define a 2-tensor $\omega = \omega_{ab}$ by $\omega_{ab} = J_a^c g_{cb}$. That is, $\omega(v, w) = g(Jv, w)$. We have

$$\begin{aligned} \omega_{ba} &= J_b^c g_{ca} = -(J_a^e J_e^d) J_b^c g_{cd} \\ &= -J_a^e (J_b^c J_e^d g_{cd}) = -J_a^e g_{be} \\ &= -J_a^e g_{eb} = -\omega_{ab}, \end{aligned}$$

using $J_a^e J_e^d = -\delta_a^d$, g Hermitian, and g symmetric. Hence $\omega_{ba} = -\omega_{ab}$, that is, ω is a 2-*form*. We call ω the *Hermitian form* of g. It is a *real* $(1,1)$-*form* on (X, J). As g is a metric, ω_{ab} is a *nondegenerate* 2-*form*, that is, $\omega^n \neq 0$ at every point, where $n = \dim_{\mathbb{C}} X$.

We can reconstruct g from J and ω by $g_{ab} = \omega_{ac} J^c_b$. Conversely, given a 2-form ω_{ab}, the tensor $g_{ab} = \omega_{ac} J^c_b$ is symmetric iff ω is of type (1,1). Then g is a metric if it is positive definite, which is an open condition on ω.
It is sometimes useful to consider the complex tensor $h_{ab} = g_{ab} + i\omega_{ab}$. One can show that h_{ab} lies in $T^{*1,0}X \otimes T^{*0,1}X$, that is, h_{ab} is of type (1,0) in the index a, and of type $(0,1)$ in the index b.
In holomorphic coordinates $(z_1, \ldots, z_n)$ with $z_a = x_a + iy_a$ we have

$$h = \sum_{a,b=1}^n A_{ab} \mathrm{d}z_a \otimes \mathrm{d}\bar{z}_b,$$

where $(A_{ab})^n_{a,b=1}$ is an $n \times n$ matrix of complex functions which is Hermitian, that is, $A_{ba} = \bar{A}_{ab}$, and positive definite, and $g = \mathrm{Re}\, h$, $\omega = \mathrm{Im}\, h$.

For example, the Euclidean metric on $\mathbb{C}^n$ is

$$\begin{aligned} h &= \sum_{a=1}^n \mathrm{d}z_a \otimes \mathrm{d}\bar{z}_a, \\ g &= \tfrac{1}{2} \sum_{a=1}^n (\mathrm{d}z_a \otimes \mathrm{d}\bar{z}_a + \mathrm{d}\bar{z}_a \otimes \mathrm{d}z_a) \\ &= \sum_{a=1}^n (\mathrm{d}x_a^2 + \mathrm{d}y_a^2) \\ \omega &= \tfrac{i}{2} \sum_{a=1}^n \mathrm{d}z_a \wedge \mathrm{d}\bar{z}_a = \sum_{a=1}^n \mathrm{d}x_a \wedge \mathrm{d}y_a. \end{aligned}$$

Kähler metrics

Definition

Let g be a Hermitian metric on a complex manifold (X, J), with Hermitian form ω. We call g *Kähler* if ω is closed, $\mathrm{d}\omega = 0$. We call (X, J, g) a *Kähler manifold*, and ω the *Kähler form*.
Then X is a $2n$-manifold and ω is a closed nondegenerate 2-form on X, that is, $\omega^n \neq 0$ at every point. So ω is a *symplectic form*, and (X, ω) a *symplectic manifold*.

We will not cover much symplectic geometry in this course.

Here is an important differential-geometric property:

Proposition 4.1

Let (X, J, g) be a Kähler manifold, with Kähler form ω, and let ∇ be the Levi-Civita connection of g. Then

$$\nabla g = \nabla J = \nabla \omega = 0.$$

So g, J, ω are *constant tensors* on (X, g). This implies that the *holonomy group* $\mathrm{Hol}(g)$ of g (which measures the constant tensors on X) is contained in $\mathrm{U}(n) \subset \mathrm{O}(2n)$.
Kähler metrics are defined by the condition $\mathrm{d}\omega = 0$, which is weak and easy to satisfy: there are lots of closed forms. Because of this, there are lots of Kähler manifolds, and examples are easy to find.
But $\mathrm{d}\omega = 0$ implies the apparently much stronger conditions $\nabla J = \nabla \omega = 0$. These mean that Kähler metrics have very good properties, for instance in their de Rham cohomology.

Sketch proof of Proposition 4.1.

We have $\nabla g = 0$ by definition of the Levi-Civita connection. Since $\omega_{ab} = J_a^c g_{cb}$ and $\nabla g = 0$, we have

$$\nabla_d \omega_{ab} = (\nabla_d J_a^c) g_{cb}.$$

Hence $\nabla\omega$ and ∇J are essentially the same, and $\nabla\omega = 0$ iff $\nabla J = 0$.

Suppose for the moment that J is only an almost complex structure. Then we can show that

$$\nabla_a \omega_{bc} = (\mathrm{d}\omega)_{abc} \oplus g_{ad} N^d_{bc},$$

where N^d_{bc} is the Nijenhuis tensor of J. So $\nabla\omega = \nabla J = 0$ iff $\mathrm{d}\omega = N = 0$. When J is a complex structure $N = 0$, and Proposition 4.1 follows. □

4.2. The Kähler class and Kähler potentials

Let (X, J, g) be a Kähler manifold with Kähler form ω. Then ω is a closed real 2-form, so it has a cohomology class $[\omega]$ in the de Rham cohomology $H^2(X;\mathbb{R})$. We call $[\omega]$ the *Kähler class* of g. Two Kähler metrics g, g' on (X, J) *lie in the same Kähler class* if $[\omega] = [\omega']$.

If $\dim_{\mathbb{C}} X = n > 0$ then $\omega^n = n!\,\mathrm{d}V_g$, where $\mathrm{d}V_g$ is the volume form of g. If X is compact then

$$[\omega]^n \cdot [X] = \int_X \omega^n = n!\,\mathrm{vol}_g(X) > 0.$$

Thus $[\omega]$ is nonzero in $H^2(X;\mathbb{R})$.

Let (X, J) be a complex manifold. Suppose $f : X \to \mathbb{R}$ is smooth. Consider the 2-form $\alpha = \mathrm{d}\mathrm{d}^c f$, with the real operator $\mathrm{d}^c = i(\bar\partial - \partial)$ as in §3.2. Then α is an exact (so closed) real 2-form, since d, d^c are real operators. But also we have

$$\begin{aligned}\mathrm{d}\mathrm{d}^c &= (\partial + \bar\partial) i(\bar\partial - \partial) \\ &= i\big[\bar\partial^2 + \partial\bar\partial - \bar\partial\partial - \partial^2\big] = 2i\partial\bar\partial,\end{aligned}$$

since $\partial^2 = \bar\partial^2 = 0$ and $\partial\bar\partial + \bar\partial\partial = 0$. So $\alpha = 2i\partial\bar\partial f$, and α is a (1,1)-form. Thus, $\alpha = \mathrm{d}\mathrm{d}^c f$ is an *exact real* $(1,1)$*-form*.

Here is a converse to this:

Lemma 4.2 (The Global $\mathrm{d}\mathrm{d}^c$-Lemma.)

Let (X, J, g) be a compact Kähler manifold, and α an exact real $(1,1)$-form on X. Then $\alpha = \mathrm{d}\mathrm{d}^c f$ for some smooth $f : X \to \mathbb{R}$.

It is necessary that X be Kähler; there exist compact complex manifolds (X, J) for which this fails.

Sketch proof.

If $\alpha = \mathrm{d}\mathrm{d}^c f$ then $\alpha \wedge \omega^{n-1} = 2n\Delta f \cdot \omega^n$, where Δ is the 'Laplacian', a second order partial differential operator. So we solve $\Delta f = (\alpha \wedge \omega^{n-1})/(2n\omega^n)$ for f by p.d.e. theory, which is possible as $\int \alpha \wedge \omega^{n-1} = 0$, and then show $\alpha = \mathrm{d}\mathrm{d}^c f$. □

Now let (X, J) be a compact complex manifold and g, g' be Kähler metrics in the same Kähler class. Then $[\omega] = [\omega']$ in $H^2(X;\mathbb{R})$, so $\omega' - \omega$ is an exact real (1,1)-form on X. Hence $\omega' - \omega = \mathrm{dd}^c f$ for some f, i.e. $\omega' = \omega + \mathrm{dd}^c f$.
Conversely, given ω and f we may define a closed real (1,1)-form $\omega' = \omega + \mathrm{dd}^c f$, and then ω' is the Kähler form of a Kähler metric g' if and only if $\omega'(v, Jv) > 0$ for all nonzero vectors v. We call f a *Kähler potential*. Note that we can never write $\omega' = \mathrm{dd}^c f$ when X is compact, as then $[\omega'] = 0$.

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In particular, if $|\mathrm{dd}^c f| < 1$, where $|\,.\,|$ is computed using g, then $\omega'(v, Jv) > 0$ for all $v \neq 0$ is automatic. So all smooth functions $f : X \to \mathbb{R}$ with $\|f\|_{C^2} < 1$ yield a new Kähler metric g' on (X, J) in the Kähler class of g; two functions $f, \tilde f$ yield the same g' iff $\tilde f - f$ is constant (for X compact). This shows that *Kähler metrics occur in infinite-dimensional families* on a fixed complex manifold. There are roughly as many Kähler metrics on X as there are smooth real functions on X.

Complex submanifolds

Suppose (X, J, g) is a Kähler metric, with Kähler form ω, and Y is a complex submanifold of X. Let $\tilde{J} = J|_Y$ be the complex structure on Y, and $\tilde{g} = g|_Y$ the restriction of g to Y as a Riemannian metric. Then $\tilde{g}(\tilde{v}, \tilde{w}) = \tilde{g}(\tilde{J}\tilde{v}, \tilde{J}\tilde{w})$ for all vector fields $\tilde{v}, \tilde{w}$ on Y follows from $g(v, w) = g(Jv, Jw)$ for all vector fields v, w on X. Hence $\tilde{g}$ is Hermitian w.r.t. $\tilde{J}$. The Hermitian form of $\tilde{g}$ is $\tilde{\omega} = \omega|_Y$. So $\mathrm{d}\tilde{\omega} = (\mathrm{d}\omega)|_Y = 0$, and $\tilde{g}$ is Kähler.
Thus, any complex submanifold of a Kähler manifold is Kähler.

4.3. The Fubini–Study metric on $\mathbb{CP}^n$

Complex projective space $\mathbb{CP}^n$ is $(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$. Define $\Pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ by $\pi : (z_0, \ldots, z_n) \mapsto [z_0, \ldots, z_n]$. The *Fubini–Study metric* on $\mathbb{CP}^n$ is the Kähler metric g with Kähler form ω, which is characterized uniquely by the equation

$$\Pi^*(\omega) = \tfrac{1}{4\pi}\mathrm{d}\mathrm{d}^c \log\bigl(\textstyle\sum_{a=0}^n |z_a|^2\bigr).$$

Equivalently, on the chart (U_b, ϕ_b) on $\mathbb{CP}^n$ mapping

$$\begin{aligned}\phi_b :&(w_1, \ldots, w_n) \longmapsto \\ &[w_1, \ldots, w_{b-1}, 1, w_b, \ldots, w_n]\end{aligned}$$

for $b = 0, \ldots, n$ we have

$$\omega = \tfrac{1}{4\pi}\mathrm{d}\mathrm{d}^c \log\bigl(1 + \textstyle\sum_{c=1}^n |w_c|^2\bigr).$$

To show these are equivalent, note that $w_c = z_{c-1}/z_b$ for $c \leqslant b$ and $w_c = z_c/z_b$ for $c > b$, and

$$\mathrm{dd}^c \log\left(|z_b|^2\right) = 0.$$

The action of $\mathrm{U}(n+1)$ on $\mathbb{C}^{n+1}$ descends to an isometry group of $\mathbb{CP}^n$, with

$$\mathbb{CP}^n \cong \mathrm{U}(n+1)/\,\mathrm{U}\,(1)\,\mathrm{U}\,(n).$$

As in §1, complex projective spaces $\mathbb{CP}^n$ have many compact complex submanifolds X, which are called *projective complex manifolds*. Any projective complex manifold is the zeroes of finitely many homogeneous polynomials on $\mathbb{C}^{n+1}$, and so may be studied using algebraic geometry.
The Fubini–Study metric g on $\mathbb{CP}^n$ restricts to a Kähler metric on X. Thus, every projective complex manifold is Kähler. This gives huge numbers of examples of compact Kähler manifolds.

4.4. Exterior forms on Kähler manifolds

Let (X, J, g) be Kähler, with Kähler form ω. Consider complex k-forms $C^\infty(\Lambda^k T^*X \otimes_{\mathbb{R}} \mathbb{C})$ on X. Using J we can decompose into (p, q)-forms for $p + q = k$, and split $\mathrm{d} = \partial + \bar\partial$. In the Kähler situation we have two more toys to play with: the Kähler form ω, and the Hodge star operator $*$ of g. On complex forms we define $*$ to be *complex antilinear*, that is, $*(\beta + i\gamma) = *_{\mathbb{R}}\beta - i *_{\mathbb{R}} \gamma$ where $*_{\mathbb{R}}$ is the Hodge star on real forms. Then $*$ takes (p, q)-forms to $(n-p, n-q)$-forms.

We can use ω to decompose (p, q)-forms further. Let $\dim_{\mathbb{C}} X = n$ and $j, k = 0, \ldots, n$ with $k + 2j \leqslant n$, and consider the linear map

$$\Lambda^k T^*X \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \Lambda^{k+2j} T^*X \otimes_{\mathbb{R}} \mathbb{C}$$

taking $\alpha \mapsto \alpha \wedge \omega^j$.

- if $0 \leqslant j < n - k$ it is injective.
- if $j = n - k$ it is an isomorphism.
- if $j > n - k$ it is surjective.

In particular, when $j = n - k + 1$ it is surjective, but not injective.

For $k = 0, \ldots, n$, call a k-form α *primitive* if $\alpha \wedge \omega^{n-k+1} = 0$. Write $\Lambda^k_0 T^*X$ for the subspace of primitive k-forms, and $\Lambda^{p,q}_0 X$ for the subspace of primitive (p, q)-forms. Then we have

$$\Lambda^k T^*X = \bigoplus_{\substack{j:0\leqslant 2j\leqslant k,\\ k\leqslant n+j}} (\Lambda_0^{k-2j} T^*X) \wedge \omega^j,$$

$$\Lambda^{p,q} X = \bigoplus_{\substack{j:0\leqslant j\leqslant p,q,\\ p+q\leqslant n+j}} (\Lambda_0^{p-j,q-j} X) \wedge \omega^j.$$

In the set up of §3.4, $(\Lambda_0^{p-j,q-j} X) \wedge \omega^j$ corresponds to an irreducible representation of $\mathrm{U}(n)$.

Operators on forms

Let (X, J, g) be a Kähler manifold, with Kähler form ω and Hodge star $*$. Define operators

$$\mathrm{d}^*, \partial^*, \bar\partial^* : C^\infty(\Lambda^k T^*X \otimes_{\mathbb{R}} \mathbb{C}) \to C^\infty(\Lambda^{k-1} T^*X \otimes_{\mathbb{R}} \mathbb{C})$$
$$\text{by} \qquad \mathrm{d}^* = - * \mathrm{d} *, \ \bar\partial^* = - * \bar\partial *, \ \partial^* = - * \partial * .$$

Define the *Lefschetz operator*

$$L : C^\infty(\Lambda^k T^*X \otimes_{\mathbb{R}} \mathbb{C}) \to C^\infty(\Lambda^{k+2} T^*X \otimes_{\mathbb{R}} \mathbb{C})$$
$$\text{by} \qquad L(\alpha) = \alpha \wedge \omega,$$

and the *dual Lefschetz operator*

$$\Lambda : C^\infty(\Lambda^k T^*X \otimes_{\mathbb{R}} \mathbb{C}) \to C^\infty(\Lambda^{k-2} T^*X \otimes_{\mathbb{R}} \mathbb{C})$$
$$\text{by} \qquad \Lambda = (-1)^k * L * .$$

The Kähler identities

Define the d, ∂ and $\bar\partial$-*Laplacians* by $\Delta_{\mathrm{d}} = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$, $\Delta_\partial = \partial\partial^* + \partial^*\partial$ and $\Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$.
Here are the *Kähler identities:*

Theorem 4.3 (The Kähler identities)

(i) $[\partial, L] = [\bar\partial, L] = [\partial^*, \Lambda] = [\bar\partial^*, \Lambda] = 0.$

(ii) $[\partial^*, L] = -i\bar\partial$, $[\bar\partial^*, L] = i\partial$, $[\Lambda, \partial] = i\bar\partial^*$, $[\Lambda, \bar\partial] = -i\partial^*$.

(iii) $\Delta_\partial = \Delta_{\bar\partial} = \frac{1}{2}\Delta_{\mathrm{d}}$.

(iv) Δ_{d} *commutes with* $*, \partial, \bar\partial, \partial^*, \bar\partial^*, L$ *and* Λ.

These are important in Hodge theory. We need $\mathrm{d}\omega = 0$ in the proof, they aren't true for general Hermitian metrics.