Questions on Lie Groups. Sheet 1

A1. Let M be a manifold and u, v, w be vector fields on M. The Jacobi identity is

$$[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0.$$

Prove the Jacobi identity in coordinates $(x^1, \ldots, x^n)$ on a coordinate patch U. Use the coordinate expression for the Lie bracket of vector fields.

In the following, $M_n(\mathbb{R})$ is the algebra of $n \times n$ real matrices, e_{ij} is the matrix in $M_n(\mathbb{R})$ which is 1 in position (i,j) and 0 elsewhere, and I is the identity matrix. If $A \in M_n(\mathbb{R})$ then A^t is the transpose of A, and $GL(n,\mathbb{R}) \subset M_n(\mathbb{R})$ is the Lie group of invertible matrices in $M_n(\mathbb{R})$.

A2. Define $O(n)$ to be the subset of matrices A in $GL(n,\mathbb{R})$ such that $AA^t = I$.

(i)* Show that $O(n)$ is a compact Lie subgroup of $GL(n,\mathbb{R})$.

(ii) Show that the Lie algebra $\mathfrak{o}(n)$ is $\{A \in M_n(\mathbb{R}) : A + A^t = 0\}$.

(iii) For $1 \le i < j \le n$, define $f_{ij} = e_{ij} - e_{ji}$. Find an expression for the Lie bracket $[f_{ij}, f_{kl}]$.

(iv) Show that the f_{ij} form a basis for $\mathfrak{o}(n)$, and hence find the dimension of $O(n)$.

A3***(a)** Let G, H be Lie groups, and let $\Phi : G \to H$ be a Lie group homomorphism. Prove carefully that $\operatorname{Ker}\Phi$ is a Lie subgroup of G, and that $\operatorname{Im}\Phi$ is a Lie subgroup of H if and only if it is *closed*.

(b) Let G be the Lie group $\mathbb{R}$, and let H be the Lie group $\mathbb{R}^2/\mathbb{Z}^2$, that is, the torus T^2. Define a Lie group homomorphism $\Phi : G \to H$ by $\Phi(t) = (t + \mathbb{Z}, \sqrt{2}t + \mathbb{Z})$. Show that $\Phi(G)$ is a subgroup of H, but it is not closed, and thus is not a submanifold of H.

A4***(a)** Prove that the vector subspace $\langle f_{12} + f_{34}, f_{13} + f_{42}, f_{14} + f_{23}\rangle$ of $\mathfrak{o}(4)$ is a Lie subalgebra of $\mathfrak{o}(4)$ isomorphic to $\mathfrak{o}(3)$.

(b) Hence show that $\mathfrak{o}(4)$ is isomorphic as a Lie algebra to $\mathfrak{o}(3) \oplus \mathfrak{o}(3)$.

(c) Let $SO(n)$ be the connected component of $O(n)$ containing the identity. For $n > 2$, define $Spin(n)$ to be the universal cover of $SO(n)$. Deduce that $Spin(4) \cong Spin(3) \times Spin(3)$ as Lie groups.

Note: Here in part **(b)**, if $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras, then the Lie algebra $\mathfrak{g} \oplus \mathfrak{h}$ is the vector space $\mathfrak{g} \oplus \mathfrak{h}$ with the Lie bracket $\big[(x_1, y_1), (x_2, y_2)\big] = \big([x_1, x_2], [y_1, y_2]\big)$, for $x_1, x_2 \in \mathfrak{g}$ and $y_1, y_2 \in \mathfrak{h}$.

Questions for practice

B1. By definition $GL(n, \mathbb{C})$ is the group of invertible $n \times n$ matrices over $\mathbb{C}$. It is a complex Lie group of complex dimension n^2, but we may also regard it as a real Lie group of real dimension $2n^2$. Define $U(n)$ by $U(n) = \big\{A \in GL(n, \mathbb{C}) : A\overline{A}^t = I\big\}$, where $\overline{A}^t$ is the transpose of the complex conjugate of A.

(i)* Show that $U(n)$ is a compact, real Lie group.

(ii) Show that the Lie algebra $\mathfrak{u}(n)$ of $U(n)$ is $\mathfrak{u}(n) = \big\{A \in M_n(\mathbb{C}) : A + \overline{A}^t = 0\big\}$, and calculate its dimension.

(iii) Show that $\mathfrak{u}(2)$ is isomorphic to $\mathbb{R} \oplus \mathfrak{o}(3)$ as a Lie algebra.