Questions on Lie Groups. Sheet 3

A1. Let $\mathfrak{sl}(n, \mathbb{C})$ be the Lie algebra of trace-free $n \times n$ complex matrices, with the usual Lie bracket $[A, B] = AB - BA$. When $n \geq 2$, $\mathfrak{sl}(n, \mathbb{C})$ is a semisimple Lie algebra over $\mathbb{C}$. Write e_{ij} for the matrix that is 1 in position (i, j) and 0 elsewhere.

(a) Let $\mathfrak{h}$ be the vector space of diagonal matrices in $\mathfrak{s}l(n, \mathbb{C})$. Then $\mathfrak{h} \cong \mathbb{C}^{n-1}$, and a basis of $\mathfrak{h}$ is $e_{11} - e_{nn}$, $e_{22} - e_{nn}, \ldots, e_{n-1\,n-1} - e_{nn}$. Show that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{sl}(n, \mathbb{C})$.

(b) Let $x \in \mathfrak{sl}(n, \mathbb{C})$ be a diagonal matrix, with all of its diagonal entries distinct. Show that $N(x) = \mathfrak{h}$.

(c) Now let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, and define $\mathfrak{h} = \langle \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \rangle$. Show by direct calculation that $\mathfrak{h}$ is a maximal abelian subalgebra of $\mathfrak{g}$, but that $\mathfrak{h}$ is not a Cartan subalgebra of $\mathfrak{g}$. This example shows that a maximal abelian subalgebra in a semisimple Lie algebra need not be a Cartan subalgebra.

For the rest of the sheet, let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$, and let $\mathfrak{h}$ be a Cartan subalgebra. Let $\Delta \subset \mathfrak{h}^*$ be the set of roots of $(\mathfrak{g}, \mathfrak{h})$ and $\mathfrak{g}_\alpha$ the root space for $\alpha \in \Delta$. Then the root space decomposition of $\mathfrak{g}$ is

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha.$$

For each $\alpha \in \Delta$, let H_α be the unique element of $\mathfrak{h}$ such that $\langle h, H_\alpha \rangle_{\mathfrak{g}} = \alpha(h)$ for all $h \in \mathfrak{h}$.

A2. Let $h \in \mathfrak{h}$ and suppose $\alpha(h) = 0$ for all $\alpha \in \Delta$. Show that h is in the centre of $\mathfrak{g}$, and deduce that $h = 0$. Hence prove that $\mathfrak{h}^*$ is generated as a vector space by Δ, and $\mathfrak{h}$ is generated by $\{H_\alpha : \alpha \in \Delta\}$.

A3. Let $\alpha \in \Delta$, and suppose $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$. For each $h \in \mathfrak{h}$, show that $\langle h, [x, y] \rangle_{\mathfrak{g}} = \langle x, y \rangle_{\mathfrak{g}} \alpha(h)$. Deduce that $[x, y] = \langle x, y \rangle_{\mathfrak{g}} H_\alpha$.

Hint: Use that fact that $\langle [x, y], z \rangle_{\mathfrak{g}} = \langle [y, z], x \rangle_{\mathfrak{g}}$.

A4. Let $\mathfrak{sl}(3,\mathbb{C})$ be the Lie algebra of trace-free 3×3 complex matrices, with the usual Lie bracket $[A,B] = AB - BA$. Then $\mathfrak{sl}(3,\mathbb{C})$ is a semisimple Lie algebra. Write e_{ij} for the matrix that is 1 in position (i,j) and 0 elsewhere. Let $\mathfrak{h}$ be the Lie subalgebra of diagonal matrices in $\mathfrak{sl}(3,\mathbb{C})$. Then $\mathfrak{h} \cong \mathbb{C}^2$, and a basis of $\mathfrak{h}$ is $e_{11} - e_{33}$, $e_{22} - e_{33}$. Question A1 shows that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{sl}(3,\mathbb{C})$.

(a) Calculate the action of $\mathrm{ad}(\mathfrak{h})$ on $\mathfrak{sl}(3,\mathbb{C})$.

(b) Hence find the roots of $\mathfrak{sl}(3,\mathbb{C})$ (write them in coordinates with respect to the given basis of $\mathfrak{h}$). Draw a diagram of the roots in $\mathbb{R}^2$.

Questions for practice

This question proves a result used in the lectures. Hand in answers to it if you like.

B1*. Let $\alpha \in \Delta$. By question A3, we can choose $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$ such that $[x,y] = H_\alpha$. Let $\beta = \langle \alpha, \alpha \rangle_{\mathfrak{g}}$. Then $\beta \neq 0$. Define a vector subspace V of $\mathfrak{g}$ by

$$V = \mathbb{C} \cdot y + \mathbb{C} \cdot H_\alpha + \sum_{k \geq 1} \mathfrak{g}_{k\alpha}.$$

(i) Write down the action of $\mathrm{ad}(H_\alpha)$ on V.

(ii) Let $d_k = \dim \mathfrak{g}_{k\alpha}$. Show that $\mathrm{Tr}\big(\mathrm{ad}(H_\alpha)|_V\big) = \beta(-1 + d_1 + 2d_2 + \cdots)$.

(iii) Show that $\mathrm{ad}(x)$ and $\mathrm{ad}(y)$ take V to V.

(iv) Deduce that $\mathrm{Tr}\big(\mathrm{ad}(H_\alpha)|_V\big) = 0$.

(v) Prove that $d_1 = 1$ and $d_j = 0$ for $j > 1$.

You have shown that if $\alpha \in \Delta$ then $\dim \mathfrak{g}_\alpha = 1$, and $k\alpha \notin \Delta$ for $k = 2, 3, 4, \ldots$.