Questions on Lie Groups. Sheet 4

A1. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$, with Cartan subalgebra $\mathfrak{h}$, set of roots Δ, and root subspaces $\mathfrak{g}_\alpha$ for $\alpha \in \Delta$.

(a) Using the root space decomposition of $\mathfrak{g}$, prove that for $x \in \mathfrak{h}$,

$$\langle x, x \rangle_{\mathfrak{g}} = \sum_{\alpha \in \Delta} \dim(g_\alpha) \alpha(x)^2.$$

(b) Deduce that the restriction of $\langle\, ,\, \rangle_{\mathfrak{g}}$ to $\mathfrak{h}$ is determined by the subset $\Delta \subset \mathfrak{h}^*$ alone,

(c) Show that $\langle\, ,\, \rangle_{\mathfrak{g}}$ is *positive definite* on $\mathfrak{h}_{\mathbb{R}}$.

A2. Consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}(4, \mathbb{C})$. A Cartan subalgebra $\mathfrak{h}$ for $\mathfrak{g}$ has dimension 3, and there are 12 roots. Identify $\mathfrak{h}$ with $\mathbb{C}^3$, written as column vectors, and $\mathfrak{h}^*$ with $\mathbb{C}^3$, written as row vectors. In one coordinate system, the 12 roots of $\mathfrak{g}, \mathfrak{h}$ are

$$\pm(1\,0\,0), \quad \pm(0\,1\,0), \quad \pm(0\,0\,1), \quad \pm(1\,-1\,0), \quad \pm(1\,0\,-1), \quad \pm(0\,1\,-1).$$

Let $\mathbf{x}$ be the column vector $(x_1\, x_2\, x_3)^t$, and $\mathbf{y}$ be the row vector $(y_1\, y_2\, y_3)$.

(i) Using question A1, show that the Killing form is given by

$$\langle \mathbf{x}, \mathbf{x} \rangle_{\mathfrak{g}} = 6(x_1^2 + x_2^2 + x_3^2) - 4(x_1x_2 + x_2x_3 + x_3x_1).$$

(ii) Deduce that the dual inner product on $\mathfrak{h}^*$ is

$$\langle \mathbf{y}, \mathbf{y} \rangle_{\mathfrak{g}} = \tfrac{1}{4}(y_1^2 + y_2^2 + y_3^2 + y_1y_2 + y_2y_3 + y_3y_1).$$

(iii) Find the lengths of all the roots. Find the angle between the roots $(1\,0\,0)$ and $(0\,1\,0)$.

(iv) The roots $(1\,0\,0)$, $(0\,-1\,0)$ and $(0\,1\,-1)$ form a *simple system.* Calculate the Cartan matrix for this simple system. Draw the Dynkin diagram.

(v)* Can you find the order of the Weyl group?

Hint: consider the mid-points of the 12 edges of a cube.

A3*. Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Prove that $\mathfrak{g}$ is *simple* if and only if the Dynkin diagram of $\mathfrak{g}$ is *connected.*

Hint: First prove that if $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$, then the Dynkin diagram of $\mathfrak{g}$ is the disjoint union of the Dynkin diagrams of $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$. Then (harder) prove that if the Dynkin diagram of $\mathfrak{g}$ splits into 2 disjoint pieces, then there is a corresponding splitting $g = \mathfrak{g}_1 \oplus \mathfrak{g}_2$.

A4*. Suppose that Γ is a Dynkin diagram with n nodes, m_1 single edges, m_2 double edges and m_3 triple edges. Let $\alpha_1, \ldots, \alpha_n$ be the roots corresponding to the nodes. Define $|\alpha_j|$ by $|\alpha_j|^2 = \langle \alpha_j, \alpha_j \rangle$. Define $\beta \in \mathfrak{h}^*$ by $\beta = \sum_{j=1}^{m} |\alpha_j|^{-1} \alpha_j$.

(a) Find a formula for $\langle \beta, \beta \rangle$ in terms of n, m_1, m_2 and m_3.

(b) Hence show that Γ cannot be the Dynkin diagram of a semisimple Lie algebra unless $m_1 + m_2 + m_3 < n$.

(c) Deduce that the Dynkin diagram of a semisimple Lie algebra must be *simply-connected.*

Questions for practice

B1. (1997 M.Sc. exam, q. 9) Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$, and $\mathfrak{h}$ a Cartan subalgebra. Let Δ be the set of roots of $(\mathfrak{g}, \mathfrak{h})$. For each $\alpha \in \Delta$, let H_α be the unique element of $\mathfrak{h}$ with $\langle H_\alpha, x \rangle = \alpha(x)$ for all $x \in \mathfrak{h}$, and let $\mathfrak{h}_\mathbb{R}$ be the real subspace of $\mathfrak{h}$ spanned by $\{H_\alpha : \alpha \in \Delta\}$. Define the *Weyl group* W of $(\mathfrak{g}, \mathfrak{h})$.

Suppose that $\mathfrak{g}$ has rank 2 and that W has order 6. Prove that the points H_α, $\alpha \in \Delta$, are the vertices of a regular hexagon in $\mathfrak{h}_\mathbb{R}$. We may choose a basis (w_1, w_2) of $\mathfrak{h}^*$ such that the roots in Δ are $\pm\alpha_1, \pm\alpha_2, \pm\alpha_3$, where

$$\alpha_1 = 2w_1, \qquad \alpha_2 = -w_1 + \sqrt{3}w_2, \qquad \text{and} \qquad \alpha_3 = -w_1 - \sqrt{3}w_2.$$

Let v_1, v_2 be the dual basis for $\mathfrak{h}$, such that $w_j(v_k) = \delta_{jk}$. Evaluate $\langle v_1, v_1 \rangle$ explicitly. Show that $H_{\alpha_1} = \frac{1}{6}v_1$, and that $\langle \alpha, \alpha \rangle = \frac{1}{3}$ for all $\alpha \in \Delta$.

[You may suppose that $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Delta$, and that $\langle .\,, . \rangle$ is positive definite on $\mathfrak{h}_\mathbb{R}$. Other standard results about root systems and the Weyl group may be used without proof but should be stated clearly.]