INTRODUCTION TO DIFFERENTIABLE MANIFOLDS

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Course by Prof. dr. R.C.A.M. van der Vorst
Solution manual Dr. G.J. Ridderbos

Introduction to differentiable manifolds
Lecture notes version 2.1, November 5, 2012

This is a self contained set of lecture notes. The notes were written by Rob van der Vorst. The solution manual is written by Guit-Jan Ridderbos. We follow the book 'Introduction to Smooth Manifolds' by John M. Lee as a reference text [1]. Additional reading and exercises are take from 'An introduction to manifolds' by Loring W. Tu [2].

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They are meant to be freely available for non-commercial use, in the sense that "free software" is free. More precisely:

CONTENTS

References

[1] John M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2003.

[2] Loring W. Tu. *An introduction to manifolds*. Universitext. Springer, New York, second edition, 2011.

I. Manifolds

1. Topological manifolds

Basically an m-dimensional (topological) manifold is a topological space M which is locally homeomorphic to $\mathbb{R}^m$. A more precise definition is:

Definition 1.1. [1] A topological space M is called an ***m-dimensional (topological) manifold***, if the following conditions hold:

(i) M is a Hausdorff space,
(ii) for any $p \in M$ there exists a neighborhood[2] U of p which is homeomorphic to an open subset $V \subset \mathbb{R}^m$, and
(iii) M has a countable basis of open sets.

Axiom (ii) is equivalent to saying that $p \in M$ has a open neighborhood $U \ni p$ homeomorphic to the open disc D^m in $\mathbb{R}^m$. We say M is ***locally homeomorphic*** to $\mathbb{R}^m$. Axiom (iii) says that M can be covered by countably many of such neighborhoods.

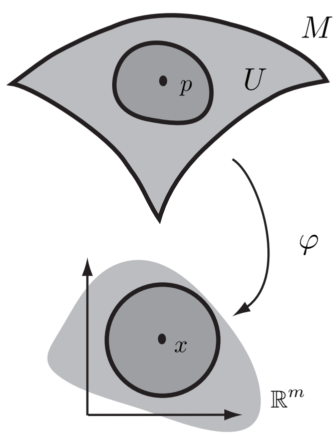

FIGURE 1. Coordinate charts (U, φ).

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[1] See Lee, pg. 3.

[2] Usually an open neighborhood U of a point $p \in M$ is an open set containing p. A neighborhood of p is a set containing an open neighborhood containing p. Here we will always assume that a neighborhood is an open set.

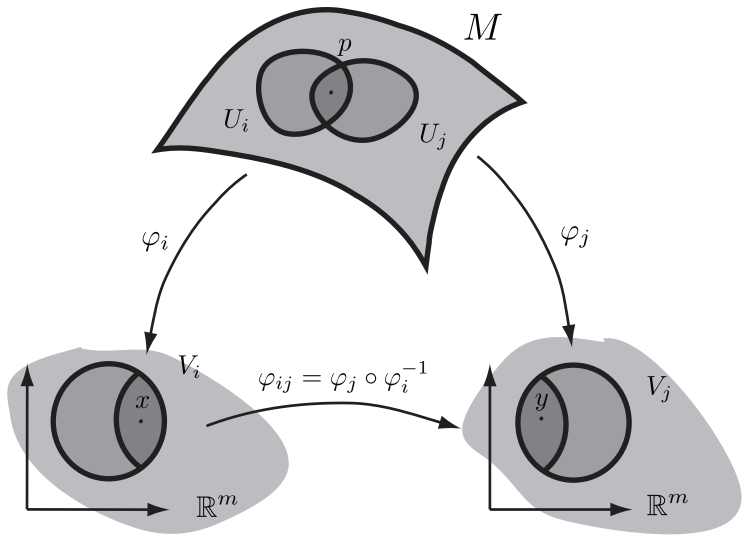

FIGURE 2. The transition maps φ_{ij}.

Recall some notions from topology: A topological space M is called ***Hausdorff*** if for any pair $p, q \in M$, there exist (open) neighborhoods $U \ni p$, and $U' \ni q$ such that $U \cap U' = \varnothing$. For a topological space M with topology τ, a collection $\beta \subset \tau$ is a ***basis*** if and only if each $U \in \tau$ can be written as union of sets in β. A basis is called a ***countable basis*** if β is a countable set.

Figure 1 displays ***coordinate charts*** (U, φ), where U are coordinate neighborhoods, or charts, and φ are (coordinate) homeomorphisms. Transitions between different choices of coordinates are called ***transitions maps*** $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}$, which are again homeomorphisms by definition. We usually write $x = \varphi(p)$, $\varphi : U \to V \subset \mathbb{R}^n$, as ***coordinates*** for U, see Figure 2, and $p = \varphi^{-1}(x)$, $\varphi^{-1} : V \to U \subset M$, as a ***parametrization*** of U. A collection $\mathcal{A} = \{(\varphi_i, U_i)\}_{i \in I}$ of coordinate charts with $M = \cup_i U_i$, is called an ***atlas*** for M.

The following Theorem gives a number of useful characteristics of topological manifolds.

Theorem 1.4. [3] *A manifold is locally connected, locally compact, and the union of countably many compact subsets. Moreover, a manifold is normal and metrizable.*

◀ **1.5** *Example.* $M = \mathbb{R}^m$; the axioms (i) and (iii) are obviously satisfied. As for (ii) we take $U = \mathbb{R}^m$, and φ the identity map. ▶

[3] Lee, 1.6, 1.8, and Boothby.

◀ **1.6** *Example.* $M = S^1 = \{p = (p_1, p_2) \in \mathbb{R}^2 \mid p_1^2 + p_2^2 = 1\}$; as before (i) and (iii) are satisfied. As for (ii) we can choose many different atlases.

(a): Consider the sets

$$U_1 = \{p \in S^1 \mid p_2 > 0\}, \quad U_2 = \{p \in S^1 \mid p_2 < 0\},$$
$$U_3 = \{p \in S^1 \mid p_1 < 0\}, \quad U_4 = \{p \in S^1 \mid p_1 > 0\}.$$

The associated coordinate maps are $\varphi_1(p) = p_1$, $\varphi_2(p) = p_1$, $\varphi_3(p) = p_2$, and $\varphi_4(p) = p_2$. For instance

$$\varphi_1^{-1}(x) = \left(x, \sqrt{1-x^2}\right),$$

and the domain is $V_1 = (-1,1)$. It is clear that φ_i and φ_i^{-1} are continuous, and therefore the maps φ_i are homeomorphisms. With these choices we have found an atlas for S^1 consisting of four charts.

(b): (Stereographic projection) Consider the two charts

$$U_1 = S^1 \backslash \{(0,1)\}, \quad \text{and} \quad U_2 = S^1 \backslash \{(0,-1)\}.$$

The coordinate mappings are given by

$$\varphi_1(p) = \frac{2p_1}{1-p_2}, \quad \text{and} \quad \varphi_2(p) = \frac{2p_1}{1+p_2},$$

which are continuous maps from U_i to $\mathbb{R}$. For example

$$\varphi_1^{-1}(x) = \Big(\frac{4x}{x^2+4}, \frac{x^2-4}{x^2+4}\Big),$$

which is continuous from $\mathbb{R}$ to U_1.

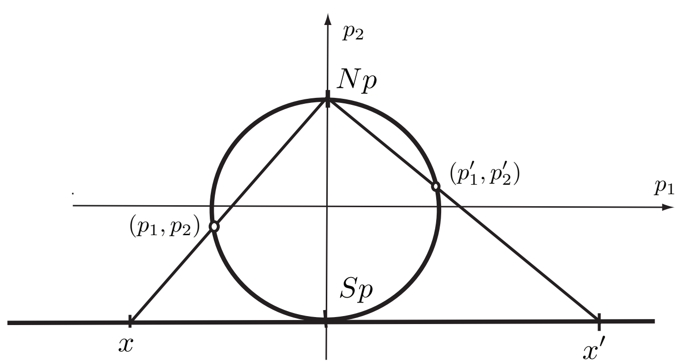

FIGURE 3. The stereographic projection describing φ_1.

(c): (Polar coordinates) Consider the two charts $U_1 = S^1 \backslash \{(1,0)\}$, and $U_2 = S^1 \backslash \{(-1,0)\}$. The homeomorphism are $\varphi_1(p) = \theta \in (0, 2\pi)$ (polar angle counter clockwise rotation), and $\varphi_2(p) = \theta \in (-\pi, \pi)$ (complex argument). For example $\varphi_1^{-1}(\theta) = (\cos(\theta), \sin(\theta))$. ▶

◀ **1.8** *Example.* $M = S^n = \{p = (p_1, \cdots p_{n+1}) \in \mathbb{R}^{n+1} \mid |p|^2 = 1\}$. Obvious extension of the choices for S^1. ▶

◀ **1.9** *Example.* (see Lee) Let $U \subset \mathbb{R}^n$ be an open set and $g : U \to \mathbb{R}^m$ a continuous function. Define

$$M = \Gamma(g) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \ : \ x \in U, \ y = g(x)\},$$

endowed with the subspace topology, see 1.17. This topological space is an n-dimensional manifold. Indeed, define $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ as $\pi(x, y) = x$, and $\varphi = \pi|_{\Gamma(g)}$, which is continuous onto U. The inverse $\varphi^{-1}(x) = (x, g(x))$ is also continuous. Therefore $\{(\Gamma(g), \varphi)\}$ is an appropriate atlas. The manifold $\Gamma(g)$ is homeomorphic to U. ▶

◀ **1.10** *Example.* $M = P\mathbb{R}^n$, the real projective spaces. Consider the following equivalence relation on points on $\mathbb{R}^{n+1} \backslash \{0\}$: For any two $x, y \in \mathbb{R}^{n+1} \backslash \{0\}$ define

$$x \sim y \text{ if there exists a } \lambda \neq 0, \text{ such that } x = \lambda y.$$

Define $P\mathbb{R}^n = \{[x] \ : \ x \in \mathbb{R}^{n+1} \backslash \{0\}\}$ as the set of equivalence classes. One can think of $P\mathbb{R}^n$ as the set of lines through the origin in $\mathbb{R}^{n+1}$. Consider the natural map $\pi : \mathbb{R}^{n+1} \backslash \{0\} \to P\mathbb{R}^n$, via $\pi(x) = [x]$. A set $U \subset P\mathbb{R}^n$ is open if $\pi^{-1}(U)$ is open in $\mathbb{R}^{n+1} \backslash \{0\}$. This makes π a quotient map and π is continuous. The equivalence relation $\sim$ is open (see [2, Sect. 7.6]) and therefore $P\mathbb{R}^n$ Hausdorff and second countable, see 1.17. Compactness of $P\mathbb{R}^n$ can be proves by showing that there exists a homeomorphism from $P\mathbb{R}^n$ to $S^n/\sim$, see [2, Sect. 7.6]. In order to verify that we are dealing with an n-dimensional manifold we need to describe an atlas for $P\mathbb{R}^n$. For $i = 1, \cdots n+1$, define $V_i \subset \mathbb{R}^{n+1} \backslash \{0\}$ as the set of points x for which $x_i \neq 0$, and define $U_i = \pi(V_i)$. Furthermore, for any $[x] \in U_i$ define

$$\varphi_i([x]) = \left(\frac{x_1}{x_i}, \cdots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_{n+1}}{x_i}\right),$$

which is continuous. This follows from the fact that $\varphi \circ \pi$ is continuous. The continuous inverse is given by

$$\varphi_i^{-1}(z_1, \cdots, z_n) = \left[(z_1, \cdots, z_{i-1}, 1, z_i, \cdots, z_n)\right].$$

These charts U_i cover $P\mathbb{R}^n$. In dimension $n = 1$ we have that $P\mathbb{R} \cong S^1$, and in the dimension $n = 2$ we obtain an immersed surface $P\mathbb{R}^2$ as shown in Figure 4. ▶

The examples 1.6 and 1.8 above are special in the sense that they are subsets of some $\mathbb{R}^m$, which, as topological spaces, are given a manifold structure.

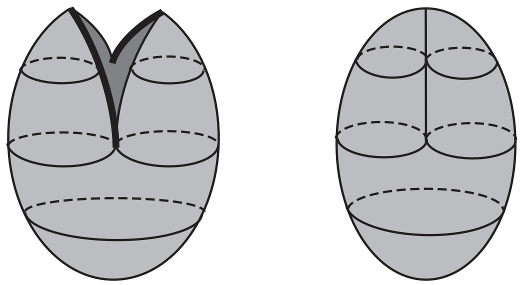

FIGURE 4. Identification of the curves indicated above yields an immersion of $P\mathbb{R}^2$ into $\mathbb{R}^3$.

Define

$$\mathbb{H}^m = \{(x_1, \cdots, x_m) \mid x_m \geq 0\},$$

as the standard Euclidean half-space.

Definition 1.12. A topological space M is called an ***m-dimensional (topological) manifold with boundary*** $\partial M \subset M$, if the following conditions hold:

(i) M is a Hausdorff space,
(ii) for any point $p \in M$ there exists a neighborhood U of p, which is homeomorphic to an open subset $V \subset \mathbb{H}^m$, and
(iii) M has a countable basis of open sets.

Axiom (ii) can be rephrased as follows, any point $p \in M$ is contained in a neighborhood U, which either homeomorphic to D^m, or to $D^m \cap \mathbb{H}^m$. The set M is locally homeomorphic to $\mathbb{R}^m$, or $\mathbb{H}^m$. The ***boundary*** $\partial M \subset M$ is a subset of M which consists of points p for which any neighborhood cannot be homeomorphic to an open subset of $\mathrm{int}\,(\mathbb{H}^m)$. In other words point $p \in \partial M$ are points that lie in the inverse image of $V \cap \partial\mathbb{H}^m$ for some chart (U, φ). Points $p \in \partial M$ are mapped to points on $\partial\mathbb{H}^m$ and ∂M is an $(m-1)$-dimensional topological manifold.

◀ **1.14** *Example.* Consider the bounded cone

$$M = C = \{p = (p_1, p_2, p_3) \in \mathbb{R}^3 \mid p_1^2 + p_2^2 = p_3^2,\ 0 \leq p_3 \leq 1\},$$

with boundary $\partial C = \{p \in C \mid p_3 = 1\}$. We can describe the cone via an atlas consisting of three charts;

$$U_1 = \{p \in C \mid p_3 < 1\},$$

with $x = (x_1, x_2) = \varphi_1(p) = (p_1, p_2 + 1)$, and

$$\varphi_1^{-1}(x) = \left(x_1, x_2 - 1, \sqrt{x_1^2 + (x_2 - 1)^2}\right),$$

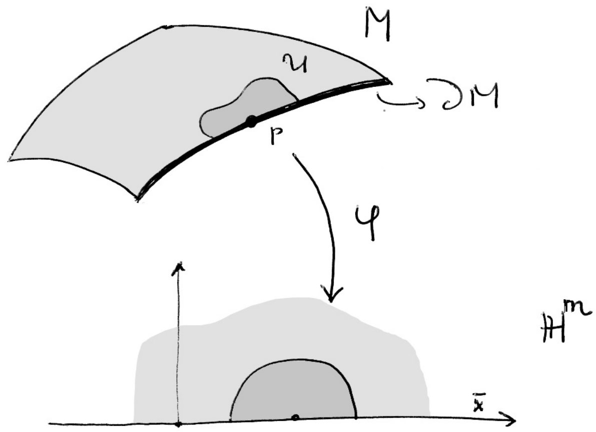

FIGURE 5. Coordinate maps for boundary points.

The other charts are given by

$$U_2 = \{p \in C \mid \frac{1}{2} < p_3 \leq 1, \quad (p_1, p_2) \neq (0, p_3)\},$$
$$U_3 = \{p \in C \mid \frac{1}{2} < p_3 \leq 1, \quad (p_1, p_2) \neq (0, -p_3)\}.$$

For instance φ_2 can be constructed as follows. Define

$$q = \psi(p) = \Big(\frac{p_1}{p_3}, \frac{p_2}{p_3}, p_3\Big), \quad \sigma(q) = \Big(\frac{2q_1}{1-q_2}, 1-q_3\Big),$$

and $x = \varphi_2(p) = (\sigma \circ \psi)(p)$, $\varphi_2(U_2) = \mathbb{R} \times [0, \frac{1}{2}) \subset \mathbb{H}^2$. The map φ_3 is defined similarly. The boundary is given by $\partial C = \varphi_2^{-1}(\mathbb{R} \times \{0\}) \cup \varphi_3^{-1}(\mathbb{R} \times \{0\})$, see Figure 6. ▶

◀ **1.16** *Example.* The open cone $M = C = \{p = (p_1, p_2, p_3) \mid p_1^2 + p_2^2 = p_3^2,\ 0 \leq p_3 < 1\}$, can be described by one coordinate chart, see U_1 above, and is therefore a manifold without boundary (non-compact), and C is homeomorphic to D^2, or $\mathbb{R}^2$, see definition below. ▶

So far we have seen manifolds and manifolds with boundary. A manifold can be either compact or non-compact, which we refer to as ***closed, or open*** manifolds respectively. Manifolds with boundary are also either compact, or non-compact. In both cases the boundary can be compact. Open subsets of a topological manifold are called ***open submanifolds***, and can be given a manifold structure again.

Let N and M be manifolds, and let $f : N \to M$ be a continuous mapping. A mapping f is called a ***homeomorphism*** between N and M if f is continuous and has a continuous inverse $f^{-1} : M \to N$. In this case the manifolds N and M are said

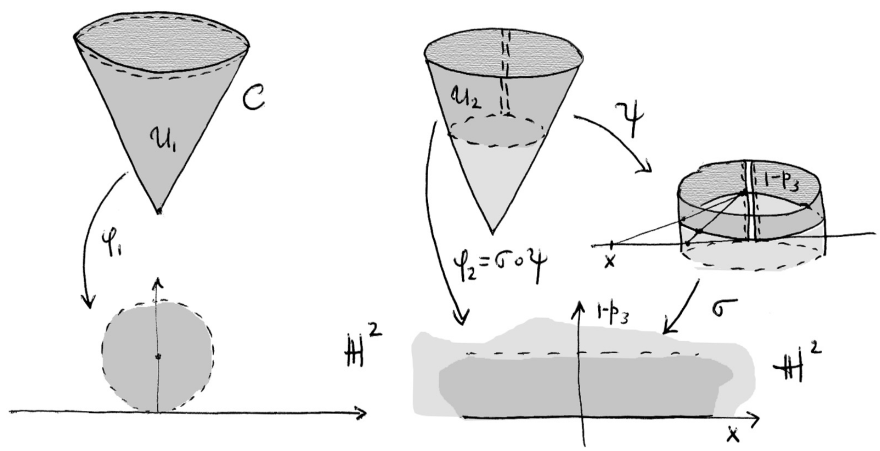

FIGURE 6. Coordinate maps for the cone C.

to be ***homeomorphic***. Using charts (U,φ), and (V,ψ) for N and M respectively, we can give a coordinate expression for f, i.e. $\tilde{f} = \psi \circ f \circ \varphi^{-1}$.

A ***(topological) embedding*** is a continuous injective mapping $f : X \to Y$, which is a homeomorphism onto its image $f(X) \subset Y$ with respect to the subspace topology. Let $f : N \to M$ be an embedding, then its image $f(N) \subset M$ is called a ***submanifold*** of M. Notice that an open submanifold is the special case when $f = i : U \hookrightarrow M$ is an inclusion mapping.

◀ **1.17** *Facts.* Recall the subspace topology. Let X be a topological space and let $S \subset X$ be any subset, then the ***subspace, or relative topology*** on S (induced by the topology on X) is defined as follows. A subset $U \subset S$ is open if there exists an open set $V \subset X$ such that $U = V \cap S$. In this case S is called a (topological) ***subspace*** of X.

For a subjective map $\pi : X \to Y$ we topologize Y via: $U \subset Y$ is open if and only if $\pi^{-1}(U)$ is open in X. In this case π is continuous and is called a ***quotient map***. For a given topological space Z the map $f : Y \to Z$ is continuous if and only if $f \circ \pi : X \to Z$ is continuous. For an equivalence relation $\sim$, the mapping $\pi : X \to X/\sim$ is a quotient map, see [1].

If $\sim$ is a open equivalence relation, i.e. the image of an open set under $\pi : X \to X/\sim$ is open, then $X/\sim$ is Hausdorff if and only if the set $\{(x,x') \in X \times X \mid x \sim x'\}$ is closed in $X \times X$. If X is second countable (countable basis of open sets), then also $X/\sim$ is second countable, see [2]. ▶

2. Differentiable manifolds and differentiable structures

A topological manifold M for which the transition maps $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}$ for all pairs φ_i, φ_j in the atlas are diffeomorphisms is called a ***differentiable, or smooth*** manifold. The transition maps are mappings between open subsets of $\mathbb{R}^m$. Diffeomorphisms between open subsets of $\mathbb{R}^m$ are C^∞-maps, whose inverses are also C^∞-maps. For two charts U_i and U_j the transitions maps are mappings:

$$\varphi_{ij} = \varphi_j \circ \varphi_i^{-1} : \ \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j),$$

and as such are homeomorphisms between these open subsets of $\mathbb{R}^m$.

Definition 2.1. A C^∞-atlas is a set of charts $\mathcal{A} = \{(U, \varphi_i)\}_{i \in I}$ such that

(i) $M = \cup_{i \in I} U_i$,

(ii) the transition maps φ_{ij} are diffeomorphisms between $\varphi_i(U_i \cap U_j)$ and $\varphi_j(U_i \cap U_j)$, for all $i \neq j$ (see Figure 2).

The charts in a C^∞-atlas are said to be ***C^∞-compatible***. Two C^∞-atlases $\mathcal{A}$ and $\mathcal{A}'$ are equivalent if $\mathcal{A} \cup \mathcal{A}'$ is again a C^∞-atlas, which defines a equivalence relation on C^∞-atlases. An equivalence class of this equivalence relation is called a ***differentiable structure*** $\mathcal{D}$ on M. The collection of all atlases associated with $\mathcal{D}$, denoted $\mathcal{A}_\mathcal{D}$, is called the ***maximal atlas*** for the differentiable structure. Figure 7 shows why compatibility of atlases defines an equivalence relation.

Definition 2.3. Let M be a topological manifold, and let $\mathcal{D}$ be a differentiable structure on M with maximal atlas $\mathcal{A}_\mathcal{D}$. Then the pair $(M, \mathcal{A}_\mathcal{D})$ is called a ***(C^∞-)differentiable manifold***.

Basically, a manifold M with a C^∞-atlas, defines a differentiable structure on M. The notion of a differentiable manifold is intuitively a difficult concept. In dimensions 1 through 3 all topological manifolds allow a differentiable structure (only one up to diffeomorphisms). Dimension 4 is the first occurrence of manifolds without a differentiable structure. Also in higher dimensions uniqueness of differentiable structures is no longer the case as the famous example by Milnor shows; S^7 has 28 different (non-diffeomorphic) differentiable structures. The first example of a manifold that allows many non-diffeomorphic differentiable structures occurs in dimension 4; exotic $\mathbb{R}^4$'s. One can also consider C^r-differentiable structures and manifolds. Smoothness will be used here for the C^∞-case.

Theorem 2.4. [4] *Let M be a topological manifold with a C^∞-atlas $\mathcal{A}$. Then there exists a unique differentiable structure $\mathcal{D}$ containing $\mathcal{A}$, i.e. $\mathcal{A} \subset \mathcal{A}_\mathcal{D}$.*

[4] Lee, Lemma 1.10.

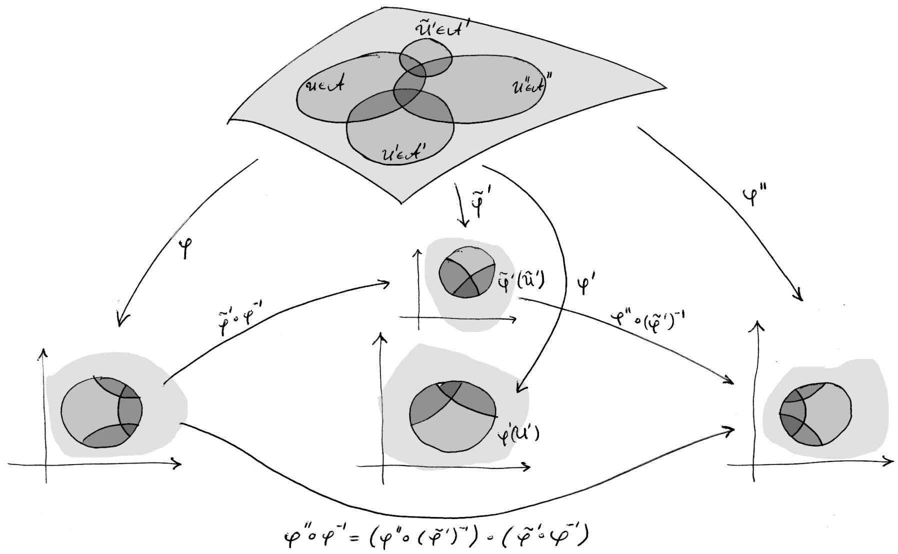

FIGURE 7. Differentiability of $\varphi'' \circ \varphi^{-1}$ is achieved via the mappings $\varphi'' \circ (\tilde{\varphi}')^{-1}$, and $\tilde{\varphi}' \circ \varphi^{-1}$, which are diffeomorphisms since $\mathcal{A} \sim \mathcal{A}'$, and $\mathcal{A}' \sim \mathcal{A}''$ by assumption. This establishes the equivalence $\mathcal{A} \sim \mathcal{A}''$.

Proof: Let $\overline{\mathcal{A}}$ be the collection of charts that are C^∞-compatible with $\mathcal{A}$. By the same reasoning as in Figure 7 we prove that all charts in $\overline{\mathcal{A}}$ are C^∞-compatible with eachother proving that $\overline{\mathcal{A}}$ is a smooth altas. Now any chart that is C^∞-compatible with any chart in $\overline{\mathcal{A}}$ is C^∞-compatible with any chart in $\mathcal{A}$ and is thus in $\overline{\mathcal{A}}$ and $\overline{\mathcal{A}}$ is a maximal atlas. Clearly any other maximal altas is contained in $\overline{\mathcal{A}}$ and therefore $\overline{\mathcal{A}} = \mathcal{A}_{\mathcal{D}}$. ■

◀ **2.5** *Remark.* In our definition of of n-dimensional differentiable manifold we use the local model over ***standard*** $\mathbb{R}^n$, i.e. on the level of coordinates we express differentiability with respect to the standard differentiable structure on $\mathbb{R}^n$, or diffeomorphic to the standard differentiable structure on $\mathbb{R}^n$. ▶

◀ **2.6** *Example.* The cone $M = C \subset \mathbb{R}^3$ as in the previous section is a differentiable manifold whose differentiable structure is generated by the one-chart atlas (C, φ) as described in the previous section. As we will see later on the cone is not a smoothly embedded submanifold. ▶

◀ **2.7** *Example.* $M = S^1$ (or more generally $M = S^n$). As in Section 1 consider S^1 with two charts via stereographic projection. We have the overlap $U_1 \cap U_2 = S^1 \backslash \{(0, \pm 1)\}$, and the transition map $\varphi_{12} = \varphi_2 \circ \varphi_1^{-1}$ given by $y = \varphi_{12}(x) = \frac{4}{x}$,

$x \neq 0, \pm\infty$, and $y \neq 0, \pm\infty$. Clearly, φ_{12} and its inverse are differentiable functions from $\varphi_1(U_1 \cap U_2) = \mathbb{R}\backslash\{0\}$ to $\varphi_2(U_1 \cap U_2) = \mathbb{R}\backslash\{0\}$. ▶

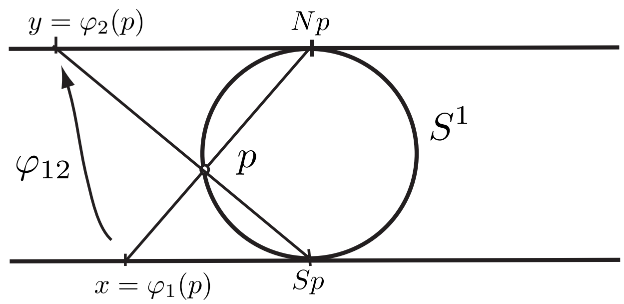

FIGURE 8. The transition maps for the stereographic projection.

◀ **2.9** *Example.* The real projective spaces $P\mathbb{R}^n$, see exercises Chapter VI. ▶

◀ **2.10** *Example.* The generalizations of projective spaces $P\mathbb{R}^n$, the so-called (k,n)-Grassmannians $G^k\mathbb{R}^n$ are examples of smooth manifolds. ▶

Theorem 2.11. [5] *Given a set M, a collection $\{U_\alpha\}_{\alpha\in A}$ of subsets, and injective mappings $\varphi_\alpha : U_\alpha \to \mathbb{R}^m$, such that the following conditions are satisfied:*

(i) $\varphi_\alpha(U_\alpha) \subset \mathbb{R}^m$ *is open for all* α*;*
(ii) $\varphi_\alpha(U_\alpha \cap U_\beta)$ *and* $\varphi_\beta(U_\alpha \cap U_\beta)$ *are open in* $\mathbb{R}^m$ *for any pair* $\alpha, \beta \in A$*;*
(iii) *for* $U_\alpha \cap U_\beta \neq \varnothing$*, the mappings* $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ *are diffeomorphisms for any pair* $\alpha, \beta \in A$*;*
(iv) *countably many sets* U_α *cover* M*.*
(v) *for any pair* $p \neq q \in M$*, either* $p, q \in U_\alpha$*, or there are disjoint sets* U_α, U_β *such that* $p \in U_\alpha$ *and* $q \in U_\beta$*.*

Then M has a unique differentiable manifold structure and $(U_\alpha, \varphi_\alpha)$ are smooth charts.

Proof: Let us give a sketch of the proof. Let sets $\varphi_\alpha^{-1}(V)$, $V \subset \mathbb{R}^n$ open, form a basis for a topology on M. Indeed, if $p \in \varphi_\alpha^{-1}(V) \cap \varphi_\beta^{-1}(W)$ then the latter is again of the same form by (ii) and (iii). Combining this with (i) and (iv) we establish a topological manifold over $\mathbb{R}^m$. Finally from (iii) we derive that $\{(U_\alpha, \varphi_\alpha)\}$ is a smooth atlas. ■

Let N and M be smooth manifolds (dimensions n and m respectively). Let $f : N \to M$ be a mapping from N to M.

[5] See Lee, Lemma 1.23.

Definition 2.12. A mapping $f : N \to M$ is said to be C^∞, or ***smooth*** if for every $p \in N$ there exist charts (U, φ) of p and (V, ψ) of $f(p)$, with $f(U) \subset V$, such that $\tilde{f} = \psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is a C^∞-mapping (from $\mathbb{R}^n$ to $\mathbb{R}^m$).

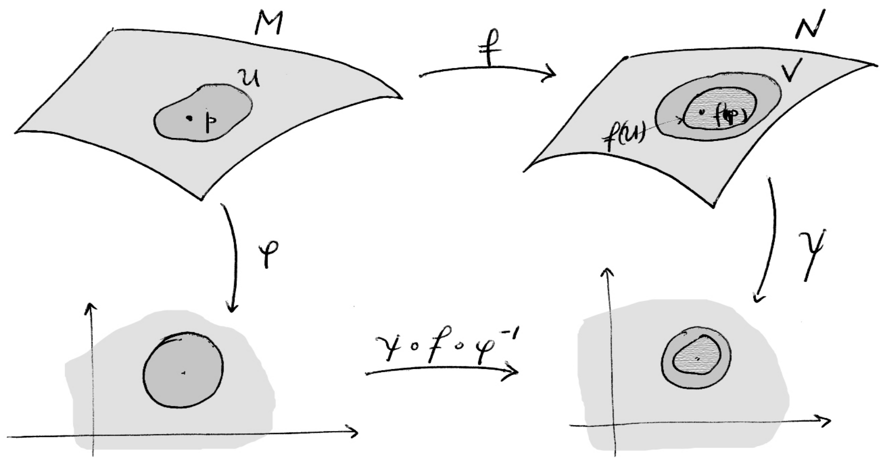

FIGURE 9. Coordinate representation for f, with $f(U) \subset V$.

The above definition also holds true for mappings defined on open subsets of N, i.e. let $W \subset N$ is an open subset, and $f : W \subset N \to M$, then smoothness on W is defined as above by restricting to pionts $p \in W$. With this definition coordinate maps $\varphi : U \to \mathbb{R}^m$ are smooth maps.

The definition of smooth mappings allows one to introduce the notion of differentiable homeomorphism, of ***diffeomorphism*** between manifolds. Let N and M be smooth manifolds. A C^∞-mapping $f : N \to M$, is called a diffeomorphism if it is a homeomorphism and also f^{-1} is a smooth mapping, in which case N and M are said to be ***diffeomorphic***. The associated differentiable structures are also called diffeomorphic. Diffeomorphic manifolds define an equivalence relation. In the definition of diffeomorphism is suffices require that f is a differentiable bijective mapping with smooth inverse (see Theorem 2.15). A mapping $f : N \to M$ is called a ***local diffeomorphism*** if for every $p \in N$ there exists a neighborhood U, with $f(U)$ open in M, such that $f : U \to f(U)$ is a diffeomorphism. A mapping $f : N \to M$ is a diffeomorphism if and only if it is a bijective local diffeomorphism.

◄ **2.14** *Example.* Consider $N = \mathbb{R}$ with atlas $(\mathbb{R}, \varphi(p) = p)$, and $M = \mathbb{R}$ with atlas $(\mathbb{R}, \psi(q) = q^3)$. Clearly these define different differentiable structures (non-compatible charts). Between N and M we consider the mapping $f(p) = p^{1/3}$,

which is a homeomorphism between N and M. The claim is that f is also a diffeomorphism. Take $U = V = \mathbb{R}$, then $\psi \circ f \circ \varphi^{-1}(p) = (p^{1/3})^3 = p$ is the identity and thus C^∞ on $\mathbb{R}$, and the same for $\varphi \circ f^{-1} \circ \psi^{-1}(q) = (q^3)^{1/3} = q$. The associated differentiable structures are diffeomorphic. In fact the above described differentiable structures correspond to defining the differential quotient via $\lim_{h\to 0} \frac{f^3(p+h)-f^3(p)}{h}$.
▶

Theorem 2.15. [6] *Let N, M be smooth manifolds with atlases $\mathcal{A}_N$ and $\mathcal{A}_M$ respectively. The following properties hold:*

(i) *Given smooth maps $f_\alpha : U_\alpha \to M$, for all $U_\alpha \in \mathcal{A}_N$, with $f_\alpha|_{U_\alpha \cap U_\beta} = f_\beta|_{U_\alpha \cap U_\beta}$ for all α, β. Then there exists a unique smooth map $f : N \to M$ such that $f|_{U_\alpha} = f_\alpha$.*

(ii) *A smooth map $f : N \to M$ between smooth manifolds is continuous.*

(iii) *Let $f : N \to M$ be continuous, and suppose that the maps $\tilde{f}_{\alpha\beta} = \psi_\beta \circ f \circ \varphi_\alpha^{-1}$, for charts $(U_\alpha, \varphi_\alpha) \in \mathcal{A}_N$, and $(V_\beta, \psi_\beta) \in \mathcal{A}_M$, are smooth on their domains for all α, β. Then f is smooth.*

Proof: Define f by $f|_{U_\alpha} = f_\alpha$, which is well-defined by the overlap conditioins. Given $p \in M$, there exists a chart $U_\alpha \ni p$ and $\tilde{f} = \tilde{f}_\alpha$ which is smooth by definition, and thus f is smooth.

For any $p \in U$ (chart) and choose $f(p) \in V$ (chart), then $\tilde{f}$ is a smooth map and $f|_U = \psi^{-1} \circ \tilde{f} \circ \varphi : U \to V$ is continuous. Continuity holds for each neighborhood of $p \in M$.

Let $p \in U_\alpha$ and $f(p) \in V_\beta$ and set $U = f^{-1}(V_\beta) \cap U_\alpha \subset U_\alpha$ which is open by continuity of f. Now $\tilde{f} = \tilde{f}_{\alpha\beta}$ on the charts $(U, \varphi_\alpha|_U)$ and (V_β, ψ_β) which proves the differentiability of f. ■

With this theorem at hand we can verify differentiability locally. In order to show that f is a diffeomorphism we need that f is a homeomorphism, or a bijection that satisfies the above local smoothness conditions, as well as for the inverse.

◀ **2.17** *Example.* Let $N = M = P\mathbb{R}^1$ and points in $P\mathbb{R}^1$ are equivalence classes $[x] = [(x_1, x_2)]$. Define the mapping $f : P\mathbb{R}^1 \to P\mathbb{R}^1$ as $[(x_1, x_2)] \mapsto [(x_1^2, x_2^2)]$. Consider the charts $U_1 = \{x_1 \neq 0\}$, and $U_2 = \{x_2 \neq 0\}$, with for example $\varphi_1(x) = \tan^{-1}(x_2/x_1)$, then $\varphi^{-1}(\theta_1) = \big([\cos(\theta_1), \sin(\theta_1)]\big)$, with $\theta_1 \in V_1 = (-\pi/2, \pi/2)$. In local coordinates on V_1 we have

$$\widetilde{f}(\theta_1) = \tan^{-1}\big(\sin^2(\theta_1)/\cos^2(\theta_1)\big),$$

[6]See Lee Lemmas 2.1, 2.2 and 2.3.

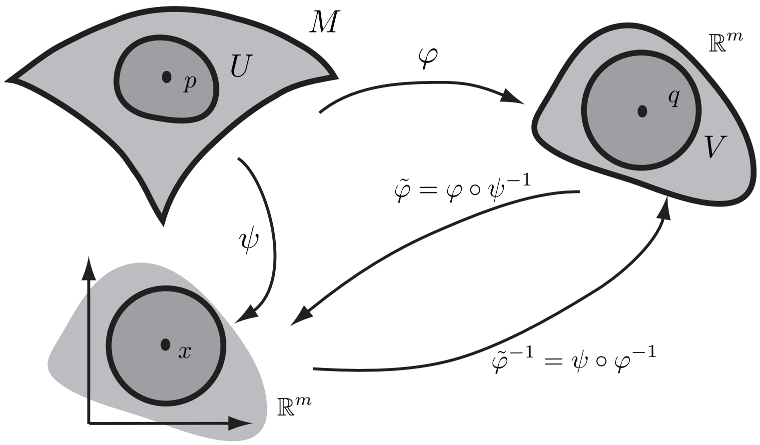

FIGURE 10. Coordinate diffeomorphisms and their local representation as smooth transition maps.

and a similar expression for V_2. Clearly, f is continuous and the local expressions also prove that f is a differentiable mapping using Theorem 2.15. ▶

The coordinate maps φ in a chart (U,φ) are diffeomorphisms. Indeed, $\varphi : U \subset M \to \mathbb{R}^n$, then if (V,ψ) is any other chart with $U \cap V \neq \varnothing$, then by the definition the transition maps $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ are smooth. For φ we then have $\tilde{\varphi} = \varphi \circ \psi^{-1}$ and $\tilde{\varphi}^{-1} = \psi \circ \varphi^{-1}$, which proves that φ is a diffeomorphism, see Figure 10. This justifies the terminology ***smooth charts***.

◀ **2.18** *Remark.* For arbitrary subsets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ a map $f : U \subset \mathbb{R}^m \to V \subset \mathbb{R}^n$ is said to be smooth at $p \in U$, if there exists an open set $U^\dagger \subset \mathbb{R}^m$ containing p, and a smooth map $f^\dagger : U^\dagger \to \mathbb{R}^m$, such that f and $f^\dagger$ coincide on $U \cap U^\dagger$. The latter is called an ***extension*** of f. A mapping $f : U \to V$ between arbitrary subsets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ is called a diffeomorphism if f maps U homeomorphically onto V and both f and f^{-1} are is smooth (as just described above). In this case U and V are said to be diffeomorphic. ▶

◀ **2.19** *Example.* An important example of a class of differentiable manifolds are appropriately chosen subsets of Euclidean space that can be given a smooth manifold structure. Let $M \subset \mathbb{R}^\ell$ be a subset such that every $p \in M$ has a neighborhood $U \ni p$ in M (open in the subspace topology, see Section 3) which is diffeomorphic to an open subset $V \subset \mathbb{R}^m$ (or, equivalently an open disc $D^m \subset \mathbb{R}^m$). In this case the set M is a smooth m-dimensional manifold. Its topology is the subsapce topology and the smooth structure is inherited from the standard smooth structure on $\mathbb{R}^\ell$, which can be described as follows. By definition a coordinate map φ is a diffeo-

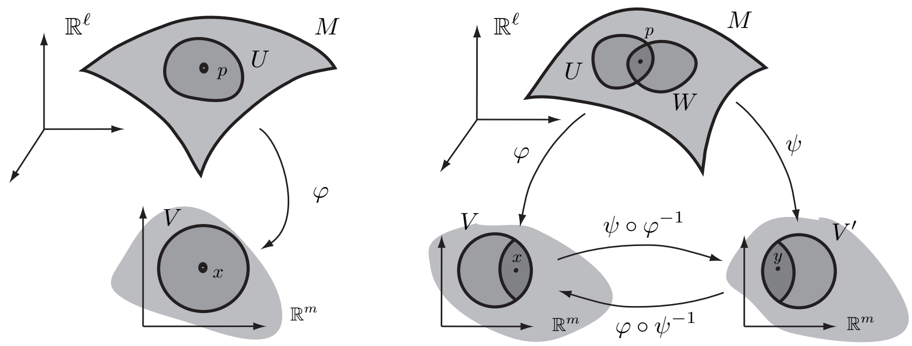

FIGURE 11. The transitions maps $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ smooth mappings establishing the smooth structure on M.

morphisms which means that $\varphi : U \to V = \varphi(U)$ is a smooth mapping (smoothness via a smooth map $\varphi^\dagger$), and for which also φ^{-1} is a smooth map. This then directly implies that the transition maps $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ are smooth mappings, see Figure 11. In some books this construction is used as the definition of a smooth manifold. ▶

◀ **2.21** *Example.* Let us consider the cone $M = C$ described in Example 1 (see also Example 2 in Section 1). We already established that C is manifold homeomorphic to $\mathbb{R}^2$, and moreover C is a differentiable manifold, whose smooth structure is defined via a one-chart atlas. However, C is not a smooth manifold with respect to the induced smooth structure as subset of $\mathbb{R}^3$. Indeed, following the definition in the above remark, we have $U = C$, and coordinate homeomorphism $\varphi(p) = (p_1, p_2) = x$. By the definition of smooth maps it easily follows that φ is smooth. The inverse is given by $\varphi^{-1}(x) = \big(x_1, x_2, \sqrt{x_1^2 + x_2^2}\big)$, which is clearly ***not*** differentiable at the cone-top $(0,0,0)$. The cone C is not a smoothly embedded submainfold of $\mathbb{R}^3$ (topological embedding). ▶

Let U, V and W be open subsets of $\mathbb{R}^n$, $\mathbb{R}^k$ and $\mathbb{R}^m$ respectively, and let $f : U \to V$ and $g : V \to W$ be smooth maps with $y = f(x)$, and $z = g(y)$. Then the ***Jacobians*** are

$$Jf|_x = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_n} \end{pmatrix}, \qquad Jg|_y = \begin{pmatrix} \frac{\partial g_1}{\partial y_1} & \cdots & \frac{\partial g_1}{\partial y_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial y_1} & \cdots & \frac{\partial g_m}{\partial y_k} \end{pmatrix},$$

and $J(g \circ f)|_x = Jg|_{y=f(x)} \cdot Jf|_x$ (chain-rule). The commutative diagram for the maps f, g and $g \circ f$ yields a commutative diagram for the Jacobians:

$$\begin{tikzcd} & V \arrow[dr, "g"] & \\ U \arrow[ur, "f"] \arrow[rr, "g\circ f"'] & & W \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{R}^k \arrow[dr, "Jg|_y"] & \\ \mathbb{R}^n \arrow[ur, "Jf|_x"] \arrow[rr, "J(g\circ f)|_x"'] & & \mathbb{R}^m \end{tikzcd}$$

For diffeomorphisms between open sets in $\mathbb{R}^n$ we have a number of important properties. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets, and let $f : U \to V$ is a diffeomorphism, then $n = m$, and $Jf|_x$ is invertible for any $x \in U$. Using the above described commutative diagrams we have that $f^{-1} \circ f$ yields that $J(f^{-1})|_{y=f(x)} \cdot Jf|_x$ is the identity on $\mathbb{R}^n$, and $f \circ f^{-1}$ yields that $Jf|_x \cdot J(f^{-1})|_{y=f(x)}$ is the identity on $\mathbb{R}^m$. Thus Jf_x has an inverse and consequently $n = m$. Conversely, if $f : U \to \mathbb{R}^n$, $U \subset \mathbb{R}^n$, open, then we have the Inverse Function Theorem;

Theorem 2.22. *If $Jf|_x : \mathbb{R}^n \to \mathbb{R}^n$ is invertible, then f is a diffeomorphism between sufficiently small neighborhoods U' and $f(U')$ of x and y respectively.*

3. Immersions, submersions and embeddings

Let N and M be smooth manifolds of dimensions n and m respectively, and let $f : N \to M$ be a smooth mapping. In local coordinates $\tilde{f} = \psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$, with respects to charts (U, φ) and (V, ψ). The ***rank*** of f at $p \in N$ is defined as the rank of $\tilde{f}$ at $\varphi(p)$, i.e. $\mathrm{rk}(f)|_p = \mathrm{rk}(J\tilde{f})|_{\varphi(p)}$, where $J\tilde{f}|_{\varphi(p)}$ is the Jacobian of f at p:

$$J\tilde{f}|_{x=\varphi(p)} = \begin{pmatrix} \frac{\partial \tilde{f}_1}{\partial x_1} & \cdots & \frac{\partial \tilde{f}_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \tilde{f}_m}{\partial x_1} & \cdots & \frac{\partial \tilde{f}_m}{\partial x_n} \end{pmatrix}$$

This definition is independent of the chosen charts, see Figure 12. Via the commutative diagram in Figure 12 we see that $\tilde{\tilde{f}} = (\psi' \circ \psi^{-1}) \circ \tilde{f} \circ (\varphi' \circ \varphi^{-1})^{-1}$, and by the chain rule $J\tilde{\tilde{f}}|_{x'} = J(\psi' \circ \psi^{-1})|_y \cdot J\tilde{f}|_x \cdot J(\varphi' \circ \varphi^{-1})^{-1}|_{x'}$. Since $\psi' \circ \psi^{-1}$ and $\varphi' \circ \varphi^{-1}$ are diffeomorphisms it easily follows that $\mathrm{rk}(J\tilde{f})|_x = \mathrm{rk}(J\tilde{\tilde{f}})|_{x'}$, which shows that our notion of rank is well-defined. If a map has constant rank for all $p \in N$ we simply write $\mathrm{rk}(f)$. These are called ***constant rank*** mappings. Let us now consider the various types of constant rank mappings between manifolds.

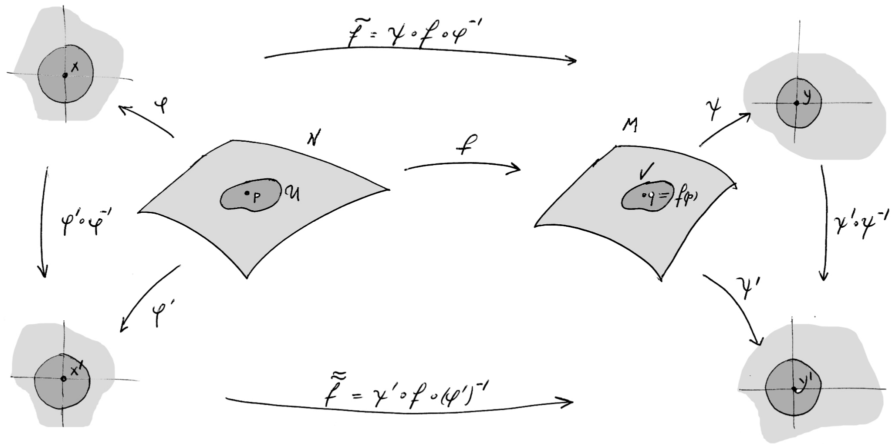

FIGURE 12. Representations of f via different coordinate charts.

Definition 3.2. A mapping $f : N \to M$ is called an ***immersion*** if $\mathrm{rk}(f) = n$, and a ***submersion*** if $\mathrm{rk}(f) = m$. An immersion that is injective,[7] or 1-1, and is a homeomorphism onto (surjective mapping[8]) its image $f(N) \subset M$, with respect to the subspace topology, is called a ***smooth embedding***.

A smooth embedding is an (injective) immersion that is a topological embedding.

Theorem 3.3. [9] *Let $f : N \to M$ be smooth with constant rank $\mathrm{rk}(f) = k$. Then for each $p \in N$, and $f(p) \in M$, there exist coordinates (U, φ) for p and (V, ψ) for $f(p)$, with $f(U) \subset V$, such that*

$$(\psi \circ f \circ \varphi^{-1})(x_1, \cdots x_k, x_{k+1}, \cdots x_n) = (x_1, \cdots x_k, 0, \cdots, 0).$$

◀ **3.4** *Example.* Let $N = (-\frac{\pi}{2}, \frac{3\pi}{2})$, and $M = \mathbb{R}^2$, and the mapping f is given by $f(t) = (\sin(2t), \cos(t))$. In Figure 13 we displayed the image of f. The Jacobian is given by

$$Jf|_t = \begin{pmatrix} 2\cos(2t) \\ -\sin(t) \end{pmatrix}$$

Clearly, $\mathrm{rk}(Jf|_t) = 1$ for all $t \in N$, and f is an injective immersion. Since N is an open manifold and $f(N) \subset M$ is a compact set with respect to the subspace

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[7]A mapping $f : N \to M$ is injective if for all $p, p' \in N$ for which $f(p) = f(p')$ it holds that $p = p'$.

[8]A mapping $f : N \to M$ is surjective if $f(N) = M$, or equivalently for every $q \in M$ there exists a $p \in N$ such that $q = f(p)$.

[9]See Lee, Theorem 7.8 and 7.13.

topology, it follows f is not a homeomorphism onto $f(N)$, and therefore is not an embedding. ▶

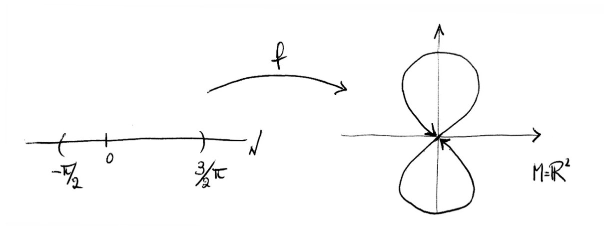

FIGURE 13. Injective parametrization of the figure eight.

◀ **3.6** *Example.* Let $N = S^1$ be defined via the atlas $\mathcal{A} = \{(U_i, \varphi_i)\}$, with $\varphi_1^{-1}(t) = ((\cos(t), \sin(t))$, and $\varphi_2^{-1}(t) = ((\sin(t), \cos(t))$, and $t \in (-\frac{\pi}{2}, \frac{3\pi}{2})$. Furthermore, let $M = \mathbb{R}^2$, and the mapping $f : N \to M$ is given in local coordinates; in U_1 as in Example 1. Restricted to $S^1 \subset \mathbb{R}^2$ the map f can also be described by

$$f(x,y) = (2xy, x).$$

This then yields for U_2 that $\tilde{f}(t) = (\sin(2t), \sin(t))$. As before $\mathrm{rk}(f) = 1$, which shows that f is an immersion of S^1. However, this immersion is not injective at the origin in $\mathbb{R}^2$, indicating the subtle differences between these two examples, see Figures 13 and 14. ▶

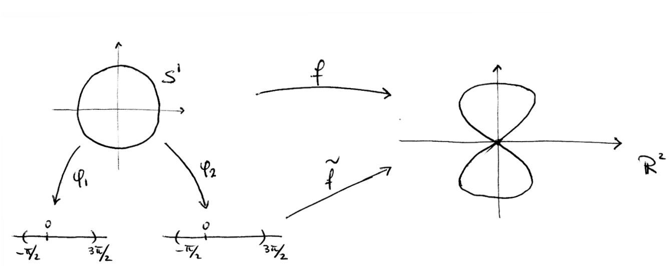

FIGURE 14. Non-injective immersion of the circle.

◀ **3.8** *Example.* Let $N = \mathbb{R}$, $M = \mathbb{R}^2$, and $f : N \to M$ defined by $f(t) = (t^2, t^3)$. We can see in Figure 15 that the origin is a special point. Indeed, $\mathrm{rk}(Jf)|_t = 1$ for all $t \neq 0$, and $\mathrm{rk}(Jf)|_t = 0$ for $t = 0$, and therefore f is not an immersion. ▶

◀ **3.10** *Example.* Consider $M = P\mathbb{R}^n$. We established $P\mathbb{R}^n$ as smooth manifolds. For $n = 1$ we can construct an embedding $f : P\mathbb{R}^1 \to \mathbb{R}^2$ as depicted in Figure 16. For $n = 2$ we find an immersion $f : P\mathbb{R}^2 \to \mathbb{R}^3$ as depicted in Figure 17. ▶

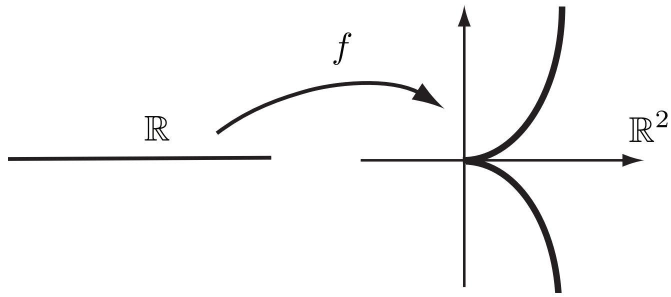

FIGURE 15. The map f fails to be an immersion at the origin.

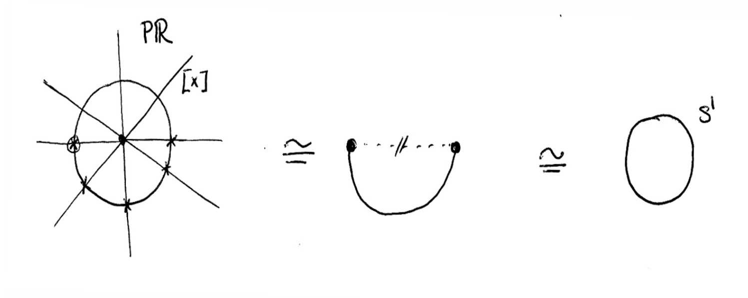

FIGURE 16. $P\mathbb{R}$ is diffeomorphic to S^1.

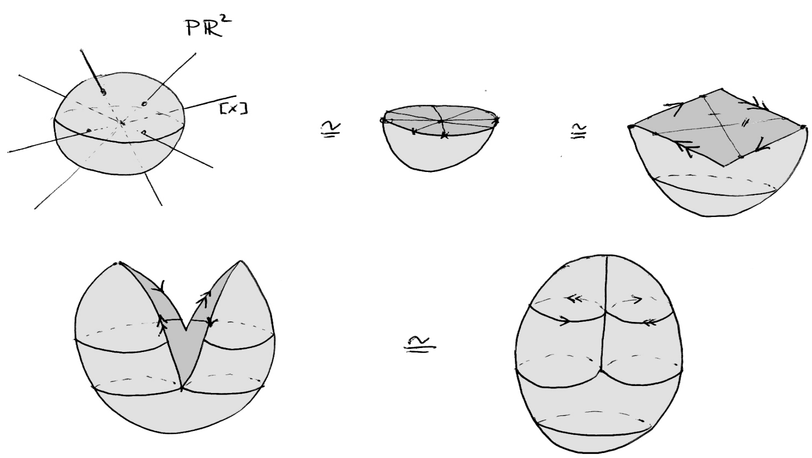

FIGURE 17. Identifications for $P\mathbb{R}^2$ giving an immersed non-orientable surface in $\mathbb{R}^3$.

◀ **3.13** *Example.* Let $N = \mathbb{R}^2$ and $M = \mathbb{R}$, then the projection mapping $f(x,y) = x$ is a submersion. Indeed, $Jf|_{(x,y)} = (1\ 0)$, and $\mathrm{rk}(Jf|_{(x,y)}) = 1$. ▶

◀ **3.14** *Example.* Let $N = M = \mathbb{R}^2$ and consider the mapping $f(x,y) = (x^2, y)^t$. The Jacobian is

$$Jf|_{(x,y)} = \begin{pmatrix} 2x & 0 \\ 0 & 1 \end{pmatrix},$$

and $\mathrm{rk}(Jf)|_{(x,y)} = 2$ for $x \neq 0$, and $\mathrm{rk}(Jf)|_{(x,y)} = 1$ for $x = 0$. See Figure 18. ▶

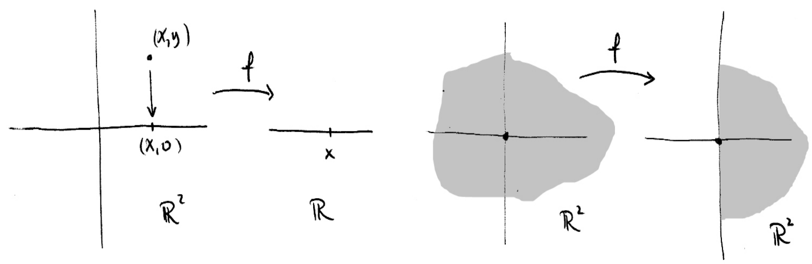

FIGURE 18. The projection (submersion), and the folding of the plane (not a submersion).

◀ **3.16** *Example.* We alter Example 2 slightly, i.e. $N = S^1 \subset \mathbb{R}^2$, and $M = \mathbb{R}^2$, and again use the atlas $\mathcal{A}$. We now consider a different map f, which, for instance on U_1, is given by $\tilde{f}(t) = (2\cos(t), \sin(t))$ (or globally by $f(x,y) = (2x, y)$). It is clear that f is an injective immersion, and since S^1 is compact it follows from Lemma 3.18 below that S^1 is homeomorphic to it image $f(S^1)$, which show that f is a smooth embedding (see also Appendix in Lee). ▶

◀ **3.17** *Example.* Let $N = \mathbb{R}$, $M = \mathbb{R}^2$, and consider the mapping $f(t) = (2\cos(t), \sin(t))$. As in the previous example f is an immersion, not injective however. Also $f(\mathbb{R}) = f(S^1)$ in the previous example. The manifold N is the universal covering of S^1 and the immersion f descends to a smooth embedding of S^1. ▶

Lemma 3.18. [10] *Let $f : N \to M$ be an injective immersion. If*

(i) *N is compact, or if*
(ii) *f is a proper map,*[11]

then f is a smooth embedding.

Proof: For (i) we argue as follows. Since N is compact any closed subset $X \subset N$ is compact, and therefore $f(X) \subset M$ is compact and thus closed; f is a ***closed***

[10] Lee, Prop. 7.4, and pg. 47.

[11] A(ny) map $f : N \to M$ is called proper if for any compact $K \subset M$, it holds that also $f^{-1}(K) \subset N$ is compact. In particular, when N is compact, continuous maps are proper.

mapping.[12] We are done if we show that f is a topological embedding. By the assumption of injectivity, $f : N \to f(N)$ is a bijection. Let $X \subset N$ be closed, then $(f^{-1})^{-1}(X) = f(X) \subset f(N)$ is closed with respect to the subspace topology, and thus $f^{-1} : f(N) \to N$ is continuous, which proves that f is a homeomorphism onto its image and thus a topological embedding.

A straightforward limiting argument shows that proper continuous mappings bewteen manifolds are closed mappings (see Exercises). ■

Let us start with defining the notion of embedded submanifolds.

Definition 3.19. *A subset $N \subset M$ is called a smooth **embedded** n**-dimensional submanifold** in M if for every $p \in N$, there exists a chart (U, φ) for M, with $p \in U$, such that*

$$\varphi(U \cap N) = \varphi(U) \cap \left(\mathbb{R}^n \times \{0\}\right) = \{x \in \varphi(U) \ : \ x_{n+1} = \cdots = x_m = 0\}.$$

*The **co-dimension** of N is defined as* $\operatorname{codim} N = \dim M - \dim N$.

The set $W = U \cap N$ in N is called a ***n*-dimensional slice, or *n*-slice** of U, see Figure 19, and (U, φ) a ***slice chart***. The associated coordinates $x = (x_1, \cdots, x_n)$ are called ***slice coordinates***. Embedded submanifolds can be characterized in terms of embeddings.

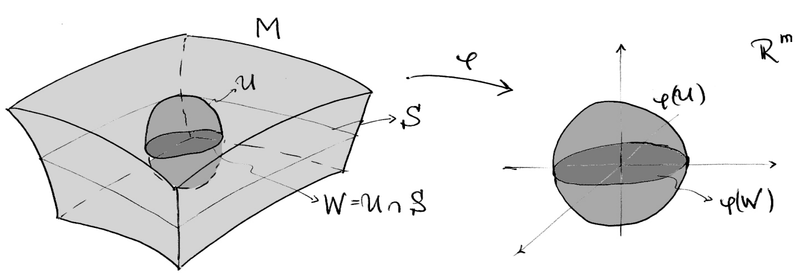

FIGURE 19. Take a k-slice W. On the left the image $\varphi(W)$ corresponds to the Eucliden subspace $\mathbb{R}^k \subset \mathbb{R}^m$.

Theorem 3.21. [13] *Let $N \subset M$ be a smooth embedded n-submanifold. Endowed with the subspace topology, N is a n-dimensional manifold with a unique (induced) smooth structure such that the inclusion map $i : N \hookrightarrow M$ is an embedding (smooth).*

---

[12] A mapping $f : N \to M$ is called ***closed*** if $f(X)$ is closed in M for any closed set $X \subset N$. Similarly, f is called ***open*** if $f(X)$ is open in M for every open set $X \subset N$.

[13] See Lee, Thm's 8.2

To get an idea let us show that N is a topological manifold. Axioms (i) and (iii) are of course satisfied. Consider the projection $\pi : \mathbb{R}^m \to \mathbb{R}^n$, and an inclusion $j : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$\pi(x_1, \cdots, x_n, x_{n+1}, \cdots, x_m) = (x_1, \cdots, x_n),$$
$$j(x_1, \cdots, x_n) = (x_1, \cdots, x_n, 0, \cdots, 0).$$

Now set $Z = (\pi \circ \varphi)(W) \subset \mathbb{R}^n$, and $\bar{\varphi} = (\pi \circ \varphi)|_W$, then $\bar{\varphi}^{-1} = (\varphi^{-1} \circ j)|_Z$, and $\bar{\varphi} : W \to Z$ is a homeomorphism. Therefore, pairs $(W, \bar{\varphi})$ are charts for N, which form an atlas for N. The inclusion $i : N \hookrightarrow M$ is a topological embedding.

Given slice charts (U, φ) and (U', φ') and associated charts $(W, \bar{\varphi})$ and $(W', \bar{\varphi}')$ for N. For the transitions maps it holds that $\bar{\varphi} \circ \bar{\varphi}^{-1} = \pi \circ \varphi' \circ \varphi^{-1} \circ j$, which are diffeomorphisms, which defines a smooth atlas for N. The inclusion $i : N \hookrightarrow M$ can be expressed in local coordinates;

$$\tilde{i} = \varphi \circ i \circ \bar{\varphi}^{-1}, \quad (x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_n, 0, \cdots, 0),$$

which is an injective immersion. Since i is also a topological embedding it is thus a smooth embedding. It remains to prove that the smooth structure is unique, see Lee Theorem 8.2.

Theorem 3.22. [14] *The image of an embedding is a smooth embedded submanifold.*

Proof: By assumption $\mathrm{rk}(f) = n$ and thus for any $p \in N$ it follows from Theorem 3.3 that

$$\widetilde{f}(x_1, \cdots, x_n) = (x_1, \cdots, x_n, 0, \cdots, 0),$$

for appropriate coordinates (U, φ) for p and (V, ψ) for $f(p)$, with $f(U) \subset V$. Consequently, $f(U)$ is a slice in V, since $\psi(f(U)$ satisfies Definition 3.19. By assumption $f(U)$ is open in $f(N)$ and thus $f(U) = A \cap f(N)$ for some open set $A \subset M$. By replacing V by $V' = A \cap V$ and restricting ψ to V', (V', ψ) is a slice chart with slice $V' \cap f(N) = V' \cap f(U)$. ■

Summarizing we conclude that embedded submanifolds are the images of smooth embeddings.

A famous result by Whitney says that considering embeddings into $\mathbb{R}^m$ is not not really a restriction for defining smooth manifolds.

Theorem 3.23. [15] *Any smooth n-dimensional manifold M can be (smoothly) embedded into* $\mathbb{R}^{2n+1}$.

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[14] See Lee, Thm's 8.3

[15] See Lee, Ch. 10.

A subset $N \subset M$ is called an ***immersed submanifold*** if N is a smooth n-dimensional manifold, and the mapping $i : N \hookrightarrow M$ is a (smooth) immersion. This means that we can endow N with an appropriate manifold topology and smooth structure, such that the natural inclusion of N into M is an immersion. If $f : N \to M$ is an injective immersion we can endow $f(N)$ with a topology and unique smooth structure; a set $U \subset f(N)$ is open if and only if $f^{-1}(U) \subset N$ is open, and the (smooth) coordinate maps are taken to be $\varphi \circ f^{-1}$, where φ's are coordinate maps for N. This way $f : N \to f(N)$ is a diffeomorphism, and $i : f(N) \hookrightarrow M$ an injective immersion via the composition $f(N) \to N \to M$. This proves:

Theorem 3.24. [16] *Immersed submanifolds are exactly the images of injective immersions.*

We should point out that embedded submanifolds are examples of immersed submanifolds, but not the other way around. For any immersion $f : N \to M$, the image $f(N)$ is called an ***immersed manifold*** in M.

In this setting Whitney established some improvements of Theorem 3.23. Namely, for dimension $n > 0$, any smooth n-dimensional manifold can be embedded into $\mathbb{R}^{2n}$ (e.g. the embedding of curves in $\mathbb{R}^2$). Also, for $n > 1$ any smooth n-dimensional manifold can be immersed into $\mathbb{R}^{2n-1}$ (e.g. the Klein bottle). In this course we will often think of smooth manifolds as embedded submanifolds of $\mathbb{R}^m$. An important tool thereby is the general version of Inverse Function Theorem,

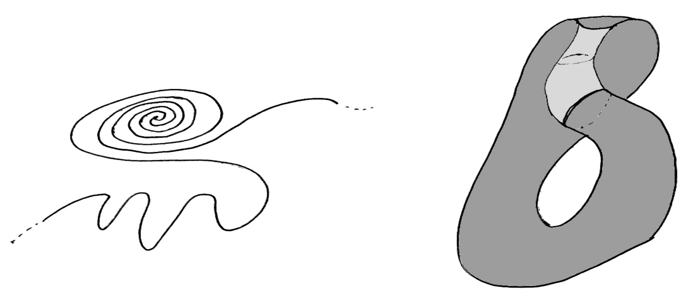

FIGURE 20. An embedding of $\mathbb{R}$ [left], and an immersion of S^2 [right] called the Klein bottle.

which can easily be derived from the 'Euclidean' version 2.22.

Theorem 3.26. [17] *Let N, M be smooth manifolds, and $f : N \to M$ is a smooth mapping. If, at some point $p \in N$, it holds that $J\tilde{f}|_{\varphi(p)=x}$ is an invertible matrix,*

[16]See Lee, Theorem 8.16.

[17]See Lee, Thm's 7.6 and 7.10.

then there exist sufficiently small neighborhoods $U_0 \ni p$, and $V_0 \in f(p)$ such that $f|_{U_0} : U_0 \to V_0$ is a diffeomorphism.

As a direct consequence of this result we have that if $f : N \to M$, with $\dim N = \dim M$, is an immersion, or submersion, then f is a local diffeomorphism. If f is a bijection, then f is a (global) diffeomorphism.

Theorem 3.27. *Let $f : N \to M$ be a constant rank mapping with $\mathrm{rk}(f) = k$. Then for each $q \in f(N)$, the level set $S = f^{-1}(q)$ is an embedded submanifold in N with co-dimension equal to k.*

Proof: Clearly, by continuity S is closed in N. By Theorem 3.3 there are coordinates (U, φ) of $p \in S$ and (V, ψ) of $f(p) = q$, such that

$$\widetilde{f}(x_1, \cdots, x_k, x_{k+1}, \cdots, x_n) = (x_1, \cdots, x_k, 0, \cdots, 0) = \psi(q) = 0.$$

The points in $S \cap U$ are characterized by $\varphi(S \cap U) = \{x \mid (x_1, \cdots, x_k) = 0\}$. Therefore, $S \cap U$ is a $(n-k)$-slice, and thus S is a smooth submanifold in M of codimension k. ∎

In particular, when $f : N \to M$ is a submersion, then for each $q \in f(N)$, the level set $S = f^{-1}(q)$ is an embedded submanifold of co-dimension $\operatorname{codim} S = m = \dim M$. In the case of maximal rank this statement can be restricted to just one level. A point $p \in N$ is called a ***regular point*** if $\mathrm{rk}(f)|_p = m = \dim M$, otherwise a point is called a ***critical point***. A value $q \in f(N)$ is called a ***regular value*** if all points $p \in f^{-1}(q)$ are regular points, otherwise a value is called a ***critical value***. If q is a regular value, then $f^{-1}(q)$ is called a ***regular level set***.

Theorem 3.28. *Let $f : N \to M$ be a smooth map. If $q \in f(N)$ is a regular value, then $S = f^{-1}(q)$ is an embedded submanifold of co-dimension equal to $\dim M$.*

Proof: Let us illustrate the last result for the important case $N = \mathbb{R}^n$ and $M = \mathbb{R}^m$. For any $p \in S = f^{-1}(q)$ the Jacobian $Jf|_p$ is surjective by assumption. Denote the kernel of $Jf|_p$ by $\ker Jf|_p \subset \mathbb{R}^n$, which has dimension $n - m$. Define

$$g : N = \mathbb{R}^n \to \mathbb{R}^{n-m} \times \mathbb{R}^m \cong \mathbb{R}^n,$$

by $g(\xi) = (L\xi, f(\xi) - q)^t$, where $L : N = \mathbb{R}^n \to \mathbb{R}^{n-m}$ is any linear map which is invertible on the subspace $\ker Jf|_p \subset \mathbb{R}^n$. Clearly, $Jg|_p = L \oplus Jf|_p$, which, by construction, is an invertible (linear) map on $\mathbb{R}^n$. Applying Theorem 2.22 (Inverse Function Theorem) to g we conclude that a sufficiently small neighborhood of U of p maps diffeomorphically onto a neighborhood V of $(L(p), 0)$. Since g is a diffeomorphism it holds that g^{-1} maps $\left(\mathbb{R}^{n-m} \times \{0\}\right) \cap V$ onto $f^{-1}(q) \cap U$ (the $0 \in \mathbb{R}^m$ corresponds to q). This exactly says that every point $p \in S$ allows an $(n-m)$-slice and is therefore an $(n-m)$-dimensional submanifold in $N = \mathbb{R}^n$ ($\operatorname{codim} S = m$). ∎

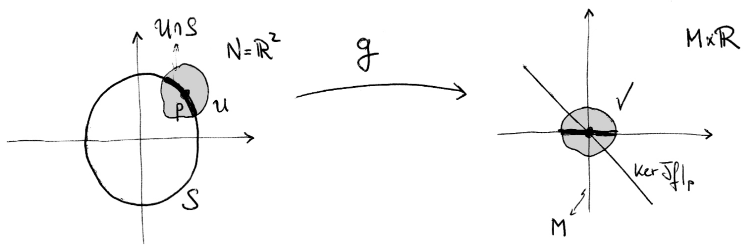

FIGURE 21. The map g yields 1-slices for the set S.

◀ **3.30** *Example.* Let us start with an explicit illustration of the above proof. Let $N = \mathbb{R}^2$, $M = \mathbb{R}$, and $f(p_1, p_2) = p_1^2 + p_2^2$. Consider the regular value $q = 2$, then $Jf|_p = (2p_1 \quad 2p_2)$, and $f^{-1}(2) = \{p \ : \ p_1^2 + p_2^2 = 2\}$, the circle with radius $\sqrt{2}$. We have $\ker Jf|_p = \mathrm{span}\{(p_1, -p_2)^t\}$, and is always isomorphic to $\mathbb{R}$. For example fix the point $(1,1) \in S$, then $\ker Jf|_p = \mathrm{span}\{(1,-1)^t\}$ and define

$$g(\xi) = \begin{pmatrix} L(\xi) \\ f(\xi) - 2 \end{pmatrix} = \begin{pmatrix} \xi_1 - \xi_2 \\ \xi_1^2 + \xi_2^2 - 2 \end{pmatrix}, \quad Jg|_p = \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}$$

where the linear map is $L = (1 \quad -1)$. The map g is a local diffeomorphism and on $S \cap U$ this map is given by

$$g(\xi_1, \sqrt{2 - \xi_1^2}) = \begin{pmatrix} \xi_1 - \sqrt{2 - \xi_1^2} \\ 0 \end{pmatrix},$$

with $\xi_1 \in (1 - \varepsilon, 1 + \varepsilon)$. The first component has derivative $1 + \frac{\xi_1}{\sqrt{2 - \xi_1^2}}$, and therefore $S \cap U$ is mapped onto a set of the form $\big(\mathbb{R} \times \{0\}\big) \cap V$. This procedure can be carried out for any point $p \in S$, see Figure 21. ▶

◀ **3.31** *Example.* Let $N = \mathbb{R}^2 \backslash \{(0,0)\} \times \mathbb{R} = \mathbb{R}^3 \backslash \{(0,0,\lambda)\}$, $M = (-1, \infty) \times (0, \infty)$, and

$$f(x,y,z) = \begin{pmatrix} x^2 + y^2 - 1 \\ 1 \end{pmatrix}, \quad \text{with} \quad Jf|_{(x,y,z)} = \begin{pmatrix} 2x & 2y & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We see immediately that $\mathrm{rk}(f) = 1$ on N. This map is not a submersion, but is of constant rank, and therefore for any value $q \in f(N) \subset M$ it holds that $S = f^{-1}(q)$ is a embedded submanifold, see Figure 22. ▶

◀ **3.33** *Example.* Let N, M as before, but now take

$$f(x,y,z) = \begin{pmatrix} x^2 + y^2 - 1 \\ z \end{pmatrix}, \quad \text{with} \quad Jf|_{(x,y,z)} = \begin{pmatrix} 2x & 2y & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

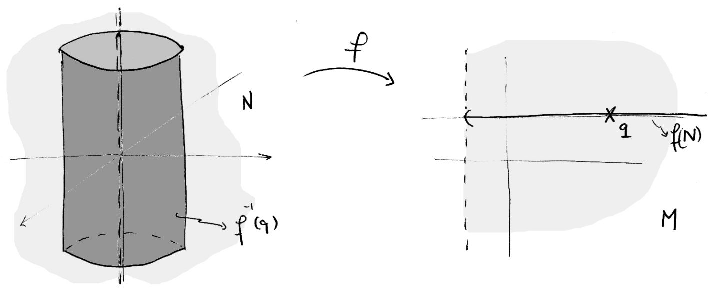

FIGURE 22. An embedding of a cylinder via a constant rank mapping.

Now $\mathrm{rk}(f) = 2$, and f is a submersion. For every $q \in M$, the set $S = f^{-1}(q)$ is a embedded submanifold, see Figure 23. ▶

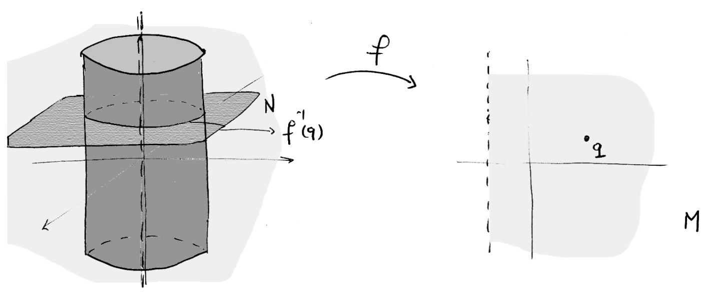

FIGURE 23. An embedding circle of an submersion.

◀ **3.35** *Example.* Let $N = \mathbb{R}^2$, $M = \mathbb{R}$, and $f(x,y) = \frac{1}{4}(x^2-1)^2 + \frac{1}{2}y^2$. The Jacobian is $Jf|_{(x,y)} = (x(x^2-1) \quad y)$. This is not a constant rank, nor a submersion. However, any $q > 0$, $q \neq \frac{1}{4}$, is a regular value since then the rank is equal to 1. Figure 24 shows the level set $f^{-1}(0)$ (not an embedded manifold), and $f^{-1}(1)$ (an embedded circle) ▶

Theorem 3.37. [18] *Let $S \subset M$ be a subset of a smooth m-dimensional manifold M. Then S is a k-dimensional smooth embedded submanifold if and if for every $p \in S$ there exists a neighborhood $U \ni p$ such that $U \cap S$ is the level set of a submersion $f : U \to \mathbb{R}^{m-k}$.*

Proof: Assume that $S \subset M$ is a smooth embedded manifold. Then for each $p \in S$ there exists a chart (U, φ), $p \in U$, such that $\varphi(S \cap U) = \{x \ : \ x_{k+1} = \cdots =$

[18]See Lee, Prop.8.12.

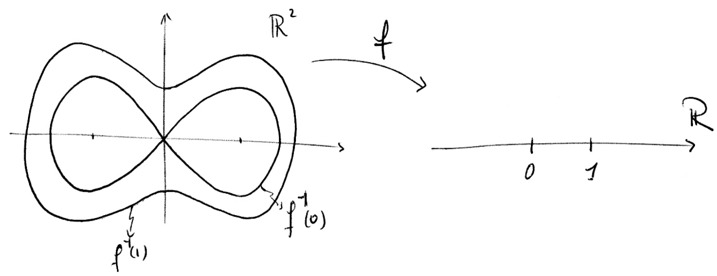

FIGURE 24. A regular and critical level set.

$x_m = 0\}$. Clearly, $S \cap U$ is the sublevel set of $f : U \to \mathbb{R}^{m-k}$ at 0, given by $\widetilde{f}(x) = (x_{k+1}, \cdots, x_m)$, which is a submersion.

Conversely, if $S \cap U = f^{-1}(0)$, for some submersion $f : U \to \mathbb{R}^{m-k}$, then by Theorem 3.28, $S \cap U$ is an embedded submanifold of U. This clearly shows that S is an embedded submanifold in M. ∎

II. Tangent and cotangent spaces

4. Tangent spaces

For a(n) (embedded) manifold $M \subset \mathbb{R}^\ell$ the tangent space T_pM at a point $p \in M$ can be pictured as a hyperplane tangent to M. In Figure 25 we consider the parametrizations $x + te_i$ in $\mathbb{R}^m$. These parametrizations yield curves $\gamma_i(t) = \varphi^{-1}(x + te_i)$ on M whose velocity vectors are given by

$$\gamma'(0) = \frac{d}{dt}\varphi^{-1}(x + te_i)\Big|_{t=0} = J\varphi^{-1}|_x(e_i).$$

The vectors $p + \gamma'(0)$ are tangent to M at p and span an m-dimensional affine linear subspace M_p of $\mathbb{R}^\ell$. Since the vectors $J\varphi^{-1}|_x(e_i)$ span T_pM the affine subspace is given by

$$M_p := p + T_pM \subset \mathbb{R}^\ell,$$

which is tangent to M at p.

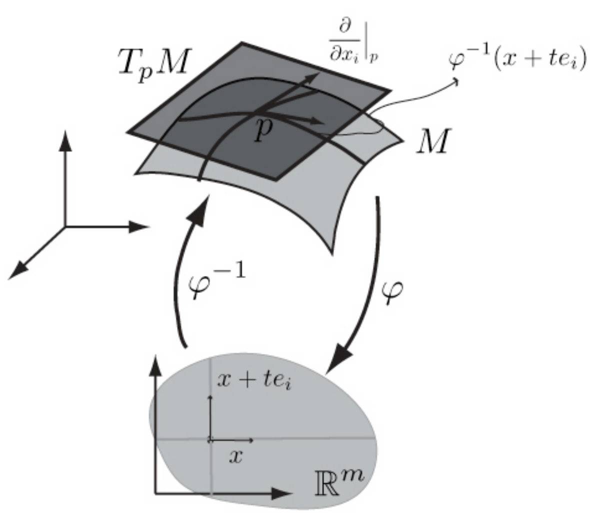

FIGURE 25. Velocity vectors of curves of M span the ‘tangent space’.

The considerations are rather intuitive in the sense that we consider only embedded manifolds, and so the tangent spaces are tangent m-dimensional affine subspaces of $\mathbb{R}^\ell$. One can also define the notion of tangent space for abstract smooth manifolds. There are many ways to do this. Let us describe one possible way (see e.g. Lee, or Abraham, Marsden and Ratiu) which is based on the above considerations.

Let $a < 0 < b$ and consider a smooth mapping $\gamma : I = (a,b) \subset \mathbb{R} \to M$, such that $\gamma(0) = p$. This mapping is called a ***(smooth) curve*** on M, and is parametrized by $t \in I$. If the mapping γ (between the manifolds $N = I$, and M) is an immersion, then γ is called an immersed curve. For such curves the 'velocity vector' $J\tilde{\gamma}|_t = (\varphi \circ \gamma)'(t)$ in $\mathbb{R}^m$ is nowhere zero. We build the concept of tangent spaces in order to define the notion velocity vector to a curve γ.

Let (U,φ) be a chart at p. Then, two curves γ and $\gamma^\dagger$ are equivalent, $\gamma^\dagger \sim \gamma$, if

$$\gamma^\dagger(0) = \gamma(0) = p, \text{ and } (\varphi \circ \gamma^\dagger)'(0) = (\varphi \circ \gamma)'(0).$$

The equivalence class of a curve γ through $p \in M$ is denoted by $[\gamma]$.

Definition 4.2. [19] At a $p \in M$ define the tangent space T_pM as the space of all equivalence classes $[\gamma]$ of curves γ through p. A tangent vector X_p, as the equivalence class of curves, is given by

$$X_p := [\gamma] = \left\{\gamma^\dagger \;:\; \gamma^\dagger(0) = \gamma(0) = p,\ (\varphi \circ \gamma^\dagger)'(0) = (\varphi \circ \gamma)'(0)\right\},$$

which is an element of T_pM.

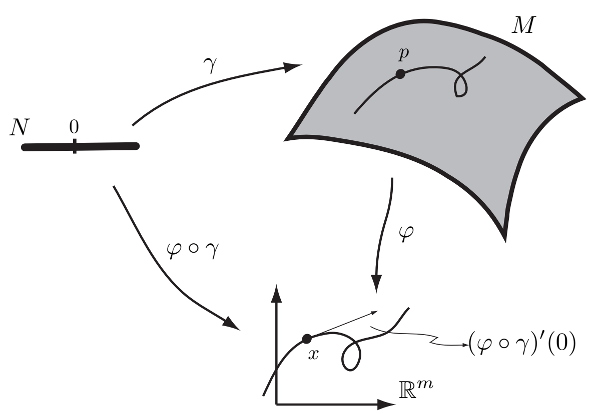

FIGURE 26. Immersed curves and velocity vectors in $\mathbb{R}^m$.

[19] Lee, Ch. 3.

The above definition does not depend on the choice of charts at $p \in M$. Let (U', φ') be another chart at $p \in M$. Then, using that $(\varphi \circ \gamma^\dagger)'(0) = (\varphi \circ \gamma)'(0)$, for $(\varphi' \circ \gamma)'(0)$ we have

$$\begin{aligned}
(\varphi' \circ \gamma)'(0) &= \Big[(\varphi' \circ \varphi^{-1}) \circ (\varphi \circ \gamma)\Big]'(0) \\
&= J(\varphi' \circ \varphi^{-1})\big|_x (\varphi \circ \gamma)'(0) \\
&= J(\varphi' \circ \varphi^{-1})\big|_x (\varphi \circ \gamma^\dagger)'(0) \\
&= \Big[(\varphi' \circ \varphi^{-1}) \circ (\varphi \circ \gamma^\dagger)\Big]'(0) = (\varphi' \circ \gamma^\dagger)'(0),
\end{aligned}$$

which proves that the equivalence relation does not depend on the particular choice of charts at $p \in M$.

One can prove that $T_pM \cong \mathbb{R}^m$. Indeed, T_pM can be given a linear structure as follows; given two equivalence classes $[\gamma_1]$ and $[\gamma_2]$, then

$$\begin{aligned}
[\gamma_1] + [\gamma_2] &:= \big\{\gamma \,:\, (\varphi \circ \gamma)'(0) = (\varphi \circ \gamma_1)'(0) + (\varphi \circ \gamma_2)'(0)\big\}, \\
\lambda[\gamma_1] &:= \big\{\gamma \,:\, (\varphi \circ \gamma)'(0) = \lambda(\varphi \circ \gamma_1)'(0)\big\}.
\end{aligned}$$

The above argument shows that these operation are well-defined, i.e. independent of the chosen chart at $p \in M$, and the operations yield non-empty equivalence classes. The mapping

$$\tau_\varphi : T_pM \to \mathbb{R}^m, \qquad \tau_\varphi\big([\gamma]\big) = (\varphi \circ \gamma)'(0),$$

is a linear isomorphism and $\tau_{\varphi'} = J(\varphi' \circ \varphi^{-1})|_x \circ \tau_\varphi$. Indeed, by considering curves $\gamma_i(x) = \varphi^{-1}(x + te_i)$, $i = 1, \ldots, m$, it follows that $[\gamma_i] \neq [\gamma_j]$, $i \neq j$, since

$$(\varphi \circ \gamma_i)'(0) = e_i \neq e_j = (\varphi \circ \gamma_j)'(0).$$

This proves the surjectivity of τ_φ. As for injectivity one argues as follows. Suppose, $(\varphi \circ \gamma)'(0) = (\varphi \circ \gamma^\dagger)'(0)$, then by definition $[\gamma] = [\gamma^\dagger]$, proving injectivity.

Given a smooth mapping $f : N \to M$ we can define how tangent vectors in T_pN are mapped to tangent vectors in T_qM, with $q = f(p)$. Choose charts (U, φ) for $p \in N$, and (V, ψ) for $q \in M$. We define the ***tangent map*** or ***pushforward of*** f as follows, see Figure 27. For a given tangent vector $X_p = [\gamma] \in T_pN$,

$$df_p = f_* : T_pN \to T_qM, \qquad f_*([\gamma]) = [f \circ \gamma].$$

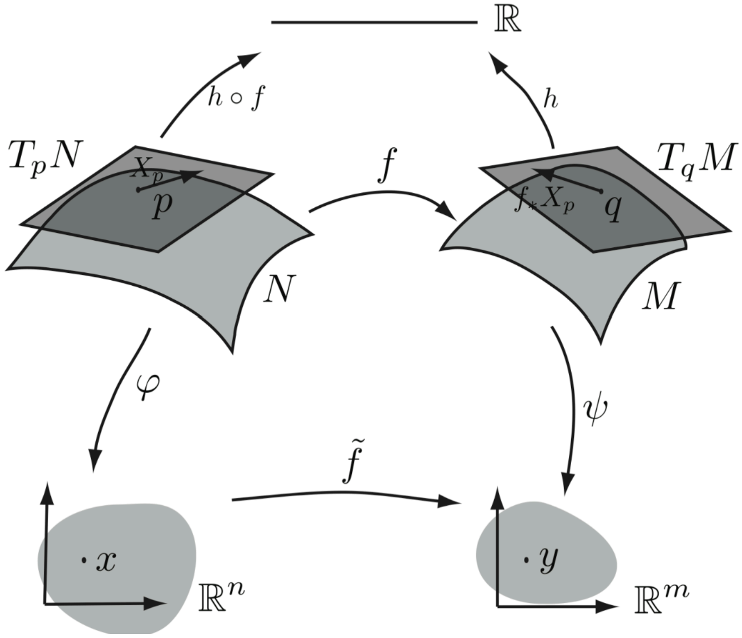

FIGURE 27. Tangent vectors in $X_p \in T_pN$ yield tangent vectors $f_*X_p \in T_qM$ under the pushforward of f.

The following commutative diagram shows that f_* is a linear map and its definition does not depend on the charts chosen at $p \in N$, or $q \in M$.

$$\begin{tikzcd} T_pN \arrow[r, "f_*"] \arrow[d, "\tau_\varphi"'] & T_qM \arrow[d, "\tau_\psi"] \\ \mathbb{R}^n \arrow[r, "J(\psi\circ f\circ\varphi^{-1})|_x"] & \mathbb{R}^m \end{tikzcd}$$

Indeed, a velocity vector $(\varphi \circ \gamma)'(0)$ is mapped to $(\psi \circ f(\gamma))'(0)$, and

$$(\psi \circ f(\gamma))'(0) = (\psi \circ f \circ \varphi^{-1} \circ \varphi \circ \gamma)'(0) = J\tilde{f}\big|_x \cdot (\varphi \circ \gamma)'(0).$$

Clearly, this mapping is linear and independent of the charts chosen.

If we apply the definition of pushforward to the coordinate mapping $\varphi : N \to \mathbb{R}^n$, then τ_φ can be identified with φ_*, and $J(\psi \circ f \circ \varphi^{-1})|_x$ with $(\psi \circ f \circ \varphi^{-1})_*$. Indeed, $\tau_\varphi([\gamma]) = (\varphi \circ \gamma)'(0)$ and $\varphi_*([\gamma]) = [\varphi \circ \gamma]$, and in $\mathbb{R}^n$ the equivalence class can be labeled by $(\varphi \circ \gamma)'(0)$. The labeling map is given as follows

$$\tau_{\mathrm{id}}([\varphi \circ \gamma]) = (\varphi \circ \gamma)'(0),$$

and is an isomorphism, and satisfies the relations

$$\tau_{\mathrm{id}} \circ \varphi_* = \tau_\varphi, \qquad \varphi_* = \tau_{\mathrm{id}}^{-1} \circ \tau_\varphi.$$

From now one we identify $T_x\mathbb{R}^n$ with $\mathbb{R}^n$ by identifying φ_* and τ_φ. This justifies the notation

$$\varphi_*([\gamma]) = [\varphi\circ\gamma] := (\varphi\circ\gamma)'(0).$$

Properties of the pushforward can be summarized as follows:

Lemma 4.5. [20] *Let $f : N \to M$, and $g : M \to P$ be smooth mappings, and let $p \in M$, then*

(i) *$f_* : T_pN \to T_{f(p)}M$, and $g_* : T_{f(p)}M \to T_{(g\circ f)(p)}P$ are linear maps (homomorphisms),*
(ii) *$(g\circ f)_* = g_*\cdot f_* : T_pN \to T_{(g\circ f)(p)}P$,*
(iii) *$(\mathrm{id})_* = \mathrm{id} : T_pN \to T_pN$,*
(iv) *if f is a diffeomorphism, then the pushforward f_* is a isomorphism from T_pN to $T_{f(p)}M$.*

Proof: We have that $f_*([\gamma]) = [f\circ\gamma]$, and $g_*([f\circ\gamma]) = [g\circ f\circ\gamma]$, which defines the mapping $(g\circ f)_*([\gamma]) = [g\circ f\circ\gamma]$. Now

$$[g\circ f\circ\gamma] = [g\circ(f\circ\gamma)] = g_*([f\circ\gamma]) = g_*(f_*([\gamma])),$$

which shows that $(g\circ f)_* = g_*\cdot f_*$. ■

A parametrization $\varphi^{-1} : \mathbb{R}^m \to M$ coming from a chart (U,φ) is a local diffeomorphism, and can be used to find a canonical basis for T_pM. Choosing local coordinates $x = (x_1,\cdots,x_n) = \varphi(p)$, and the standard basis vectors e_i for $\mathbb{R}^m$, we define

$$\frac{\partial}{\partial x_i}\Big|_p := \varphi_*^{-1}(e_i).$$

By definition $\frac{\partial}{\partial x_i}\big|_p \in T_pM$, and since the vectors e_i form a basis for $\mathbb{R}^m$, the vectors $\frac{\partial}{\partial x_i}\big|_p$ form a basis for T_pM. An arbitrary tangent vector $X_p \in T_pM$ can now be written with respect to the basis $\{\frac{\partial}{\partial x_i}\big|_p\}$:

$$X_p = \varphi_*^{-1}(X_ie_i) = X_i\frac{\partial}{\partial x_i}\Big|_p.$$

where the notation $X_i\frac{\partial}{\partial x_i}\big|_p = \sum_i X_i\frac{\partial}{\partial x_i}\big|_p$ denotes the Einstein summation convention, and (X_i) is a vector in $\mathbb{R}^m$!

We now define the directional derivative of a smooth function $h : M \to \mathbb{R}$ in the direction of $X_p \in T_pM$ by

$$X_ph := h_*X_p = [h\circ\gamma].$$

[20] Lee, Lemma 3.5.

In fact we have that $X_p h = (h\circ\gamma)'(0)$, with $X_p = [\gamma]$. For the basis vectors of T_pM this yields the following. Let $\gamma_i(t) = \varphi^{-1}(x+te_i)$, and $X_p = \frac{\partial}{\partial x_i}\Big|_p$, then

$$X_p h = \frac{\partial}{\partial x_i}\Big|_p h = \big((h\circ\varphi^{-1})\circ(\varphi\circ\gamma_i)\big)'(0) = \frac{\partial\tilde{h}}{\partial x_i}, \tag{2}$$

in local coordinates, which explains the notation for tangent vectors. In particular, for general tangent vectors X_p, $X_p h = X_i \frac{\partial\tilde{h}}{\partial x_i}$.

Let go back to the curve $\gamma : N = (a,b) \to M$ and express velocity vectors. Consider the chart (N,id) for N, with coordinates t, then

$$\frac{d}{dt}\Big|_{t=0} := \mathrm{id}_*^{-1}(1).$$

We have the following commuting diagrams:

$$\begin{array}{ccc} N & \xrightarrow{\gamma} & M \\ {\scriptstyle\mathrm{id}}\downarrow & & \downarrow{\scriptstyle\varphi} \\ \mathbb{R} & \xrightarrow{\tilde{\gamma}=\varphi\circ\gamma} & \mathbb{R}^m \end{array} \qquad\qquad \begin{array}{ccc} T_tN & \xrightarrow{\gamma_*} & T_pM \\ {\scriptstyle\mathrm{id}_*}\downarrow & & \downarrow{\scriptstyle\varphi_*} \\ \mathbb{R} & \xrightarrow{\tilde{\gamma}_*} & \mathbb{R}^m \end{array}$$

We now define

$$\gamma'(0) = \gamma_*\Big(\frac{d}{dt}\Big|_{t=0}\Big) = \varphi_*^{-1}\big((\varphi\circ\gamma)'(0)\big) \in T_pM,$$

by using the second commuting diagram.

If we take a closer look at Figure 27 we can see that using different charts at $p\in M$, or equivalently, considering a change of coordinates leads to the following relation. For the charts (U,φ) and (U',φ') we have local coordinates $x=\varphi(p)$ and $x'=\varphi'(p)$, and $p\in U\cap U'$. This yields two different basis for T_pM, namely $\big\{\frac{\partial}{\partial x_i}\big|_p\big\}$, and $\big\{\frac{\partial}{\partial x_i'}\big|_p\big\}$. Consider the identity mapping $f=\mathrm{id} : M\to M$, and the push-forward yields the identity on T_pM. If we use the different choices of coordinates as described above we obtain

$$\Big(\mathrm{id}_*\frac{\partial}{\partial x_i}\Big|_p\Big)h = (\varphi'\circ\varphi^{-1})_*\frac{\partial\tilde{h}}{\partial x_j'} = \frac{\partial x_j'}{\partial x_i}\frac{\partial\tilde{h}}{\partial x_j'}.$$

In terms of the different basis for T_pM this gives

$$\frac{\partial}{\partial x_i}\Big|_p = \frac{\partial x_j'}{\partial x_i}\frac{\partial}{\partial x_j'}\Big|_p.$$

Let us prove this formula by looking a slightly more general situation. Let N,M be smooth manifolds and $f : N\to M$ a smooth mapping. Let $q=f(p)$ and (V,ψ)

is a chart for M containing q. The vectors $\frac{\partial}{\partial y_j}\big|_q$ form a basis for T_qM. If we write $X_p = X_i \frac{\partial}{\partial x_i}\big|_p$, then for the basis vectors $\frac{\partial}{\partial x_i}\big|_p$ we have

$$\begin{aligned}
\Big(f_* \frac{\partial}{\partial x_i}\Big|_p\Big)h &= \frac{\partial}{\partial x_i}\Big|_p (h \circ f) = \frac{\partial}{\partial x_i}\big(\widetilde{h \circ f}\big) = \frac{\partial}{\partial x_i}\big(h \circ f \circ \varphi^{-1}\big) \\
&= \frac{\partial}{\partial x_i}\big(\tilde{h} \circ \psi \circ f \circ \varphi^{-1}\big) = \frac{\partial}{\partial x_i}\big(\tilde{h} \circ \tilde{f}\big) \\
&= \frac{\partial \tilde{h}}{\partial y_j}\frac{\partial \tilde{f}_j}{\partial x_i} = \Big(\frac{\partial \tilde{f}_j}{\partial x_i}\frac{\partial}{\partial y_j}\Big|_q\Big)h
\end{aligned}$$

which implies that T_pN is mapped to T_qM under the map f_*. In general a tangent vector $X_i \frac{\partial}{\partial x_i}\big|_p$ is pushed forward to a tangent vector

$$Y_j \frac{\partial}{\partial y_j}\Big|_q = \left[\frac{\partial \tilde{f}_j}{\partial x_i} X_i\right] \frac{\partial}{\partial y_j}\Big|_q \in T_qM, \tag{3}$$

expressed in local coordinates. By taking $N = M$ and $f = \mathrm{id}$ we obtain the above change of variables formula.

5. Cotangent spaces

In linear algebra it is often useful to study the space of linear functions on a given vector space V. This space is denoted by V^* and called the ***dual vector space*** to V — again a linear vector space. So the elements of V^* are linear functions $\theta : V \to \mathbb{R}$. As opposed to vectors $v \in V$, the elements, or vectors in V^* are called ***covectors***.

Lemma 5.1. *Let V be a n-dimensional vector space with basis $\{v_1, \cdots, v_n\}$, then the there exist covectors $\{\theta^1, \cdots, \theta^n\}$ such that*

$$\theta^i \cdot v_j := \theta^i(v_j) = \delta^i_j, \quad \text{Kronecker delta,}$$

and the covectors $\{\theta^1, \cdots, \theta^n\}$ form a basis for V^*.

This procedure can also be applied to the tangent spaces T_pM described in the previous chapter.

Definition 5.2. Let M be a smooth m-dimensional manifold, and let T_pM be the tangent space at some $p \in M$. The the ***cotangent space*** T_p^*M is defined as the dual vector space of T_pM, i.e.

$$T_p^*M := (T_pM)^*.$$

By definition the cotangent space T_p^*M is also m-dimensional and it has a canonical basis as described in Lemma 5.1. As before have the canonical basis vectors $\frac{\partial}{\partial x_i}|_p$ for T_pM, the associated basis vectors for T_p^*M are denoted $dx^i|_p$. Let us now describe this dual basis for T_p^*M and explain the notation.

The covectors $dx^i|_p$ are called ***differentials*** and we show now that these are indeed related dh_p. Let $h : M \to \mathbb{R}$ be a smooth function, then $h_* : T_pM \to \mathbb{R}$ and $h_* \in T_p^*M$. Since the differentials $dx^i|_p$ form a basis of T_p^*M we have that $h_* = \lambda^i dx^i|_p$, and therefore

$$h_* \frac{\partial}{\partial x_i}\Big|_p = \lambda^j dx^j|_p \cdot \frac{\partial}{\partial x_i}\Big|_p = \lambda^j \delta_i^j = \lambda^i = \frac{\partial \tilde{h}}{\partial x_i},$$

and thus

$$dh_p = h_* = \frac{\partial \tilde{h}}{\partial x_i} dx^i\Big|_p. \tag{4}$$

Choose h such that h_* satisfies the identity in Lemma 5.1, i.e. let $\tilde{h} = x_i$ ($h = x_i \circ \varphi$ = $\langle \varphi, e_i \rangle$). These linear functions h of course span T_p^*M, and

$$h_* = (x_i \circ \varphi)_* = d(x_i \circ \varphi)_p = dx^i|_p.$$

Cotangent vectors are of the form

$$\theta_p = \theta^i dx^i\Big|_p.$$

The pairing between a tangent vector X_p and a cotangent vector θ_p is expressed component wise as follows:

$$\theta_p \cdot X_p = \theta^i X_j \delta_j^i = \theta^i X_i.$$

In the case of tangent spaces a mapping $f : N \to M$ pushes forward to a linear $f_*; T_pN \to T_qM$ for each $p \in N$. For cotangent spaces one expects a similar construction. Let $q = f(p)$, then for a given cotangent vector $\theta_q \in T_q^*M$ define

$$(f^*\theta_q) \cdot X_p = \theta_q \cdot (f_* X_p) \in T_p^*N,$$

for any tangent vector $X_p \in T_pN$. The homomorphism $f^* : T_q^*M \to T_p^*N$, defined by $\theta_q \mapsto f^*\theta_q$ is called the ***pullback*** of f at p. It is a straightforward consequence from linear algebra that f^* defined above is indeed the dual homomorphism of f_*, also called the adjoint, or transpose (see Lee, Ch. 6, for more details, and compare the the definition of the transpose of a matrix). If we expand the definition of pullback in local coordinates, using (3), we obtain

$$\begin{aligned}\Big(f^*dy^j\big|_q\Big)\cdot X_i\frac{\partial}{\partial x_i}\Big|_p &= dy^j\big|_q\cdot f_*\Big(X_i\frac{\partial}{\partial x_i}\Big|_p\Big)\\ &= dy^j\big|_q\left[\frac{\partial \tilde f_j}{\partial x_i}X_i\right]\frac{\partial}{\partial y_j}\Big|_q = \frac{\partial \tilde f_j}{\partial x_i}X_i\end{aligned}$$

Using this relation we obtain that

$$\Big(f^*\sigma^j dy^j\big|_q\Big)\cdot X_i\frac{\partial}{\partial x_i}\Big|_p = \sigma^j\frac{\partial \tilde f_j}{\partial x_i}X_i = \sigma^j\frac{\partial \tilde f_j}{\partial x_i}dx^i\big|_p X_i\frac{\partial}{\partial x_i}\Big|_p,$$

which produces the local formula

(5) $$f^*\sigma^j dy^j\big|_q = \left[\sigma^j\frac{\partial \tilde f_j}{\partial x_i}\right]dx^i\big|_p = \left[\sigma^j\circ\tilde f\frac{\partial \tilde f_j}{\partial x_i}\right]_x dx^i\big|_p.$$

Lemma 5.3. *Let $f: N\to M$, and $g: M\to P$ be smooth mappings, and let $p\in M$, then*

(i) *$f^*: T^*_{f(p)}M\to T^*_pN$, and $g^*: T^*_{(g\circ f)(p)}P\to T^*_{f(p)}M$ are linear maps (homomorphisms),*
(ii) *$(g\circ f)^* = f^*\cdot g^*: T^*_{(g\circ f)(p)}P\to T^*_pN$,*
(iii) *$(\mathrm{id})^* = \mathrm{id}: T^*_pN\to T^*_pN$,*
(iv) *if f is a diffeomorphism, then the pullback f^* is a isomorphism from $T^*_{f(p)}M$ to T^*_pN.*

Proof: By definition of the pullbacks of f and g we have

$$f^*\theta_q\cdot X_p = \theta_q\cdot f_*(X_p),\quad g^*\omega_{g(q)}Y_q = \omega_{g(q)}\cdot g_*(Y_q).$$

For the composition $g\circ f$ it holds that $(g\circ f)^*\omega_{(g\circ f)(p)}\cdot X_p = \omega_{(g\circ f)(p)}\cdot(g\circ f)_*(X_p)$. Using Lemma 4.5 we obtain

$$\begin{aligned}(g\circ f)^*\omega_{(g\circ f)(p)}\cdot X_p &= \omega_{(g\circ f)(p)}\cdot g_*\big(f_*(X_p)\big)\\ &= g^*\omega_{(g\circ f)(p)}\cdot f_*(X_p) = f^*\big(g^*\omega_{(g\circ f)(p)}\big)\cdot X_p,\end{aligned}$$

which proves that $(g\circ f)^* = f^*\cdot g^*$. ∎

Now consider the coordinate mapping $\varphi: U\subset M\to\mathbb{R}^m$, which is local diffeomorphism. Using Lemma 5.3 we then obtain an isomorphism $\varphi^*:\mathbb{R}^m\to T^*_pM$, which justifies the notation

$$dx^i|_p = \varphi^*(e_i).$$

6. **Vector bundles**

The abstract notion of vector bundle consists of topological spaces E (total space) and M (the base) and a projection $\pi : E \to M$ (surjective). To be more precise:

Definition 6.1. A triple (E, M, π) is called a ***real vector bundle*** of rank k over M if

(i) for each $p \in M$, the set $E_p = \pi^{-1}(p)$ is a real k-dimensional linear vector space, called the ***fiber over*** p, such that

(ii) for every $p \in M$ there exists a open neighborhood $U \ni p$, and a homeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$;

  (a) $\big(\pi \circ \Phi^{-1}\big)(p, \xi) = p$, for all $\xi \in \mathbb{R}^k$;

  (b) $\xi \mapsto \Phi^{-1}(p, \xi)$ is a vector space isomorphism between $\mathbb{R}^k$ and E_p.

The homeomorphism Φ is called a ***local trivialization*** of the bundle.

If there is no ambiguity about the base space M we often denote a vector bundle by E for short. Another way to denote a vector bundle that is common in the literature is $\pi : E \to M$. It is clear from the above definition that if M is a topological manifold then so is E. Indeed, via the homeomorphisms Φ it follows that E

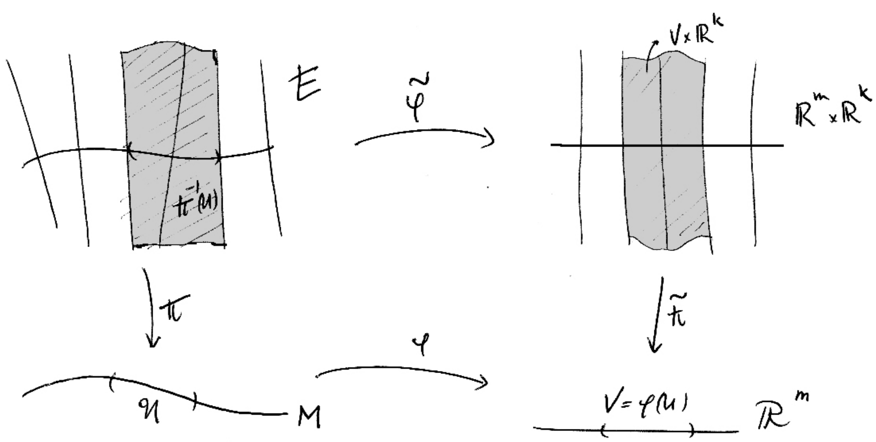

FIGURE 28. Charts in a vector bundle E over M.

is Hausdorff and has a countable basis of open set. Define $\widetilde{\varphi} = \big(\varphi \times \mathrm{Id}_{\mathbb{R}^k}\big) \circ \Phi$, and $\widetilde{\varphi} : \pi^{-1}(U) \to V \times \mathbb{R}^k$ is a homeomorphism. Figure 28 explains the choice of bundle charts for E related to charts for M.

For two trivializations $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ and $\Psi : \pi^{-1}(V) \to V \times \mathbb{R}^k$, we have that the transition map $\Psi \circ \Phi^{-1} : (U \cap V) \times \mathbb{R}^k \to (U \cap V) \times \mathbb{R}^k$ has the following form

$$\Psi \circ \Phi^{-1}(p,\xi) = (p, \tau(p)\xi)),$$

where $\tau : U \cap V \to \mathrm{Gl}(k,\mathbb{R})$ is continuous. It is clear from the assumptions that

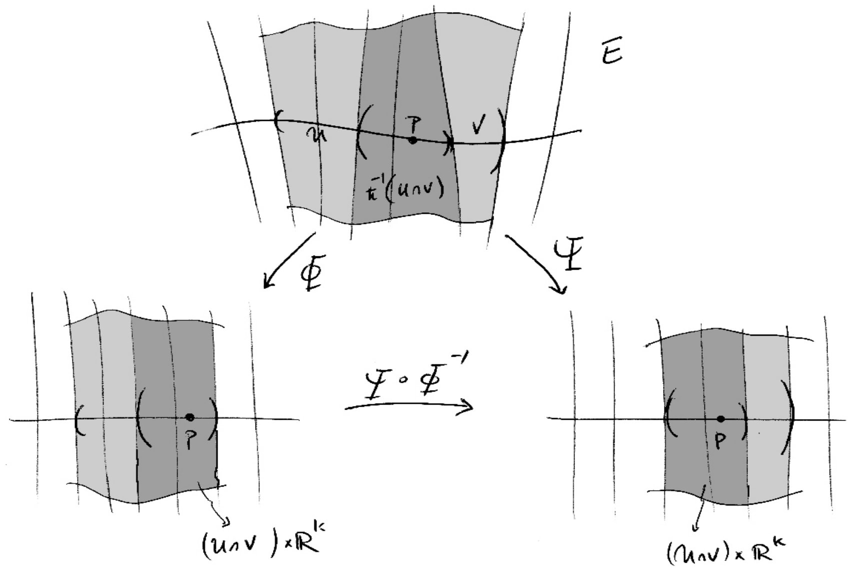

FIGURE 29. Transition mappings for local trivializations.

$\Psi \circ \Phi^{-1}(p,\xi) = (p, \sigma(p,\xi))$. By assumption we also have that $\xi \mapsto \Phi^{-1}(p,\xi) = A(p)\xi$, and $\xi \mapsto \Psi^{-1}(p,\xi) = B(p)\xi$. Now,

$$\Psi \circ \Phi^{-1}(p,\xi) = \Psi(A(p)\xi) = (p, \sigma(p,\xi)),$$

and $\Psi^{-1}(p,\sigma) = A(p)\xi = B(p)\sigma$. Therefore $\sigma(p,\xi) = B^{-1}A\xi =: \tau(p)\xi$. Continuity is clear from the assumptions in Definition 6.1.

If both E and M are smooth manifolds and π is a smooth projection, such that the local trivializations can be chosen to be diffeomorphisms, then (E,M,π) is called a ***smooth vector bundle***. In this case the maps τ are smooth. The following result allows us to construct smooth vector bundles and is important for the special vector bundles used in this course.

Theorem 6.4. [21] *Let M be a smooth manifold, and let $\{E_p\}_{p\in M}$ be a family of k-dimensional real vector spaces parametrized by $p \in M$. Define $E = \bigsqcup_{p\in M} E_p$, and $\pi : E \to M$ as the mapping that maps E_p to $p \in M$. Assume there exists*

(i) *an open covering $\{U_\alpha\}_{\alpha\in A}$ for M;*

[21]See Lee, Lemma 5.5.

(ii) *for each* $\alpha \in A$*, a bijection* $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$*, such that* $\xi \mapsto \Phi_\alpha^{-1}(p,\xi)$ *is a vector space isomorphism between* $\mathbb{R}^k$ *and* E_p*;*

(iii) *for each* $\alpha, \beta \in I$*, with* $U_\alpha \cap U_\beta \neq \varnothing$*, smooth mappings* $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Gl}(k,\mathbb{R})$ *such that*

$$\Phi_\beta \circ \Phi_\alpha^{-1}(p,\xi) = (p, \tau_{\alpha\beta}(p)\xi).$$

Then E *has a unique differentiable (manifold) structure making* (E,M,π) *a smooth vector bundle of rank* k *over* M*.*

Proof: The proof of this theorem goes by verifying the hypotheses in Theorem 2.11. Let us propose charts for E. Let (V_p, φ_p), $V_p \subset U_\alpha$, be a smooth chart for M containing p. As before define $\widetilde{\varphi}_p : \pi^{-1}(V_p) \to \widetilde{V}_p \times \mathbb{R}^k$. We can show, using Theorem 2.11, that $(\pi^{-1}(V_p), \widetilde{\varphi}_p)$, $p \in M$ are smooth charts, which gives a unique differentiable manifold structure. For details of the proof see Lee, Lemma 5.5. ∎

◀ **6.5** *Example.* Consider the set E defined as

$$E = \big\{(p_1,p_2,\xi_1,\xi_2) \mid p_1 = \cos(\theta), p_2 = \sin(\theta),\ \cos(\theta/2)\xi_1 + \sin(\theta/2)\xi_2 = 0\big\}.$$

Clearly, E is a smooth vector bundle over S^1 of rank 1, embedded in $\mathbb{R}^4$. For example if $U = S^1 \backslash \{(1,0)\}$ ($\theta \in (0,2\pi)$), then $\Phi(\pi^{-1}(U)) = (p_1,p_2,\xi_1)$ is a local trivialization. This bundle is called the Möbius strip.

To show that E is a smooth vector bundle we need to verify the conditions in Theorem 6.4. Consider a second chart $U' = S^1 \backslash \{(-1,0)\}$ ($\theta \in (-\pi,\pi)$), and the trivialization $\Psi(\pi^{-1}(U')) = (p_1,p_2,\xi_2)$. For the Φ we have that

$$\Phi^{-1}(p_1,p_2,\xi_1) = \Big(p_1,p_2,\xi_1, -\frac{\cos(\theta/2)}{\sin(\theta/2)}\xi_1\Big),$$

and thus

$$\Psi \circ \Phi^{-1}(p_1,p_2,\xi_1) = \Big(p_1,p_2, -\frac{\cos(\theta/2)}{\sin(\theta/2)}\xi_1\Big),$$

which gives that $\tau(p)$ is represented by $\tau = -\frac{\cos(\theta/2)}{\sin(\theta/2)}$, for $\theta \in (0,\pi)$, which invertible and smooth in p. ▶

Mappings between smooth vector bundles are called ***bundle maps***, and are defined as follows. Given vector bundles $\pi : E \to M$ and $\pi' : E' \to M'$ and smooth mappings $F : E \to E'$ and $f : M \to M'$, such that the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ M & \xrightarrow{f} & M' \end{array}$$

and $F|_{E_p} : E_p \to E'_{f(p)}$ is linear, then the pair (F, f) is a called a smooth bundle map.

A ***section***, or ***cross section*** of a bundle E is a continuous mapping $\sigma : M \to E$, such that $\pi \circ \sigma = \mathrm{Id}_M$. A section is smooth if σ is a smooth mapping. The space of

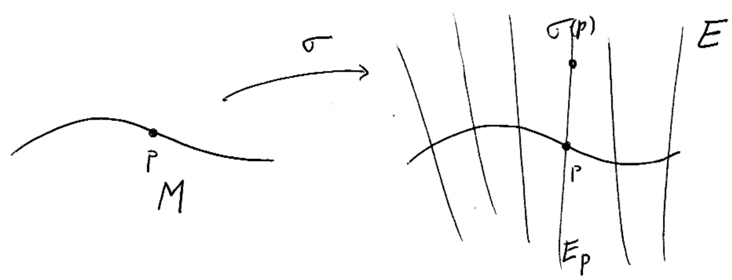

FIGURE 30. A cross section σ in a bundle E.

smooth section in E is denoted by $\mathcal{E}(M)$. The ***zero section*** is a mapping $\sigma : M \to E$ such that $\sigma(p) = 0 \in E_p$ for all $p \in M$.

◀ **6.7** *Remark.* By identifying M with the trivial bundle $E_0 = M \times \{0\}$, a section is a bundle map from $E_0 = M \times \{0\}$ to E. Indeed, define $\sigma' : E^0 \to E$ by $\sigma'(p, 0) = \sigma(p)$, and let $f = \mathrm{Id}_M$. Then $\pi \circ \sigma' = \pi \circ \sigma = \mathrm{Id}_M$. ▶

6.1. The tangent bundle and vector fields

The disjoint union of tangent spaces

$$TM := \bigsqcup_{p \in M} T_p M$$

is called the ***tangent bundle*** of M. We show now that TM is a fact a smooth vector bundle over M.

Theorem 6.8. *The tangent bundle TM is smooth vector bundle over M of rank m, and as such TM is a smooth $2m$-dimensional manifold.*

Proof: Let (U, φ) be a smooth chart for $p \in M$. Define

$$\Phi\Big(X_i \frac{\partial}{\partial x_i}\Big|_p\Big) = \big(p, (X_i)\big),$$

which clearly is a bijective map between $\pi^{-1}(U)$ and $U \times \mathbb{R}^m$. Moreover, $X_p = (X_i) \mapsto \Phi^{-1}(p, X_p) = X_i \frac{\partial}{\partial x_i}\Big|_p$ is vector space isomorphism. Let $(U_\alpha, \varphi_\alpha)$ be a covering of smooth charts of M. Let $x = \varphi_\alpha(p)$, and $x' = \varphi_\beta(p)$. From our previous considerations we easily see that

$$\Phi_\beta \circ \Phi_\alpha^{-1}(p, X_p) = (p, \tau_{\alpha\beta}(p) X_p),$$

where $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Gl}(m,\mathbb{R})$. It remains to show that $\tau_{\alpha\beta}$ is smooth. We have $\widetilde{\varphi}_\beta \circ \widetilde{\varphi}_\alpha^{-1} = (\varphi_\beta \times \mathrm{Id}) \circ \Phi_\beta \circ \Phi_\alpha^{-1} \circ (\varphi_\alpha^{-1} \times \mathrm{Id})$. Using the change of coordinates formula derived in Section 4 we have that

$$\widetilde{\varphi}_\beta \circ \widetilde{\varphi}_\alpha^{-1}\big(x, (X_i)\big) = \left(x'_1(x), \cdots, x'_m(x), \Big(\frac{\partial x'_j}{\partial x_i} X_j\Big)\right),$$

which proves the smoothness of $\tau_{\alpha\beta}$. Theorem 6.4 can be applied now showing that TM is asmooth vector bundle over M. From the charts $(\pi^{-1}(U), \widetilde{\varphi})$ we conclude that TM is a smooth $2m$-dimensional manifold. ■

Definition 6.9. A ***smooth (tangent) vector field*** is a smooth mapping

$$X : M \to TM,$$

with the property that $\pi \circ X = \mathrm{id}_M$. In other words X is a smooth (cross) section in the vector bundle TM, see Figure 31. The space of smooth vector fields on M is denoted by $\mathcal{F}(M)$.

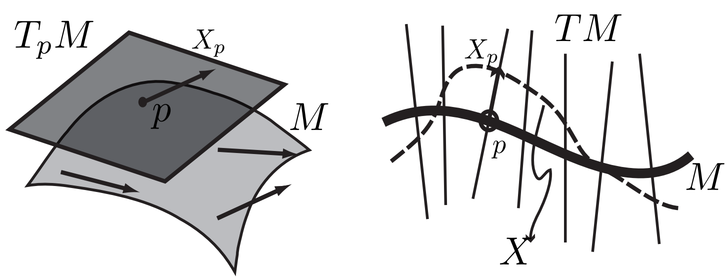

FIGURE 31. A smooth vector field X on a manifold M [right], and as a 'curve', or section in the vector bundle TM [left].

For a chart (U, φ) a vector field X can be expressed as follows

$$X = X_i \frac{\partial}{\partial x_i}\Big|_p,$$

where $X_i : U \to \mathbb{R}$. Smoothness of vector fields can be described in terms of the component functions X_i.

Lemma 6.11. *A mapping $X : M \to TM$ is a smooth vector field at $p \in U$ if and only if the coordinate functions $X_i : U \to \mathbb{R}$ are smooth.*

Proof: In standard coordinates X is given by

$$\widetilde{X}(x) = (x_1, \cdots, x_m, \widetilde{X}_1(x), \cdots, \widetilde{X}_m(x)),$$

which immediately shows that smoothness of X is equivalent to the smoothness of the coordinate functions X_i. ■

In Section 4 we introduced the notion of push forward of a mapping $f : N \to M$. Under submersions and immersion a vector field $X : N \to TN$ does not necessarily push forward to a vector field on M. If f is a diffeomorphism, then $f_*X = Y$ is a vector field on M. We remark that the definition of f_* also allows use to restate definitions about the rank of a map in terms of the differential, or pushforward f_* at a point $p \in N$.

6.2. The cotangent bundle and differential 1-forms

The disjoint union

$$T^*M := \bigsqcup_{p\in M} T_p^*M$$

is called ***cotangent bundle*** of M.

Theorem 6.12. [22] *The cotangent bundle T^*M is a smooth vector bundle over M of rank m, and as such T^*M is a smooth $2m$-dimensional manifold.*

Proof: The proof is more identical to the proof for TM, and is left to the reader as an exercise. ∎

The differential $dh : M \to T^*M$ is an example of a smooth function. The above consideration give the coordinate wise expression for dh.

Definition 6.13. A ***smooth covector field*** is a smooth mapping

$$\theta : \ M \to T^*M,$$

with the property that $\pi \circ \theta = \mathrm{id}_M$. In order words θ is a smooth section in T^*M. The space of smooth covector fields on M is denoted by $\mathcal{F}^*(M)$. Usually the space $\mathcal{F}^*(M)$ is denoted by $\Gamma^1(M)$, and covector fields are referred to a ***(smooth) differential 1-form*** on M.

For a chart (U, φ) a covector field θ can be expressed as follows

$$\theta = \theta^i dx^i\big|_p,$$

where $\theta^i : U \to \mathbb{R}$. Smoothness of a covector fields can be described in terms of the component functions θ^i.

Lemma 6.14. *A covector field θ is smooth at $p \in U$ if and only if the coordinate functions $\theta^i : U \to \mathbb{R}$ are smooth.*

Proof: See proof of Lemma 6.11. ∎

We saw in the previous section that for arbitrary mappings $f : N \to M$, a vector field $X \in \mathcal{F}(N)$ does not necessarily push forward to a vector on M under f_*. The reason is that surjectivity and injectivity are both needed to guarantee this, which

[22]See Lee, Proposition 6.5.

requires f to be a diffeomorphism. In the case of covector fields or differential 1-forms the situation is completely opposite, because the pullback acts in the opposite direction and is therefore onto the target space and uniquely defined. To be more precise; given a 1-form $\theta \in \Lambda^1(M)$ we define a 1-form $f^*\theta \in \Lambda^1(N)$ as follows

$$(f^*\theta)_p = f^*\theta_{f(p)}.$$

Theorem 6.15. [23] *The above defined pullback $f^*\theta$ of θ under a smooth mapping $f : N \to M$ is a smooth covector field, or differential 1-form on N.*

Proof: Write $\theta \in \Lambda^1(M)$ is local coordinates; $\theta = \theta^i dy^i\big|_q$. By Formula (5) we then obtain

$$f^*\theta^i dy^i\big|_{f(p)} = \left[\theta^i\big|_y \frac{\partial \tilde{f}_i}{\partial x_j}\right] dx^j\big|_p = \left[\theta^i \circ \tilde{f} \frac{\partial \tilde{f}_i}{\partial x_j}\right]_x dx^j\big|_p,$$

which proves that $f^*\theta \in \Lambda^1(N)$. ∎

Given a mapping $g : M \to \mathbb{R}$, the differential, or push forward $g_* = dg$ defines an element in T_p^*M. In local coordinates we have $dg\big|_p = \frac{\partial \widetilde{g}}{\partial x^i} dx_i\big|_p$, and thus defines a smooth covector field on M; $dg \in \Lambda^1(M)$. Given a smooth mapping $f : N \to M$, it follows from (5) that

$$f^* dg = f^* \frac{\partial \widetilde{g}}{\partial x_i} dx^i\big|_{f(p)} = \frac{\partial \widetilde{g}}{\partial x^i} \circ \widetilde{f} \frac{\partial \widetilde{f_i}}{\partial x_j} dy^j\big|_p.$$

By applying the same to the 1-form $d(g \circ f)$ we obtain the following identities

$$f^* dg = d(g \circ f), \qquad f^*(g\theta) = (g \circ f) f^*\theta. \tag{6}$$

Using the formulas in (6) we can also obtain (5) in a rather straightforward way. Let $g = \psi_j = \langle \psi, e_j \rangle = y_j$, and $\omega = dg = dy^j|_q$ in local coordinates, then

$$f^*\big((\sigma^j \circ \psi)\omega\big) = \big(\sigma^j \circ \psi \circ f\big) f^* dg = \big(\sigma^j \circ \psi \circ f\big) d(g \circ f) = \left[\sigma^j \circ \tilde{f} \frac{\partial \tilde{f}_j}{\partial x_i}\right]_x dx^i\big|_p,$$

where the last step follows from (4).

Definition 6.16. A differential 1-form $\theta \in \Lambda^1(N)$ is called an ***exact*** 1-form if there exists a smooth function $g : N \to \mathbb{R}$, such that $\theta = dg$.

---

[23] Lee, Proposition 6.13.

The notation for tangent vectors was motivated by the fact that functions on a manifold can be differentiated in tangent directions. The notation for the cotangent vectors was partly motivated as the 'reciprocal' of the partial derivative. The introduction of line integral will give an even better motivation for the notation for cotangent vectors. Let $N = \mathbb{R}$, and θ a 1-form on N given in local coordinates by $\theta_t = h(t)dt$, which can be identified with a function h. The notation makes sense because θ can be integrated over any interval $[a,b] \subset \mathbb{R}$:

$$\int_{[a,b]} \theta := \int_a^b h(t)dt.$$

Let $M = \mathbb{R}$, and consider a mapping $f : M = \mathbb{R} \to N = \mathbb{R}$, which satisfies $f'(t) > 0$. Then $t = f(s)$ is an appropriate change of variables. Let $[c,d] = f([a,b])$, then

$$\int_{[c,d]} f^*\theta = \int_c^d h(f(s))f'(s)ds = \int_a^b h(t)dt = \int_{[a,b]} \theta,$$

which is the change of variables formula for integrals. We can use this now to define the line integral over a curve γ on a manifold N.

Definition 6.17. Let $\gamma : [a,b] \subset \mathbb{R} \to N$, and let θ be a 1-form on N and $\gamma^*\theta$ the pullback of θ, which is a 1-form on $\mathbb{R}$. Denote the image of γ in N also by γ, then

$$\int_\gamma \theta := \int_{[a,b]} \gamma^*\theta = \int_{[a,b]} \theta^i(\gamma(t))\gamma_i'(t)dt,$$ [24]

the expression in local coordinates.

The latter can be seen by combining some of the notion introduced above:

$$\gamma^*\theta \cdot \frac{d}{dt} = (\gamma^*\theta)_t = \theta_{\gamma(t)} \cdot \gamma_* \frac{d}{dt} = \theta_\gamma(t) \cdot \gamma'(t).$$

Therefore, $\gamma^*\theta = (\gamma^*\theta)_t dt = \theta_{\gamma(t)} \cdot \gamma'(t)dt = \theta^i(\gamma(t))\gamma_i'(t)dt$, and

$$\int_\gamma \theta = \int_{[a,b]} \gamma^*\theta = \int_{[a,b]} \theta_{\gamma(t)} \cdot \gamma'(t)dt = \int_{[a,b]} \theta^i(\gamma(t))\gamma_i'(t)dt.$$

If γ' is nowhere zero then the map $\gamma : [a,b] \to N$ is either an immersion or embedding. For example in the embedded case this gives an embedded submanifold $\gamma \subset N$ with boundary $\partial\gamma = \{\gamma(a), \gamma(b)\}$. Let $\theta = dg$ be an exact 1-form, then

$$\int_\gamma dg = g\big|_{\partial\gamma} = g(\gamma(b)) - g(\gamma(a)).$$

Indeed,

$$\int_\gamma dg = \int_{[a,b]} \gamma^* dg = \int_{[a,b]} d(g \circ \gamma) = \int_a^b (g \circ \gamma)'(t)dt = g(\gamma(b)) - g(\gamma(a)).$$

---

[24] The expressions $\gamma_i(t) \in \mathbb{R}$ are the components of $\gamma'(t) \in T_{\gamma(t)}M$.

This identity is called the ***Fundamental Theorem for Line Integrals*** and is a special case of the Stokes Theorem (see Section 16).

III. Tensors and differential forms

7. Tensors and tensor products

In the previous chapter we encountered linear functions on vector spaces, linear functions on tangent spaces to be precise. In this chapter we extend to notion of linear functions on vector spaces to multilinear functions.

Definition 7.1. *Let $V_1, \cdots, V_r$, and W be real vector spaces. A mapping $T : V_1 \times \cdots \times V_r \to W$ is called a **multilinear mapping** if*

$$T(v_1, \cdots, \lambda v_i + \mu v_i', \cdots, v_r) = \lambda T(v_1, \cdots, v_i, \cdots v_r) + \mu T(v_1, \cdots, v_i', \cdots v_r), \quad \forall i,$$

and for all $\lambda, \mu \in \mathbb{R}$ i.e. f is linear in each variable v_i separately.

Now consider the special case that $W = \mathbb{R}$, then T becomes a ***multilinear function, or form***, and a generalization of linear functions. If in addition $V_1 = \cdots = V_r = V$, then

$$T : V \times \cdots \times V \to \mathbb{R},$$

is a multilinear function on V, and is called a ***covariant r-tensor*** on V. The number of copies r is called the rank of T. The space of covariant r-tensors on V is denoted by $T^r(V)$, which clearly is a real vector space using the multilinearity property in Definition 7.1. In particular we have that $T^0(V) \cong \mathbb{R}$, $T^1(V) = V^*$, and $T^2(V)$ is the space of bilinear forms on V. If we consider the case $V_1 = \cdots = V_r = V^*$, then

$$T : V^* \times \cdots \times V^* \to \mathbb{R},$$

is a multilinear function on V^*, and is called a ***contravariant r-tensor*** on V. The space of contravariant r-tensors on V is denoted by $T_r(V)$. Here we have that $T_0(V) \cong \mathbb{R}$, and $T_1(V) = (V^*)^* \cong V$.

◀ **7.2** *Example.* The cross product on $\mathbb{R}^3$ is an example of a multilinear (bilinear) function mapping not to $\mathbb{R}$ to $\mathbb{R}^3$. Let $x, y \in \mathbb{R}^3$, then

$$T(x, y) = x \times y \in \mathbb{R}^3,$$

which clearly is a bilinear function on $\mathbb{R}^3$. ▶

Since multilinear functions on V can be multiplied, i.e. given vector spaces V,W and tensors $T \in T^r(V)$, and $S \in T^s(W)$, the multilinear function

$$R(v_1,\cdots v_r,w_1,\cdots,w_s) = T(v_1,\cdots v_r)S(w_1,\cdots,w_s)$$

is well defined and is a multilnear function on $V^r \times W^s$. This brings us to the following definition. Let $T \in T^r(V)$, and $S \in T^s(W)$, then

$$T \otimes S : V^r \times W^s \to \mathbb{R},$$

is given by

$$T \otimes S(v_1,\cdots,v_r,w_1,\cdots w_s) = T(v_1,\cdots,v_r)S(w_1,\cdots w_s).$$

This product is called the ***tensor product***. By taking $V = W$, $T \otimes S$ is a covariant $(r+s)$-tensor on V, which is a element of the space $T^{r+s}(V)$ and $\otimes : T^r(V) \times T^s(V) \to T^{r+s}(V)$.

Lemma 7.3. *Let $T \in T^r(V)$, $S,S' \in T^s(V)$, and $R \in T^t(V)$, then*

(i) $(T \otimes S) \otimes R = T \otimes (S \otimes R)$ *(associative),*
(ii) $T \otimes (S + S') = T \otimes S + T \otimes S'$ *(distributive),*
(iii) $T \otimes S \neq S \otimes T$ *(non-commutative).*

The tensor product is also defined for contravariant tensors and mixed tensors. As a special case of the latter we also have the product between covariant and contravariant tensors.

◀ **7.4** *Example.* The last property can easily be seen by the following example. Let $V = \mathbb{R}^2$, and $T,S \in T^1(\mathbb{R}^2)$, given by $T(v) = v_1 + v_2$, and $S(w) = w_1 - w_2$, then

$$T \otimes S(1,1,1,0) = 2 \neq 0 = S \otimes T(1,1,1,0),$$

which shows that $\otimes$ is not commutative in general. ▶

The following theorem shows that the tensor product can be used to build the tensor space $T^r(V)$ from elementary building blocks.

Theorem 7.5. [25] *Let $\{v_1,\cdots,v_n\}$ be a basis for V, and let $\{\theta^1,\cdots,\theta^n\}$ be the dual basis for V^*. Then the set*

$$\mathcal{B} = \left\{\theta^{i_1} \otimes \cdots \otimes \theta^{i_r} \;:\; 1 \leq i_1,\cdots,i_r \leq n\right\},$$

is a basis for the n^r-dimensional vector space $T^r(V)$.

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[25] See Lee, Prop. 11.2.

Proof: Compute

$$\begin{aligned} T_{i_1\cdots i_r}\theta^{i_1}\otimes\cdots\otimes\theta^{i_r}(v_{j_1},\cdots,v_{j_r}) &= T_{i_1\cdots i_r}\theta^{i_1}(v_{j_1})\cdots\theta^{i_r}(v_{j_r}) \\ &= T_{i_1\cdots i_r}\delta^{j_1}_{j_1}\cdots\delta^{j_r}_{j_r} = T_{j_1\cdots j_r} \\ &= T(v_{j_1},\cdots,v_{j_r}), \end{aligned}$$

which shows by using the multilinearity of tensors that T can be expanded in the basis $\mathcal{B}$ as follows;

$$T = T_{i_1\cdots i_r}\theta^{i_1}\otimes\cdots\otimes\theta^{i_r},$$

where $T_{i_1\cdots i_r} = T(v_{i_1},\cdots,v_{i_r})$, the components of the tensor T. Linear independence follows form the same calculation. ■

◀ **7.6** *Example.* Consider the the 2-tensors $T(x,y) = x_1y_1 + x_2y_2$, $T'(x,y) = x_1y_2 + x_2y_2$ and $T'' = x_1y_1 + x_2y_2 + x_1y_2$ on $\mathbb{R}^2$. With respect to the standard bases $\theta^1(x) = x_1$, $\theta^2(x) = x_1$, and

$$\theta^1\otimes\theta^1(x,y) = x_1y_1, \quad \theta^1\otimes\theta^2(x,y) = x_1y_2,$$
$$\theta^1\otimes\theta^2(x,y) = x_2y_1, \quad \text{and} \quad \theta^2\otimes\theta^2(x,y) = x_2y_2.$$

Using this the components of T are given by $T_{11} = 1$, $T_{12} = 0$, $T_{21} = 0$, and $T_{22} = 1$. Also notice that $T' = S\otimes S'$, where $S(x) = x_1 + x_2$, and $S'(y) = y_2$. Observe that not every tensor $T \in T^2(\mathbb{R}^2)$ is of the form $T = S\otimes S'$. For example $T'' \neq S\otimes S'$, for any $S, S' \in T^1(\mathbb{R}^2)$. ▶

In Lee, Ch. 11, the notion of tensor product between arbitrary vector spaces is explained. Here we will discuss a simplified version of the abstract theory. Let V and W be two (finite dimensional) real vector spaces, with bases $\{v_1,\cdots v_n\}$ and $\{w_1,\cdots w_m\}$ respectively, and for their dual spaces V^* and W^* we have the dual bases $\{\theta^1,\cdots,\theta^n\}$ and $\{\sigma^1,\cdots,\sigma^m\}$ respectively. If we use the identification $\{V^*\}^* \cong V$, and $\{W^*\}^* \cong W$ we can define $V\otimes W$ as follows:

Definition 7.7. The tensor product of V and W is the real vector space of (finite) linear combinations

$$V\otimes W := \Big\{\lambda^{ij}v_i\otimes w_j \;:\; \lambda^{ij}\in\mathbb{R}\Big\} = \Big[\big\{v_i\otimes w_j\big\}_{i,j}\Big],$$

where $v_i\otimes w_j(v^*,w^*) := v^*(v_i)w^*(w_j)$, using the identification $v_i(v^*) := v^*(v_i)$, and $w_j(w^*) := w^*(w_j)$, with $(v^*,w^*)\in V^*\times W^*$.

To get a feeling of what the tensor product of two vector spaces represents consider the tensor product of the dual spaces V^* and W^*. We obtain the real vector space of (finite) linear combinations

$$V^* \otimes W^* := \left\{\lambda_{ij}\theta^i \otimes \sigma^j \ : \ \lambda_{ij} \in \mathbb{R}\right\} = \left[\{\theta^i \otimes \sigma^j\}_{i,j}\right],$$

where $\theta^i \otimes \sigma^j(v,w) = \theta^i(v)\sigma^j(w)$ for any $(v,w) \in V \times W$. One can show that $V^* \otimes W^*$ is isomorphic to space of bilinear maps from $V \times W$ to $\mathbb{R}$. In particular elements $v^* \otimes w^*$ all lie in $V^* \otimes W^*$, but not all elements in $V^* \otimes W^*$ are of this form. The isomorphism is easily seen as follows. Let $v = \xi_i v_i$, and $w = \eta_j w_j$, then for a given bilinear form b it holds that $b(v,w) = \xi_i \eta_j b(v_i, w_j)$. By definition of dual basis we have that $\xi_i \eta_j = \theta^i(v)\sigma^j(w) = \theta^i \otimes \sigma^j(v,w)$, which shows the isomorphism by setting $\lambda_{ij} = b(v_i, w_j)$.

In the case $V^* \otimes W$ the tensors represent linear maps from V to W. Indeed, from the previous we know that elements in $V^* \otimes W$ represent bilinear maps from $V \times W^*$ to $\mathbb{R}$. For an element $b \in V^* \otimes W$ this means that $b(v,\cdot) : W^* \to \mathbb{R}$, and thus $b(v,\cdot) \in (W^*)^* \cong W$.

◀ **7.8** *Example.* Consider vectors $a \in V$ and $b^* \in W$, then $a^* \otimes (b^*)^*$ can be identified with a matrix, i.e $a^* \otimes (b^*)^*(v,\cdot) = a^*(v)(b^*)^*(\cdot) \cong a^*(v)b$. For example let $a^*(v) = a_1 v_1 + a_2 v_2 + a_3 v_3$, and

$$Av = a^*(v)b = \begin{pmatrix} a_1b_1v_1 + a_2b_1v_2 + a_3b_1v_3 \\ a_1b_2v_1 + a_2b_2v_2 + a_3b_2v_3 \end{pmatrix} = \begin{pmatrix} a_1b_1 & a_2b_1 & a_3b_1 \\ a_1b_2 & a_2b_2 & a_3b_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

Symbolically we can write

$$A = a^* \otimes b = \begin{pmatrix} a_1b_1 & a_2b_1 & a_3b_1 \\ a_1b_2 & a_2b_2 & a_3b_2 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

which shows how a vector and covector can be 'tensored' to become a matrix. Note that it also holds that $A = (a \cdot b^*)^* = b \cdot a^*$. ▶

Lemma 7.9. *We have that*

(i) *$V \otimes W$ and $W \otimes V$ are isomorphic;*
(ii) *$(U \otimes V) \otimes W$ and $U \otimes (V \otimes W)$ are isomorphic.*

With the notion of tensor product of vector spaces at hand we now conclude that the above describe tensor spaces $T^r(V)$ and $V_r(V)$ are given as follows;

$$T^r(V) = \underbrace{V^* \otimes \cdots \otimes V^*}_{r \text{ times}}, \qquad T_r(V) = \underbrace{V \otimes \cdots \otimes V}_{r \text{ times}}.$$

By considering tensor products of V's and V^*'s we obtain the tensor space of mixed tensors;

$$T^r_s(V) := \underbrace{V^* \otimes \cdots \otimes V^*}_{r \text{ times}} \otimes \underbrace{V \otimes \cdots \otimes V}_{s \text{ times}}.$$

Elements in this space are called (r,s)***-mixed tensors*** on V — r copies of V^*, and s copies of V. Of course the tensor product described above is defined in general for tensors $T \in T^r_s(V)$, and $S \in T^{r'}_{s'}(V)$:

$$\otimes : \ T^r_s(V) \times T^{r'}_{s'}(V) \to T^{r+r'}_{s+s'}(V).$$

The analogue of Theorem 7.5 can also be established for mixed tensors. In the next sections we will see various special classes of covariant, contravariant and mixed tensors.

◀ **7.10** *Example.* The inner product on a vector space V is an example of a covariant 2-tensor. This is also an example of a symmetric tensor. ▶

◀ **7.11** *Example.* The determinant of n vectors in $\mathbb{R}^n$ is an example of covariant n-tensor on $\mathbb{R}^n$. The determinant is skew-symmetric, and an example of an alternating tensor. ▶

If $f : V \to W$ is a linear mapping between vector spaces and T is an covariant tensor on W we can define concept of pullback of T. Let $T \in T^r(W)$, then $f^*T \in T^r(V)$ is defined as follows:

$$f^*T(v_1, \cdots v_r) = T(f(v_1), \cdots, f(v_r)),$$

and $f^* : T^r(W) \to T^r(V)$ is a linear mapping. Indeed, $f^*(T+S) = T \circ f + S \circ f = f^*T + f^*S$, and $f^*\lambda T = \lambda T \circ f = \lambda f^*T$. If we represent f by a matrix A with respect to bases $\{v_i\}$ and $\{w_j\}$ for V and W respectively, then the matrix for the linear f^* is given by

$$\underbrace{A^* \otimes \cdots \otimes A^*}_{r \text{ times}},$$

with respect to the bases $\{\theta^{i_1} \otimes \cdots \otimes \theta^{i_r}\}$ and $\{\sigma^{j_1} \otimes \cdots \otimes \sigma^{j_r}\}$ for $T^r(W)$ and $T^r(V)$ respectively.

◀ **7.12** *Remark.* The direct sum

$$T^*(V) = \bigoplus_{r=0}^{\infty} T^r(V),$$

consisting of finite sums of covariant tensors is called the ***covariant tensor algebra*** of V with multiplication given by the tensor product $\otimes$. Similarly, one defines the contravariant tensor algebra

$$T_*(V) = \bigoplus_{r=0}^{\infty} T_r(V).$$

For mixed tensors we have

$$T(V) = \bigoplus_{r,s=0}^{\infty} T_s^r(V),$$

which is called the ***tensor algebra of mixed tensor*** of V. Clearly, $T^*(V)$ and $T_*(V)$ subalgebras of $T(V)$. ▶

8. Symmetric and alternating tensors

There are two special classes of tensors which play an important role in the analysis of differentiable manifolds. The first class we describe are symmetric tensors. We restrict here to covariant tensors.

Definition 8.1. A covariant r-tensor T on a vector space V is called ***symmetric*** if

$$T(v_1,\cdots,v_i,\cdots,v_j,\cdots,v_r) = T(v_1,\cdots,v_j,\cdots,v_i,\cdots,v_r),$$

for any pair of indices $i \leq j$. The set of symmetric covariant r-tensors on V is denoted by $\Sigma^r(V) \subset T^r(V)$, which is a (vector) subspace of $T^r(V)$.

If $\mathbf{a} \in S_r$ is a permutation, then define

$$^{\mathbf{a}}T(v_1,\cdots,v_r) = T(v_{\mathbf{a}(1)},\cdots,v_{\mathbf{a}(r)}),$$

where $\mathbf{a}(\{1,\cdots,r\}) = \{\mathbf{a}(1),\cdots,\mathbf{a}(r)\}$. From this notation we have that for two permutations $\mathbf{a},\mathbf{b} \in S_r$, $^{\mathbf{b}}(^{\mathbf{a}}T) = {}^{\mathbf{ba}}T$. Define

$$\text{Sym } T = \frac{1}{r!}\sum_{\mathbf{a}\in S_r} {}^{\mathbf{a}}T.$$

It is straightforward to see that for any tensor $T \in T^r(V)$, Sym T is a symmetric. Moreover, a tensor T is symmetric if and only if Sym $T = T$. For that reason Sym T is called the ***(tensor) symmetrization***.

◀ **8.2** *Example.* Let $T,T' \in T^2(\mathbb{R}^2)$ be defined as follows: $T(x,y) = x_1y_2$, and $T'(x,y) = x_1y_1$. Clearly, T is not symmetric and T' is. We have that

$$\begin{aligned}\text{Sym } T(x,y) &= \frac{1}{2}T(x,y) + \frac{1}{2}T(y,x)\\ &= \frac{1}{2}x_1y_2 + \frac{1}{2}y_1x_2,\end{aligned}$$

which clearly is symmetric. If we do the same thing for T' we obtain:

$$\begin{aligned}\mathrm{Sym}\ T'(x,y) &= \frac{1}{2}T'(x,y)+\frac{1}{2}T'(y,x)\\ &= \frac{1}{2}x_1y_1+\frac{1}{2}y_1x_1 = T'(x,y),\end{aligned}$$

showing that operation Sym applied to symmetric tensors produces the same tensor again. ▶

Using symmetrization we can define the ***symmetric product***. Let $S \in \Sigma^r(V)$ and $T \in \Sigma^s(V)$ be symmetric tensors, then

$$S \cdot T = \mathrm{Sym}\ (S \otimes T).$$

The symmetric product of symmetric tensors is commutative which follows directly from the definition:

$$S \cdot T(v_1,\cdots,v_{r+s}) = \frac{1}{(r+s)!}\sum_{\mathbf{a}\in S_{r+s}} S(v_{\mathbf{a}(1)},\cdots,v_{\mathbf{a}(r)})T(v_{\mathbf{a}(r+1)},\cdots,v_{\mathbf{a}(r+s)}).$$

◀ **8.3** *Example.* Consider the 2-tensors $S(x) = x_1 + x_2$, and $T(y) = y_2$. Now $S \otimes T(x,y) = x_1y_2 + x_2y_2$, and $T \otimes S(x,y) = x_2y_1 + x_2y_2$, which clearly gives that $S \otimes T \neq T \otimes S$. Now compute

$$\begin{aligned}\mathrm{Sym}\ (S\otimes T)(x,y) &= \frac{1}{2}x_1y_2+\frac{1}{2}x_2y_2+\frac{1}{2}y_1x_2+\frac{1}{2}x_2y_2\\ &= \frac{1}{2}x_1y_2+\frac{1}{2}x_2y_1+x_2y_2 = S\cdot T(x,y).\end{aligned}$$

Similarly,

$$\begin{aligned}\mathrm{Sym}\ (T\otimes S)(x,y) &= \frac{1}{2}x_2y_1+\frac{1}{2}x_2y_2+\frac{1}{2}y_2x_1+\frac{1}{2}x_2y_2\\ &= \frac{1}{2}x_1y_2+\frac{1}{2}x_2y_1+x_2y_2 = T\cdot S(x,y),\end{aligned}$$

which gives that $S \cdot T = T \cdot S$. ▶

Lemma 8.4. *Let $\{v_1,\cdots,v_n\}$ be a basis for V, and let $\{\theta^1,\cdots,\theta^n\}$ be the dual basis for V^*. Then the set*

$$\mathcal{B}_\Sigma = \left\{\theta^{i_1}\cdots\theta^{i_r}\ :\ 1 \le i_1 \le \cdots \le i_r \le n\right\},$$

is a basis for the (sub)space $\Sigma^r(V)$ of symmetric r-tensors. Moreover, $\dim\Sigma^r(V) = \begin{pmatrix} n+r-1 \\ r \end{pmatrix} = \frac{(n+r-1)!}{r!(n-1)!}$.

Proof: Proving that $\mathcal{B}_\Sigma$ is a basis follows from Theorem 7.5, see also Lemma 8.10. It remains to establish the dimension of $\Sigma^r(V)$. Note that the elements in the basis are given by multi-indices $(i_1, \cdots, i_r)$, satisfying the property that $1 \leq i_1 \leq \cdots \leq i_r \leq n$. This means choosing r integers satisfying this restriction. To do this redefine $j_k := i_k + k - 1$. The integers j range from 1 through $n + r - 1$. Now choose r integers out of $n + r - 1$, i.e. $\begin{pmatrix} n+r-1 \\ r \end{pmatrix}$ combinations $(j_1, \cdots, j_r)$, which are in one-to-one correspondence with $(i_1, \cdots, i_r)$. ∎

Another important class of tensors are alternating tensors and are defined as follows.

Definition 8.5. A covariant r-tensor T on a vector space V is called ***alternating*** if

$$T(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_r) = -T(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_r),$$

for any pair of indices $i \leq j$. The set of alternating covariant r-tensors on V is denoted by $\Lambda^r(V) \subset T^r(V)$, which is a (vector) subspace of $T^r(V)$.

As before we define

$$\mathrm{Alt}\, T = \frac{1}{r!} \sum_{\mathbf{a} \in S_r} (-1)^{\mathbf{a}}\, {}^{\mathbf{a}}T,$$

where $(-1)^{\mathbf{a}}$ is $+1$ for even permutations, and -1 for odd permutations. We say that Alt T is the ***alternating projection*** of a tensor T, and Alt T is of course a alternating tensor.

◀ **8.6** *Example.* Let $T, T' \in T^2(\mathbb{R}^2)$ be defined as follows: $T(x,y) = x_1 y_2$, and $T'(x,y) = x_1 y_2 - x_2 y_1$. Clearly, T is not alternating and $T'(x,y) = -T'(y,x)$ is alternating. We have that

$$\begin{aligned} \mathrm{Alt}\, T(x,y) &= \frac{1}{2} T(x,y) - \frac{1}{2} T(y,x) \\ &= \frac{1}{2} x_1 y_2 - \frac{1}{2} y_1 x_2 = \frac{1}{2} T'(x,y), \end{aligned}$$

which clearly is alternating. If we do the same thing for T' we obtain:

$$\begin{aligned} \mathrm{Alt}\, T'(x,y) &= \frac{1}{2} T'(x,y) - \frac{1}{2} T'(y,x) \\ &= \frac{1}{2} x_1 y_2 - \frac{1}{2} x_2 y_1 - \frac{1}{2} y_1 x_2 + \frac{1}{2} y_2 x_1 = T'(x,y), \end{aligned}$$

showing that operation Alt applied to alternating tensors produces the same tensor again. Notice that $T'(x,y) = \det(x,y)$. ▶

This brings us to the fundamental product of alternating tensors called the ***wedge product***. Let $S \in \Lambda^r(V)$ and $T \in \Lambda^s(V)$ be symmetric tensors, then

$$S \wedge T = \frac{(r+s)!}{r!s!} \text{Alt}\,(S \otimes T).$$

The wedge product of alternating tensors is anti-commutative which follows directly from the definition:

$$S \wedge T(v_1, \cdots, v_{r+s}) = \frac{1}{r!s!} \sum_{\mathbf{a} \in S_{r+s}} (-1)^{\mathbf{a}} S(v_{\mathbf{a}(1)}, \cdots, v_{\mathbf{a}(r)}) T(v_{\mathbf{a}(r+1)}, \cdots, v_{\mathbf{a}(r+s)}).$$

In the special case of the wedge of two covectors $\theta, \omega \in V^*$ gives

$$\theta \wedge \omega = \theta \otimes \omega - \omega \otimes \theta.$$

In particular we have that

(i) $(T \wedge S) \wedge R = T \wedge (S \wedge R)$;
(ii) $(T + T') \wedge S = T \wedge S + T' \wedge S$;
(iii) $T \wedge S = (-1)^{rs} S \wedge T$, for $T \in \Lambda^r(V)$ and $S \in \Lambda^s(V)$;
(iv) $T \wedge T = 0$.

The latter is a direct consequence of the definition of Alt . In order to prove these properties we have the following lemma.

Lemma 8.7. *Let $T \in T^r(V)$ and $S \in T^s(V)$, then*

$$\text{Alt}\,(T \otimes S) = \text{Alt}\,((\text{Alt}\,T) \otimes S) = \text{Alt}\,(T \otimes \text{Alt}\,S).$$

Proof: Let $G \cong S_r$ be the subgroup of S_{r+s} consisting of permutations that only permute the element $\{1, \cdots, r\}$. For $\mathbf{a} \in G$, we have $\mathbf{a}' \in S_r$. Now ${}^{\mathbf{a}}(T \otimes S) = {}^{\mathbf{a}'}T \otimes S$, and thus

$$\frac{1}{r!} \sum_{\mathbf{a} \in G} (-1)^{\mathbf{a}}\, {}^{\mathbf{a}}(T \otimes S) = (\text{Alt}\,T) \otimes S.$$

For the right cosets $\{\mathbf{ba} \,:\, \mathbf{a} \in G\}$ we have

$$\begin{aligned} \sum_{\mathbf{a} \in G} (-1)^{\mathbf{ba}}\, {}^{\mathbf{ba}}(T \otimes S) &= (-1)^{\mathbf{b}}\, {}^{\mathbf{b}}\Big(\sum_{\mathbf{a} \in G} (-1)^{\mathbf{a}}\, {}^{\mathbf{a}}(T \otimes S) \Big) \\ &= r!(-1)^{\mathbf{b}}\, {}^{\mathbf{b}}\Big((\text{Alt}\,T) \otimes S \Big). \end{aligned}$$

Taking the sum over all right cosets with the factor $\frac{1}{(r+s)!}$ gives

$$\begin{aligned} \text{Alt}\,(T \otimes S) &= \frac{1}{(r+s)!} \sum_{\mathbf{b}} \sum_{\mathbf{a} \in G} (-1)^{\mathbf{ba}}\, {}^{\mathbf{ba}}(T \otimes S) \\ &= \frac{r!}{(r+s)!} \sum_{\mathbf{b}} (-1)^{\mathbf{b}}\, {}^{\mathbf{b}}\Big((\text{Alt}\,T) \otimes S \Big) = \text{Alt}\,((\text{Alt}\,T) \otimes S), \end{aligned}$$

where the latter equality is due to the fact that $r!$ terms are identical under the definition of Alt $((\text{Alt } T) \otimes S)$. ∎

Property (i) can now be proved as follows. Clearly Alt $(\text{Alt } (T \otimes S) - T \otimes S) = 0$, and thus from Lemma 8.7 we have that

$$0 = \text{Alt } ((\text{Alt } (T \otimes S) - T \otimes S) \otimes R) = \text{Alt } (\text{Alt } (T \otimes S) \otimes R) - \text{Alt } (T \otimes S \otimes R).$$

By definition

$$\begin{aligned} (T \wedge S) \wedge R &= \frac{(r+s+t)!}{(r+s)!t!} \text{Alt } ((T \wedge S) \otimes R) \\ &= \frac{(r+s+t)!}{(r+s)!t!} \text{Alt } \Big(\Big(\frac{(r+s)!}{r!s!} \text{Alt } (T \otimes S)\Big) \otimes R\Big) \\ &= \frac{(r+s+t)!}{r!s!t!} \text{Alt } (T \otimes S \otimes R). \end{aligned}$$

The same formula holds for $T \wedge (S \wedge R)$, which prove associativity. More generally it holds that for $T_i \in \Lambda^{r_i}(V)$

$$T_1 \wedge \cdots \wedge T_k = \frac{(r_1 + \cdots + r_k)!}{r_1! \cdots r_k!} \text{Alt } (T_1 \otimes \cdots \otimes T_k).$$

Property (iii) can be seen as follows. Each term in $T \wedge S$ can be found in $S \wedge T$. This can be done by linking the permutations $\mathbf{a}$ and $\mathbf{a}'$. To be more precise, how many permutation of two elements are needed to change

$$\mathbf{a} \leftrightarrow (i_1, \cdots, i_r, j_{r+1}, \cdots, j_{r+s}) \quad \text{into} \quad \mathbf{a}' \leftrightarrow (j_{r+1}, \cdots, j_{r+s}, i_1, \cdots, i_r).$$

This clearly requires rs permutations of two elements, which shows Property (iii).

◀ **8.8** *Example.* Consider the 2-tensors $S(x) = x_1 + x_2$, and $T(y) = y_2$. As before $S \otimes T(x,y) = x_1y_2 + x_2y_2$, and $T \otimes S(x,y) = x_2y_1 + x_2y_2$. Now compute

$$\begin{aligned} \text{Alt } (S \otimes T)(x,y) &= \frac{1}{2}x_1y_2 + \frac{1}{2}x_2y_2 - \frac{1}{2}y_1x_2 - \frac{1}{2}x_2y_2 \\ &= \frac{1}{2}x_1y_2 - \frac{1}{2}x_2y_1 = 2\Big(S \wedge T(x,y)\Big). \end{aligned}$$

Similarly,

$$\begin{aligned} \text{Alt } (T \otimes S)(x,y) &= \frac{1}{2}x_2y_1 + \frac{1}{2}x_2y_2 - \frac{1}{2}y_2x_1 - \frac{1}{2}x_2y_2 \\ &= -\frac{1}{2}x_1y_2 + \frac{1}{2}x_2y_1 = -2\Big(T \wedge S(x,y)\Big), \end{aligned}$$

which gives that $S \wedge T = -T \wedge S$. Note that if $T = e_1^*$, i.e. $T(x) = x_1$, and $S = e_2^*$, i.e. $S(x) = x_2$, then

$$T \wedge S(x,y) = x_1y_2 - x_2y_1 = \det(x,y).$$

▶

◀ **8.9** *Remark.* Some authors use the more logical definition

$$S\bar{\wedge}T = \text{Alt}\,(S\otimes T),$$

which is in accordance with the definition of the symmetric product. This definition is usually called the alt convention for the wedge product, and our definition is usually referred to as the determinant convention. For computational purposes the determinant convention is more appropriate. ▶

If $\{e_1^*,\cdots,e_n^*\}$ is the standard dual basis for $(\mathbb{R}^n)^*$, then for vectors $a_1,\cdots,a_n\in\mathbb{R}^n$,

$$\det(a_1,\cdots,a_n) = e_1^*\wedge\cdots\wedge e_n^*(a_1,\cdots,a_n).$$

Using the multilinearity the more general statement reads

$$\beta^1\wedge\cdots\wedge\beta^n(a_1,\cdots,a_n) = \det\big(\beta^i(a_j)\big), \tag{7}$$

where β^i are co-vectors.

The alternating tensor $\det = e_1^*\wedge\cdots\wedge e_n^*$ is called the determinant function on $\mathbb{R}^n$. If $f:V\to W$ is a linear map between vector spaces then the pullback $f^*T\in\Lambda^r(V)$ of any alternating tensor $T\in\Lambda^r(W)$ is given via the relation:

$$f^*T(v_1,\cdots,v_r) = T\big(f(v_1),\cdots,f(v_r)\big),\qquad f^*:\Lambda^r(W)\to\Lambda^r(V).$$

In particular, $f^*(T\wedge S) = (f^*T)\wedge f^*(S)$. As a special case we have that if $f:V\to V$, linear, and $\dim V = n$, then

$$f^*T = \det(f)T, \tag{8}$$

for any alternating tensor $T\in\Lambda^n(V)$. This can be seen as follows. By multilinearity we verify the above relation for the vectors $\{e_i\}$. We have that

$$\begin{aligned} f^*T(e_1,\cdots,e_n) &= T(f(e_1),\cdots,f(e_n)) \\ &= T(f_1,\cdots,f_n) = c\det(f_1,\cdots,f_n) = c\det(f), \end{aligned}$$

where we use the fact that $\Lambda^n(V)\cong\mathbb{R}$ (see below). On the other hand

$$\begin{aligned} \det(f)T(e_1,\cdots e_n) &= \det(f)c\cdot\det(e_1,\cdots,e_n) \\ &= c\det(f), \end{aligned}$$

which proves (8).

Lemma 8.10. *Let $\{\theta^1,\cdots,\theta^n\}$ be a basis for V^*, then the set*

$$\mathcal{B}_\Lambda = \big\{\theta^{i_1}\wedge\cdots\wedge\theta^{i_r}\;:\;1\le i_1<\cdots<i_r\le n\big\},$$

is a basis for $\Lambda^r(V)$, and $\dim\Lambda^r(V) = \frac{n!}{(n-r)!r!}$. In particular, $\dim\Lambda^r(V) = 0$ for $r>n$.

Proof: From Theorem 7.5 we know that any alternating tensor $T \in \Lambda^r(V)$ can be written as

$$T = T_{j_1 \cdots j_r} \theta^{j_1} \otimes \cdots \otimes \theta^{j_r}.$$

We have that Alt $T = T$, and so

$$T = T_{j_1 \cdots j_r} \mathrm{Alt}\,(\theta^{j_1} \otimes \cdots \otimes \theta^{j_r}) = \frac{1}{r!} T_{j_1 \cdots j_r} \theta^{j_1} \wedge \cdots \wedge \theta^{j_r}$$

In the expansion the terms with $j_k = j_\ell$ are zero since $\theta^{j_k} \wedge \theta^{j_\ell} = 0$. If we order the indices in increasing order we obtain

$$T = \pm \frac{1}{r!} T_{i_1 \cdots i_r} \theta^{i_1} \wedge \cdots \wedge \theta^{i_r},$$

which show that $\mathcal{B}_\Lambda$ spans $\Lambda^r(V)$.

Linear independence can be proved as follows. Let $0 = \lambda_{i_1 \cdots i_r} \theta^{i_1} \wedge \cdots \wedge \theta^{i_r}$, and thus $\lambda_{i_1 \cdots i_r} = \theta^{i_1} \wedge \cdots \wedge \theta^{i_r}(v_{i_1}, \cdots, v_{i_r}) = 0$, which proves linear independence.

It is immediately clear that $\mathcal{B}_\Lambda$ consists of $\begin{pmatrix} n \\ r \end{pmatrix}$ elements. ■

As we mentioned before the operation Alt is called the alternating projection. As a matter of fact Sym is also a projection.

Lemma 8.11. *Some of the basic properties can be listed as follows;*

(i) Sym *and* Alt *are projections on* $T^r(V)$, *i.e.* $\mathrm{Sym}^2 = \mathrm{Sym}$, *and* $\mathrm{Alt}^2 = \mathrm{Alt}$*;*

(ii) T *is symmetric if and only if* $\mathrm{Sym}\, T = T$, *and* T *is alternating if and only if* $\mathrm{Alt}\, T = T$*;*

(iii) $\mathrm{Sym}(T^r(V)) = \Sigma^r(V)$, *and* $\mathrm{Alt}(T^r(V)) = \Lambda^r(V)$*;*

(iv) $\mathrm{Sym} \circ \mathrm{Alt} = \mathrm{Alt} \circ \mathrm{Sym} = 0$, *i.e. if* $T \in \Lambda^r(V)$, *then* $\mathrm{Sym}\, T = 0$, *and if* $T \in \Sigma^r(V)$, *then* $\mathrm{Alt}\, T = 0$*;*

(v) *let* $f : V \to W$, *then* Sym *and* Alt *commute with* $f^* : T^r(W) \to T^r(V)$, *i.e.* $\mathrm{Sym} \circ f^* = f^* \circ \mathrm{Sym}$, *and* $\mathrm{Alt} \circ f^* = f^* \circ \mathrm{Alt}$.

9. Tensor bundles and tensor fields

Generalizations of tangent spaces and cotangent spaces are given by the tensor spaces

$$T^r(T_pM), \qquad T_s(T_pM), \quad \text{and} \qquad T^r_s(T_pM),$$

where $T^r(T_pM) = T_r(T_p^*M)$. As before we can introduce the tensor bundles:

$$T^rM = \bigsqcup_{p\in M} T^r(T_pM),$$

$$T_sM = \bigsqcup_{p\in M} T_s(T_pM),$$

$$T_s^rM = \bigsqcup_{p\in M} T_s^r(T_pM),$$

which are called the ***covariant*** r***-tensor bundle, contravariant*** s***-tensor bundle***, and the ***mixed*** (r,s)***-tensor bundle*** on M. As for the tangent and cotangent bundle the tensor bundles are also smooth manifolds. In particular, $T^1M = T^*M$, and $T_1M = TM$. Recalling the symmetric and alternating tensors as introduced in the previous section we also define the tensor bundles Σ^rM and Λ^rM.

Theorem 9.1. *The tensor bundles T^rM, T_rM and T_s^rM are smooth vector bundles.*

Proof: Using Section 6 the theorem is proved by choosing appropriate local trivializations Φ. For coordinates $x = \varphi_\alpha(p)$ and $x' = \varphi_\beta(p)$ we recall that

$$\frac{\partial}{\partial x_i}\Big|_p = \frac{\partial x'_j}{\partial x_i}\frac{\partial}{\partial x'_j}\Big|_p, \quad dx'^j|_p = \frac{\partial x'_j}{\partial x_i}dx^i|_p.$$

For a covariant tensor $T \in T^rM$ this implies the following. The components are defined in local coordinates by

$$T = T_{i_1\cdots i_r}dx^{i_1}\otimes\cdots\otimes dx^{i_r}, \quad T_{i_1\cdots i_r} = T\Big(\frac{\partial}{\partial x_{i_1}},\cdots,\frac{\partial}{\partial x_{i_r}}\Big).$$

The change of coordinates $x \to x'$ then gives

$$T_{i_1\cdots i_r} = T\Big(\frac{\partial x'_{j_1}}{\partial x_{i_1}}\frac{\partial}{\partial x'_{j_1}},\cdots,\frac{\partial x'_{j_r}}{\partial x_{i_r}}\frac{\partial}{\partial x'_{j_r}}\Big) = T'_{j_1\cdots j_r}\frac{\partial x'_{j_1}}{\partial x_{i_1}}\cdots\frac{\partial x'_{j_r}}{\partial x_{i_r}}.$$

Define

$$\Phi(T_{i_1\cdots i_r}dx^{i_1}\otimes\cdots\otimes dx^{i_r}) = \big(p,(T_{i_1\cdots i_r})\big),$$

then

$$\Phi\circ\Phi'^{-1}\big(p,(T'_{j_1\cdots j_r})\big) = \left(p,\Big(T'_{j_1\cdots j_r}\frac{\partial x'_{j_1}}{\partial x_{i_1}}\cdots\frac{\partial x'_{j_r}}{\partial x_{i_r}}\Big)\right).$$

The products of the partial derivatives are smooth functions. We can now apply Theorem 6.4 as we did in Theorems 6.8 and 6.12. ■

On tensor bundles we also have the natural projection

$$\pi : T_s^rM \to M,$$

defined by $\pi(p,T) = p$. A ***smooth section*** in T_s^rM is a smooth mapping

$$\sigma : M \to T_s^rM,$$

such that $\pi \circ \sigma = \mathrm{id}_M$. The space of smooth sections in $T^r_s M$ is denoted by $\mathcal{F}^r_s(M)$. For the co- and contravariant tensors these spaces are denoted by $\mathcal{F}^r(M)$ and $\mathcal{F}_s(M)$ respectively. Smooth sections in these tensor bundles are also called ***smooth tensor fields***. Clearly, vector fields and 1-forms are examples of tensor fields. Sections in tensor bundles can be expressed in coordinates as follows:

$$\sigma = \begin{cases} \sigma_{i_1\cdots i_r} dx^{i_1} \otimes \cdots \otimes dx^{i_r}, & \sigma \in \mathcal{F}^r(M), \\ \sigma^{j_1\cdots j_s} \frac{\partial}{\partial x_{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{j_s}}, & \sigma \in \mathcal{F}_s(M), \\ \sigma^{j_1\cdots j_s}_{i_1\cdots i_r} dx_{i_1} \otimes \cdots \otimes dx^{i_r} \otimes \frac{\partial}{\partial x_{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{j_s}}, & \sigma \in \mathcal{F}^r_s(M). \end{cases}$$

Tensor fields are often denoted by the component functions. The tensor and tensor fields in this course are, except for vector fields, all covariant tensors and covariant tensor fields. Smoothness of covariant tensor fields can be described in terms of the component functions $\sigma_{i_1\cdots i_r}$.

Lemma 9.2. [26] *A covariant tensor field σ is smooth at $p \in U$ if and only if*

(i) *the coordinate functions $\sigma_{i_1\cdots i_r} : U \to \mathbb{R}$ are smooth, or equivalently if and only if*

(ii) *for smooth vector fields $X_1, \cdots, X_r$ defined on any open set $U \subset M$, then the function $\sigma(X_1, \cdots, X_r) : U \to \mathbb{R}$, given by*

$$\sigma(X_1, \cdots, X_r)(p) = \sigma_p(X_1(p), \cdots, X_r(p)),$$

is smooth.

The same equivalences hold for contravariant and mixed tensor fields.

Proof: Use the identities in the proof of Theorem 9.1 and then the proof goes as Lemma 6.11. ■

◀ **9.3** *Example.* Let $M = \mathbb{R}^2$ and let $\sigma = dx^1 \otimes dx^1 + x_1^2 dx^2 \otimes dx^2$. If $X = \xi^1(x)\frac{\partial}{\partial x_1} + \xi^2(x)\frac{\partial}{\partial x_2}$ and $Y = \eta^1(x)\frac{\partial}{\partial x_1} + \eta^2(x)\frac{\partial}{\partial x_2}$ are arbitrary smooth vector fields on $T_x\mathbb{R}^2 \cong \mathbb{R}^2$, then

$$\sigma(X,Y) = \xi_1(x)\eta_1(x) + x_1^2 \xi_2(x)\eta_2(x),$$

which clearly is a smooth function in $x \in \mathbb{R}^2$. ▶

For covariant tensors we can also define the notion of pullback of a mapping f between smooth manifolds. Let $f : N \to M$ be a smooth mappings, then the pullback $f^* : T^r(T_{f(p)}M) \to T^r(T_pN)$ is defined as

$$(f^*T)(X_1, \cdots X_r) := T(f_*X_1, \cdots, f_*X_r),$$

where $T \in T^r(T_{f(p)}M)$, and $X_1, \cdots, X_r \in T_pN$. We have the following properties.

[26] See Lee, Lemma 11.6.

Lemma 9.4. [27] *Let $f : N \to M$, $g : M \to P$ be smooth mappings, and let $p \in N$, $S \in T^r(T_{f(p)}M)$, and $T \in T^{r'}(T_{f(p)}M)$, then:*

(i) $f^* : T^r(T_{f(p)}M) \to T^r(T_pN)$ *is linear;*
(ii) $f^*(S \otimes T) = f^*S \otimes f^*T$*;*
(iii) $(g \circ f)^* = f^* \circ g^* : T^r(T_{(g\circ f)(p)}P) \to T^r(T_pM)$*;*
(iv) $\mathrm{id}_M^* S = S$*;*
(v) $f^* : T^rM \to T^rN$ *is a smooth bundle map.*

Proof: Combine the proof of Lemma 5.3 with the identities in the proof of Theorem 9.1. ■

◀ **9.5** *Example.* Let us continue with the previous example and let $N = M = \mathbb{R}^2$. Consider the mapping $f : N \to M$ defined by

$$f(x) = (2x_1, x_2^3 - x_1).$$

For given tangent vectors $X, Y \in T_xN$, given by $X = \xi_1 \frac{\partial}{\partial x_1} + \xi_2 \frac{\partial}{\partial x_2}$ and $Y = \eta_1 \frac{\partial}{\partial x_1} + \eta_2 \frac{\partial}{\partial x_2}$, we can compute the pushforward

$$f_* = \begin{pmatrix} 2 & 0 \\ -1 & 3x_2^2 \end{pmatrix}, \quad \text{and}$$

$$\begin{aligned} f_*X &= 2\xi_1 \frac{\partial}{\partial y_1} + (-\xi_1 + 3x_2^2\xi_2)\frac{\partial}{\partial y_2}, \\ f_*Y &= 2\eta_1 \frac{\partial}{\partial y_1} + (-\eta_1 + 3x_2^2\eta_2)\frac{\partial}{\partial y_2}. \end{aligned}$$

Let σ be given by $\sigma = dy^1 \otimes dy^1 + y_1^2 dy^2 \otimes dy^2$. This then yields

$$\begin{aligned} \sigma(f_*X, f_*Y) &= 4\xi_1\eta_1 + 4x_1^2(\xi_1 - 3x_1^2\xi_2)(\eta_1 - 3x_1^2\eta_2) \\ &= 4(1 + x_1^2)\xi_1\eta_1 - 12x_1^2x_2^2\xi_1\eta_2 \\ &\quad -12x_1^2x_2^2\xi_2\eta_1 + 36x_1^2x_2^4x_1^2\xi_2\eta_2. \end{aligned}$$

We have to point out here that f_*X and f_*Y are tangent vectors in T_xN and not necessarily vector fields, although we can use this calculation to compute $f^*\sigma$, which clearly is a smooth 2-tensor field on T^2N. ▶

[27] See Lee, Proposition 11.8.

◀ **9.6** *Example.* A different way of computing $f^*\sigma$ is via a local representation of σ directly. We have $\sigma = dy^1\otimes dy^1 + y_1^2 dy^2\otimes dy^2$, and

$$\begin{aligned} f^*\sigma &= d(2x_1)\otimes d(2x_1) + 4x_1^2 d(x_2^3 - x_1)\otimes d(x_2^3-x_1) \\ &= 4dx^1\otimes dx^1 + 4x_1^2\big(3x_2^2dx^2 - dx^1\big)\otimes\big(3x_2^2dx^2-dx^1\big) \\ &= 4(1+x_1^2)dx^1\otimes dx^1 - 12x_1^2x_2^2dx^1\otimes dx^2 \\ &\quad -12x_1^2x_2^2dx^2\otimes dx^1 + 36x_1^2x_2^4x_1^2dx^2\otimes dx^2. \end{aligned}$$

Here we used the fact that computing the differential of a mapping to $\mathbb{R}$ produces the pushforward to a 1-form on N. ▶

◀ **9.7** *Example.* If we perform the previous calculation for an arbitrary 2-tensor

$$\sigma = a_{11}dy^1\otimes dy^1 + a_{12}dy^1\otimes dy^2 + a_{21}dy^2\otimes dy^1 + a_{22}dy^2\otimes dy^2.$$

Then,

$$\begin{aligned} f^*\sigma &= (4a_{11}-2a_{12}-2a_{21}+a_{22})dx^1\otimes dx^1 + (6a_{12}-3a_{22})x_2^2dx^1\otimes dx^2 \\ &\quad +(6a_{21}-3a_{22})x_2^2dx^2\otimes dx^1 + 9x_2^4a_{22}dx^2\otimes dx^2, \end{aligned}$$

which produces the following matrix if we identify $T^2(T_xN)$ and $T^2(T_yM)$ with $\mathbb{R}^4$:

$$f^* = \begin{pmatrix} 4 & -2 & -2 & 1 \\ 0 & 6x_2^2 & 0 & -3x_2^2 \\ 0 & 0 & 6x_2^2 & -3x_2^2 \\ 0 & 0 & 0 & 9x_2^4 \end{pmatrix},$$

which clearly equal to the tensor product of the matrices $(Jf)^*$, i.e.

$$f^* = (Jf)^*\otimes(Jf)^* = \begin{pmatrix} 2 & -1 \\ 0 & 3x_2^2 \end{pmatrix}\otimes\begin{pmatrix} 2 & -1 \\ 0 & 3x_2^2 \end{pmatrix}.$$

This example show how to interpret $f^* : T^2(T_yM)\to T^2(T_xN)$ as a linear mapping. ▶

As for smooth 1-forms this operation extends to smooth covariant tensor fields: $(f^*\sigma)_p = f^*(\sigma_{f(p)})$, $\sigma\in\mathcal{T}^r(N)$, which in coordinates reads

$$(f^*\sigma)_p(X_1,\cdots X_r) := \sigma_{f(p)}(f_*X_1,\cdots,f_*X_r),$$

for tangent vectors $X_1,\cdots,X_r\in T_pN$.

Lemma 9.8. [28] *Let $f: N\to M$, $g: M\to P$ be smooth mappings, and let $h\in C^\infty(M)$, $\sigma\in\mathcal{T}^r(M)$, and $\tau\in\mathcal{T}^r(N)$, then:*

(i) *$f^*:\mathcal{T}^r(M)\to\mathcal{T}^r(N)$ is linear;*

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[28] See Lee, Proposition 11.9.

(ii) $f^*(h\sigma) = (h \circ f) f^*\sigma$;
(iii) $f^*(\sigma \otimes \tau) = f^*\sigma \otimes f^*\tau$;
(iv) $f^*\sigma$ *is a smooth covariant tensor field;*
(v) $(g \circ f)^* = f^* \circ g^*$;
(vi) $\mathrm{id}_M^*\sigma = \sigma$;

Proof: Combine the Lemmas 9.4 and 9.2. ∎

10. Differential forms

A special class of covariant tensor bundles and associated bundle sections are the so-called alternating tensor bundles. Let $\Lambda^r(T_pM) \subset T^r(T_pM)$ be the space alternating tensors on T_pM. We know from Section 8 that a basis for $\Lambda^r(T_pM)$ is given by

$$\left\{ dx^{i_1} \wedge \cdots \wedge dx^{i_r} \ : \ 1 \le i_1, \cdots, i_r \le m \right\},$$

and $\dim \Lambda^r(T_pM) = \frac{m!}{r!(m-r)!}$. The associated tensor bundles of atlernating covariant tensors is denoted by $\Lambda^r M$. Smooth sections in $\Lambda^r M$ are called ***differential r-forms***, and the space of smooth sections is denoted by $\Gamma^r(M) \subset \mathcal{F}^r(M)$. In particular $\Gamma^0(M) = C^\infty(M)$, and $\Gamma^1(M) = \mathcal{F}^*(M)$. In terms of components a differential r-form, or r-form for short, is given by

$$\sigma_{i_1\cdots i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r},$$

and the components $\sigma_{i_1\cdots i_r}$ are smooth functions. An r-form σ acts on vector fields $X_1, \cdots, X_r$ as follows:

$$\begin{aligned} \sigma(X_1, \cdots, X_r) &= \sum_{\mathbf{a} \in S_r} (-1)^{\mathbf{a}} \sigma_{i_1\cdots i_r} dx_{i_1}(X_{\mathbf{a}(1)}) \cdots dx_{i_r}(X_{\mathbf{a}(r)}) \\ &= \sum_{\mathbf{a} \in S_r} (-1)^{\mathbf{a}} \sigma_{i_1\cdots i_r} X_{\mathbf{a}(1)}^{i_1} \cdots X_{\mathbf{a}(r)}^{i_r}. \end{aligned}$$

◀ **10.1** *Example.* Let $M = \mathbb{R}^3$, and $\sigma = dx \wedge dz$. Then for vector fields

$$X_1 = X_1^1 \frac{\partial}{\partial x} + X_1^2 \frac{\partial}{\partial y} + X_1^3 \frac{\partial}{\partial z},$$

and

$$X_2 = X_2^1 \frac{\partial}{\partial x} + X_2^2 \frac{\partial}{\partial y} + X_2^3 \frac{\partial}{\partial z},$$

we have that

$$\sigma(X_1, X_2) = X_1^1 X_2^3 - X_1^3 X_2^1.$$

▶

An important notion that comes up in studying differential forms is the notion of contracting an r-form. Given an r-form $\sigma \in \Gamma^r(M)$ and a vector field $X \in \mathcal{F}(M)$, then

$$i_X\sigma := \sigma(X, \cdot, \cdots, \cdot),$$

is called the ***contraction with*** X, and is a differential $(r-1)$-form on M. Another notation for this is $i_X\sigma = X \lrcorner \sigma$. Contraction is a linear mapping

$$i_X : \Gamma^r(M) \to \Gamma^{r-1}(M).$$

Contraction is also linear in X, i.e. for vector fields X, Y it holds that

$$i_{X+Y}\sigma = i_X\sigma + i_Y\sigma, \qquad i_{\lambda X}\sigma = \lambda \cdot i_X\sigma.$$

Lemma 10.2. [29] *Let $\sigma \in \Gamma^r(M)$ and $X \in \mathcal{F}(M)$ a smooth vector field, then*

(i) $i_X\sigma \in \Gamma^{r-1}(M)$ *(smooth $(r-1)$-form);*
(ii) $i_X \circ i_X = 0$*;*
(iii) i_X *is an **anti-derivation**, i.e. for $\sigma \in \Gamma^r(M)$ and $\omega \in \Gamma^s(M)$,*

$$i_X(\sigma \wedge \omega) = (i_X\sigma) \wedge \omega + (-1)^r \sigma \wedge (i_X\omega).$$

A direct consequence of (iii) is that if $\sigma = \sigma_1 \wedge \cdots \wedge \sigma_r$, where $\sigma_i \in \Gamma^1(M) = \mathcal{F}^*(M)$, then

$$i_X\sigma = (-1)^{i-1}\sigma_i(X)\sigma_1 \wedge \cdots \widehat{\sigma}_i \wedge \cdots \wedge \sigma_r, \tag{10}$$

where the hat indicates that σ_i is to be omitted, and we use the summation convention.

◀ **10.3** *Example.* Let $\sigma = x_2 dx^1 \wedge dx^3$ be a 2-form on $\mathbb{R}^3$, and $X = x_1^2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + (x_1 + x_2)\frac{\partial}{\partial x_3}$ a given vector field on $\mathbb{R}^3$. If $Y = Y^1 \frac{\partial}{\partial y_1} + Y^2 \frac{\partial}{\partial y_2} + Y^3 \frac{\partial}{\partial y_3}$ is an arbitrary vector fields then

$$\begin{aligned}(i_X\sigma)(Y) = \sigma(X,Y) &= dx^1(X)dx^3(Y) - dx^1(Y)dx^3(X) \\ &= x_1^2 Y^3 - (x_1 + x_2)Y^1,\end{aligned}$$

which gives that

$$i_X\sigma = x_1^2 dx^3 - (x_1 + x_2)dx^1.$$

▶

[29] See Lee, Lemma 13.11.

Since σ is a 2-form the calculation using X,Y is still doable. For higher forms this becomes to involved. If we use the multilinearity of forms we can give a simple procedure for computing $i_X\sigma$ using Formula (10).

◀ **10.4** *Example.* Let $\sigma = dx^1\wedge dx^2\wedge dx^3$ be a 3-form on $\mathbb{R}^3$, and X the vector field as given in the previous example. By linearity

$$i_X\sigma = i_{X_1}\sigma + i_{X_2}\sigma + i_{X_3}\sigma,$$

where $X_1 = x_1^2\frac{\partial}{\partial x_1}$, $X_2 = x_3\frac{\partial}{\partial x_2}$, and $X_3 = (x_1+x_2)\frac{\partial}{\partial x_3}$. This composition is chosen so that X is decomposed in vector fields in the basis directions. Now

$$\begin{aligned} i_{X_1}\sigma &= dx^1(X_1)dx^2\wedge dx^3 = x_1^2dx^2\wedge dx^3,\\ i_{X_2}\sigma &= -dx^2(X_2)dx^1\wedge dx^3 = -x_3dx^1\wedge dx^3,\\ i_{X_3}\sigma &= dx^2(X_3)dx^1\wedge dx^2 = (x_1+x_2)dx^1\wedge dx^2,\end{aligned}$$

which gives

$$i_X\sigma = x_1^2dx^2\wedge dx^3 - x_3dx^1\wedge dx^3 + (x_1+x_2)dx^1\wedge dx^2,$$

a 2-form on $\mathbb{R}^3$. One should now verify that the same answer is obtained by computing $\sigma(X,Y,Z)$. ▶

For completeness we recall that for a smooth mapping $f: N\to M$, the pullback of a r-form σ is given by

$$(f^*\sigma)_p(X_1,\cdots,X_r) = f^*\sigma_{f(p)}(X_1,\cdots,X_r) = \sigma_{f(p)}(f_*X_1,\cdots,f_*X_r).$$

We recall that for a mapping $h: M\to\mathbb{R}$, then pushforward, or differential of h $dh_p = h_*\in T_p^*M$. In coordinates $dh_p = \frac{\partial\tilde{h}}{\partial x_i}dx^i|_p$, and thus the mapping $p\mapsto dh_p$ is a smooth section in $\Lambda^1(M)$, and therefore a differential 1-form, with component $\sigma_i = \frac{\partial\tilde{h}}{\partial x_i}$ (in local coordinates).

If $f: N\to M$ is a mapping between m-dimensional manifolds with charts (U,φ), and (V,ψ) respectively, and $f(U)\subset V$. Set $x=\varphi(p)$, and $y=\psi(q)$, then

$$f^*\big(\sigma dy^1\wedge\cdots\wedge dy^m\big) = (\sigma\circ f)\det\big(J\tilde{f}|_x\big)dx^1\wedge\cdots\wedge dx^m. \tag{11}$$

This can be proved as follows. From the definition of the wedge product and Lemma 9.8 it follows that $f^*(dy^1\wedge\cdots\wedge dy^m) = f^*dy^1\wedge\cdots\wedge f^*dy^m$, and $f^*dy^j = \frac{\partial\widetilde{f}_j}{\partial x_i}dx^i = dF^j$, where $F=\psi\circ f$ and $\widetilde{f} = \psi\circ f\circ\varphi^{-1}$. Now

$$\begin{aligned} f^*(dy^1\wedge\cdots\wedge dy^m) &= f^*dy^1\wedge\cdots\wedge f^*dy^m\\ &= dF^1\wedge\cdots\wedge dF^m,\end{aligned}$$

and furthermore, using (7),

$$dF^1 \wedge \cdots \wedge dF^m\Big(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_m}\Big) = \det\left(dF^i\Big(\frac{\partial}{\partial x_j}\Big)\right) = \det\Big(\frac{\partial \widetilde{f}_j}{\partial x_j}\Big),$$

which proves the above claim.

As a consequence of this a change of coordinates $\widetilde{f} = \psi \circ \varphi^{-1}$ yields

$$f^*(dy^1 \wedge \cdots \wedge dy^m) = \det\big(J\tilde{f}|_x\big) dx^1 \wedge \cdots \wedge dx^m. \tag{12}$$

◀ **10.5** *Example.* Consider $\sigma = dx \wedge dy$ on $\mathbb{R}^2$, and mapping f given by $x = r\cos(\theta)$ and $y = r\sin(\theta)$. The map f the identity mapping that maps $\mathbb{R}^2$ in Cartesian coordinates to $\mathbb{R}^2$ in polar coordinates (consider the chart $U = \{(r, \theta) \;:\; r > 0,\ 0 < \theta < 2\pi\}$). As before we can compute the pullback of σ to $\mathbb{R}^2$ with polar coordinates:

$$\begin{aligned}
\sigma = dx \wedge dy &= d(r\cos(\theta)) \wedge d(r\sin(\theta)) \\
&= (\cos(\theta)dr - r\sin(\theta)d\theta) \wedge (\sin(\theta)dr + r\cos(\theta)d\theta) \\
&= r\cos^2(\theta)dr \wedge d\theta - r\sin^2(\theta)d\theta \wedge dr \\
&= rdr \wedge d\theta.
\end{aligned}$$

Of course the same can be obtained using (12). ▶

◀ **10.6** *Remark.* If we define

$$\Gamma(M) = \bigoplus_{r=0}^{\infty} \Gamma^r(M),$$

which is an associative, anti-commutative graded algebra, then $f^* : \Gamma(N) \to \Gamma(M)$ is a algebra homomorphism. ▶

11. Orientations

In order to explain orientations on manifolds we first start with orientations of finite-dimensional vector spaces. Let V be a real m-dimensional vector space. Two ordered basis $\{v_1, \cdots, v_m\}$ and $\{v_1', \cdots, v_m'\}$ are said to be ***consistently oriented*** if the transition matrix $A = (a_{ij})$, defined by the relation

$$v_i = a_{ij} v_j',$$

has positive determinant. This notion defines an equivalence relation on ordered bases, and there are exactly two equivalence classes. An ***orientation*** for V is a choice of an equivalence class of order bases. Given an ordered basis $\{v_1, \cdots, v_m\}$ the orientation is determined by the class $\mathcal{O} = [v_1, \cdots, v_m]$. The pair $(V, \mathcal{O})$ is called an ***oriented vector space***. For a given orientation any ordered basis that has the same orientation is called ***positively oriented***, and otherwise ***negatively oriented***.

Lemma 11.1. *Let* $0 \neq \theta \in \Lambda^m(V)$*, then the set of all ordered bases* $\{v_1, \cdots, v_m\}$ *for which* $\theta(v_1, \cdots, v_m) > 0$ *is an orientation for* V.

Proof: Let $\{v_i'\}$ be any ordered basis and $v_i = a_{ij}v_j'$. Then $\theta(v_1, \cdots, v_m) = \det(A)\theta(v_1', \cdots, v_m') > 0$, which proves that the set of bases $\{v_i\}$ for which it holds that $\theta(v_1, \cdots, v_m) > 0$, characterizes an orientation for V. ∎

Let us now describe orientations for smooth manifolds M. We will assume that $\dim M = m \geq 1$ here. For each point $p \in M$ we can choose an orientation $\mathcal{O}_p$ for the tangent space T_pM, making $(T_pM, \mathcal{O}_p)$ oriented vector spaces. The collection $\{\mathcal{O}_p\}_{p\in M\}}$ is called a pointwise orientation. Without any relation between these choices this concept is not very useful. The following definition relates the choices of orientations of T_pM, which leads to the concept of orientation of a smooth manifold.

Definition 11.2. A smooth m-dimensional manifold M with a pointwise orientation $\{\mathcal{O}_p\}_{p\in M}$, $\mathcal{O}_p = \big[X_1, \cdots, X_m\big]$, is (positively) ***oriented*** if for each point in M there exists a neighborhood U and a diffeomorphism φ, mapping from U to an open subset of $\mathbb{R}^m$, such that

$$\big[\varphi_*(X_1), \cdots, \varphi_*(X_m)\big]\Big|_p = [e_1, \cdots, e_m], \quad \forall p \in U, \tag{13}$$

where $[e_1, \cdots, e_m]$ is the standard orientation of $\mathbb{R}^m$. In this case the pointwise orientation $\{\mathcal{O}_p\}_{p\in M}$ is said to be consistently oriented. The choice of $\{\mathcal{O}_p\}_{p\in M}$ is called an ***orientation*** on M, denoted by $\mathcal{O} = \{\mathcal{O}_p\}_{p\in M}$.

If a consistent choice of orientations $\mathcal{O}_p$ does not exist we say that a manifold is ***non-orientable***.

◀ **11.3** *Remark.* If we look at a single chart (U, φ) we can choose orientations $\mathcal{O}_p$ that are consistently oriented for $p \in \mathcal{O}_p$. The choices $X_i = \varphi_*^{-1}(e_i) = \frac{\partial}{\partial x_i}|_p$ are ordered basis for T_pM. By definition $\varphi_*(X_i) = e_i$, and thus by this choice we obtain a consistent orientation for all $p \in U$. This procedure can be repeated for each chart in an atlas for M. In order to get a globally consistent ordering we need to worry about the overlaps between charts. ▶

◀ **11.4** *Example.* Let $M = S^1$ be the circle in $\mathbb{R}^2$, i.e. $M = \{p = (p_1, p_2) : p_1^2 + p_2^2 = 1\}$. The circle is an orientable manifold and we can find a orientation as follows. Consider the stereographic charts $U_1 = S^1 \backslash Np$ and $U_2 = S^1 \backslash Sp$, and the associated mappings

$$\varphi_1(p) = \frac{2p_1}{1-p_2}, \qquad \varphi_2(p) = \frac{2p_1}{1+p_2}$$

$$\varphi_1^{-1}(x) = \Big(\frac{4x}{x^2+4}, \frac{x^2-4}{x^2+4}\Big), \qquad \varphi_2^{-1}(x) = \Big(\frac{4x}{x^2+4}, \frac{4-x^2}{x^2+4}\Big).$$

Let us start with a choice of orientation and verify its validity. Choose $X(p) = (-p_2, p_1)$, then

$$(\varphi_1)_*(X) = J\varphi_1|_p(X) = \Big(\frac{2}{1-p_2} \ \frac{2p_1}{(1-p_2)^2}\Big)\begin{pmatrix} -p_2 \\ p_1 \end{pmatrix} = \frac{2}{1-p_2} > 0,$$

on U_1 (standard orientation). If we carry out the same calculation for U_2 we obtain $(\varphi_2)_*(X) = J\varphi_2|_p(X) = \frac{-2}{1+p_2} < 0$ on U_2, with corresponds to the opposite orientation. Instead by choosing $\tilde{\varphi}_2(p) = \varphi_2(-p_1, p_2)$ we obtain

$$(\tilde{\varphi}_2)_*(X) = J\tilde{\varphi}_2|_p(X) = \frac{2}{1+p_2} > 0,$$

which chows that $X(p)$ is defines an orientation $\mathcal{O}$ on S^1. The choice of vectors $X(p)$ does not have to depend continuously on p in order to satisfy the definition of orientation, as we will explain now.

For $p \in U_1$ choose the canonical vectors $X(p) = \frac{\partial}{\partial x}|_p$, i.e.

$$\frac{\partial}{\partial x}\Big|_p = J\varphi_1^{-1}(1) = \begin{pmatrix} \frac{16-4x^2}{(x^2+4)^2} \\ \frac{16x}{(x^2+4)^2} \end{pmatrix}.$$

In terms of p this gives $X(p) = \frac{1}{2}(1-p_2)\begin{pmatrix} -p_2 \\ p_1 \end{pmatrix}$. By definition $(\varphi_1)_*(X) = 1$ for $p \in U_1$. For $p = Np$ we choose $X(p) = (-1 \quad 0)^t$, so $X(p)$ is defined for $p \in S^1$ and is not a continuous function of p! It remains to verify that $(\tilde{\varphi}_2)_*(X) > 0$ for some neighborhood of Np. First, at $p = Np$,

$$J\tilde{\varphi}_2|_p \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \frac{2}{1+p_2} = 1 > 0.$$

The choice of $X(p)$ at $p = Np$ comes from the canonical choice $\frac{\partial}{\partial x}\Big|_p$ with respect to $\tilde{\varphi}_2$. Secondly,

$$J\tilde{\varphi}_2|_p(X(p)) = \frac{1}{2}(1-p_2)\Big(\frac{-2}{1+p_2} \ \frac{2p_1}{(1+p_2)^2}\Big)\begin{pmatrix} -p_2 \\ p_1 \end{pmatrix} = \frac{1-p_2}{1+p_2},$$

for all $p \in U_2 \backslash Np$. Note that $\tilde{\varphi}_2 \circ \varphi_1^{-1}(x) = \frac{-4}{x}$, and

$$J\tilde{\varphi}_2|_p(X(p)) = J\tilde{\varphi}_2|_p(J\varphi_1^{-1}|_p(1)) = J(\tilde{\varphi}_2 \circ \varphi_1^{-1})|_p(1) = \frac{4}{x^2},$$

which shows that consistency at overlapping charts can be achieved if the transition mappings have positive determinant. Basically, the above formula describes the change of basis as was carried out in the case of vector spaces. ▶

What the above example shows us is that if we choose $X(p) = \frac{\partial}{\partial x_i}|_p$ for all charts, it remains to be verified if we have the proper collection of charts, i.e. does $\big[\{J\varphi_\beta \circ J\varphi_\alpha^{-1}(e_i)\}\big] = [\{e_i\}]$ hold? This is equivalent to having $\det(J\varphi_\beta \circ J\varphi_\alpha^{-1}) > 0$. These observations lead to the following theorem.

Theorem 11.5. *A smooth manifold M is orientable if and only if there exists an atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha\}$ such that for any $p \in M$*

$$\det(J\varphi_{\alpha\beta}|_x) > 0, \qquad \varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1},$$

for any pair α, β for which $x = \varphi_\alpha(p)$, and $p \in U_\alpha \cap U_\beta$.

Proof: Obvious from the previous example and Lemma 11.1. ■

As we have seen for vector spaces m-forms can be used to define an orientation on a vector space. This concept can also be used for orientations on manifolds.

Theorem 11.6. [30] *Let M be a smooth m-dimensional manifold. A nowhere vanishing differential m-form $\theta \in \Gamma^m(M)$ determines a unique orientation $\mathcal{O}$ on M for which θ_p is positively orientated at each T_pM. Conversely, given an orientation $\mathcal{O}$ on M, then there exists a nowhere vanishing m-form $\theta \in \Gamma^m(M)$ that is positively oriented at each T_pM.*

Proof: Let $\theta \in \Gamma^m(M)$ be a nowhere vanishing m-form, then in local coordinates (U, φ), $\theta = f dx^1 \wedge \cdots \wedge dx^m$. For the canonical bases for T_pM we have

$$\theta\Big(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_m}\Big) = f \neq 0.$$

Without loss of generality can be assumed to be positive, otherwise change φ by means of $x_1 \to -x_1$. Now let $\{\frac{\partial}{\partial y_i}\}$ be canonical bases for T_pM with respect to a chart (V, ψ). As before $\theta = g dy^1 \wedge \cdots \wedge dy^m$, $g > 0$ on V. Assume that $U \cap V \neq \varnothing$, then at the overlap we have

$$\begin{aligned} 0 < g &= \theta\Big(\frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_m}\Big) = \det\Big(J(\varphi \circ \psi^{-1})\Big)\theta\Big(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_m}\Big) \\ &= f \det\Big(J(\varphi \circ \psi^{-1})\Big), \end{aligned}$$

which shows that $\det\Big(J(\varphi \circ \psi^{-1})\Big) > 0$, and thus M is orientable.

For the converse we construct non-vanishing m-forms on each chart (U, φ). Via a partition of unity we can construct a smooth m-form on M, see Lee. ■

[30] See Lee, Prop. 13.4.

This theorem implies in particular that non-orientable m-dimensional manifolds do not admit a nowhere vanishing m-form, or volume form.

◀ **11.7** *Example.* Consider the Möbius strip M. The Möbius strip is an example of a non-orientable manifold. Let us parametrize the Möbius strip as an embedded manifold in $\mathbb{R}^3$:

$$g : \mathbb{R} \times (-1,1) \to \mathbb{R}^3, \quad g(\theta, r) = \begin{pmatrix} r\sin(\theta/2)\cos(\theta) + \cos(\theta) \\ r\sin(\theta/2)\sin(\theta) + \sin(\theta) \\ r\cos(\theta/2) \end{pmatrix},$$

where g is a smooth embedding when regarded as a mapping from $\mathbb{R}/2\pi\mathbb{Z} \times (-1,1) \to \mathbb{R}^3$. Let us assume that the Möbius strip M is oriented, then by the above theorem there exists a nonwhere vanishing 2-form σ which can be given as follows

$$\sigma = a(x,y,z)dy \wedge dz + b(x,y,z)dz \wedge dx + c(x,y,z)dx \wedge dy,$$

where $(x,y,z) = g(\theta, r)$. Since g is a smooth embedding (parametrization) the pullback form $g^*\sigma = \rho(\theta, r)d\theta \wedge dr$ is a nowhere vanishing 2-form on $\mathbb{R}/2\pi\mathbb{Z} \times (-1,1)$. In particular, this means that ρ is 2π-periodic in θ. Notice that $\sigma = i_X dx \wedge dy \wedge dz$, where

$$X = a(x,y,z)\frac{\partial}{\partial x} + b(x,y,z)\frac{\partial}{\partial y} + c(x,y,z)\frac{\partial}{\partial z}.$$

For vector fields $\xi = \xi_1 \frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial y} + \xi_3 \frac{\partial}{\partial z}$ and $\eta = \eta_1 \frac{\partial}{\partial x} + \eta_2 \frac{\partial}{\partial y} + \eta_3 \frac{\partial}{\partial z}$ we then have that

$$\sigma(\xi, \eta) = i_X dx \wedge dy \wedge dz(\xi, \eta) = X \cdot (\xi \times \eta),$$

where $\xi \times \eta$ is the cross product of ξ and η. The condition that θ is nowhere vanishing can be translated to $\mathbb{R}^2$ as follows:

$$g^*\sigma(e_1, e_2) = \sigma(g_*(e_1), g_*(e_2)).$$

Since $g_*(e_1) \times g_*(e_2)|_{\theta=0} = -g_*(e_1) \times g_*(e_2)|_{\theta=2\pi}$, the pullback $\mathrm{g}^*\sigma$ form cannot be nowhere vanishing on $\mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}$, which is a contradiction. This proves that the Möbius band is a non-orientable manifold. ▶

Let N, M be oriented manifolds of the same dimension, and let $f : N \to M$ be a smooth mapping. A mapping is called ***orientation preserving*** at a point $p \in N$ if f_* maps positively oriented bases of T_pN to positively oriented bases in $T_{f(p)}M$. A mapping is called ***orientation reversing*** at $p \in N$ if f_* maps positively oriented bases of T_pN to negatively oriented bases in $T_{f(p)}M$.

Let us now look at manifolds with boundary; $(M, \partial M)$. For a point $p \in \partial M$ we distinguish three types of tangent vectors:

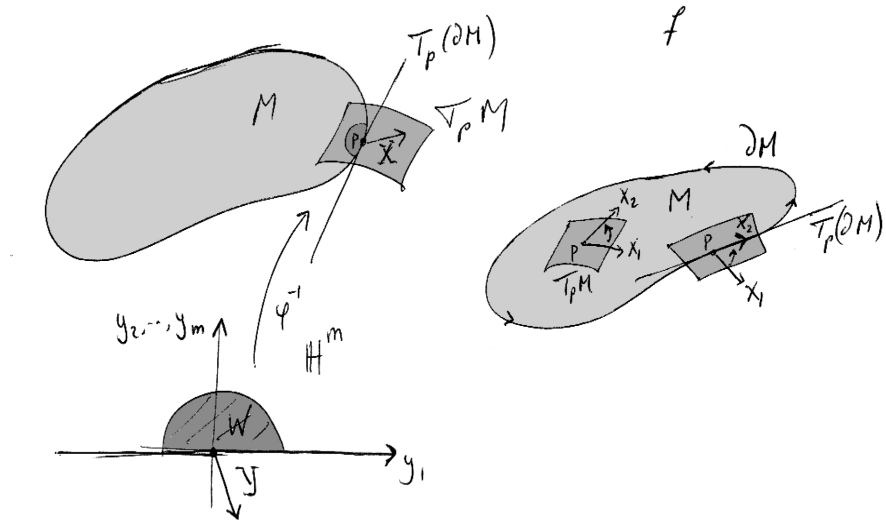

FIGURE 32. Out(in)ward vectors and the induced orientation to ∂M.

(i) tangent boundary vectors $X \in T_p(\partial M) \subset T_pM$, which form an $(m-1)$-dimensional subspace of T_pM;
(ii) outward vectors; let $\varphi^{-1} : W \subset \mathbb{H}^m \to M$, then $X \in T_pM$ is called an outward vector if $\varphi_*^{-1}(Y) = X$, for some $Y = (y_1, \cdots, y_m)$ with $y_1 < 0$;
(iii) inward vectors; let $\varphi^{-1} : W \subset \mathbb{H}^m \to M$, then $X \in T_pM$ is called an inward vector if $\varphi_*^{-1}(Y) = X$, for some $Y = (y_1, \cdots, y_m)$ with $y_1 > 0$.

Using this concept we can now introduce the notion of induced orientation on ∂M. Let $p \in \partial M$ and choose a basis $\{X_1, \cdots, X_m\}$ for T_pM such that $[X_1, \cdots, X_m] = \mathcal{O}_p$, $\{X_2, \cdots, X_m\}$ are tangent boundary vectors, and X_1 is an outward vector. In this case $[X_2, \cdots, X_m] = (\partial\mathcal{O})_p$ determines an orientation for $T_p(\partial M)$, which is consistent, and therefore $\partial\mathcal{O} = \{(\partial\mathcal{O})_p\}_{p\in\partial M}$ is an orientation on ∂M induced by $\mathcal{O}$. Thus for an oriented manifold M, with orientation $\mathcal{O}$, ∂M has an orientation $\partial\mathcal{O}$, called the ***induced orientation on*** ∂M.

◀ **11.8** *Example.* Any open set $M \subset \mathbb{R}^m$ (or $\mathbb{H}^m$) is an orientable manifold. ▶

◀ **11.9** *Example.* Consider a smooth embedded co-dimension 1 manifold

$$M = \{p \in R^{m+1} \ : \ f(p) = 0\}, \qquad f : \mathbb{R}^{m+1} \to \mathbb{R}, \quad \mathrm{rk}(f)|_p, \ p \in M.$$

Then M is an orientable manifold. Indeed, $M = \partial N$, where $N = \{p \in \mathbb{R}^{m+1} \ : \ f(p) > 0\}$, which is an open set in $\mathbb{R}^{m+1}$ and thus an oriented manifold. Since $M = \partial N$ the manifold M inherits an orientation from M and hence it is orientable. ▶

IV. Integration on manifolds

12. Integrating m-forms on $\mathbb{R}^m$

We start off integration of m-forms by considering m-forms on $\mathbb{R}^m$.

Definition 12.1. A subset $D \subset \mathbb{R}^m$ is called a domain of integration if

(i) D is bounded, and
(ii) ∂D has m-dimensional Lebesgue measure $d\mu = dx_1 \cdots dx_m$ equal to zero.

In particular any finite union or intersection of open or closed rectangles is a domain of integration. Any bounded[31] continuous function f on D is integrable, i.e.

$$-\infty < \int_D f dx_1 \cdots dx_m < \infty.$$

Since $\Lambda^m(\mathbb{R}^m) \cong \mathbb{R}$, a smooth m-form on $\mathbb{R}^m$ is given by

$$\omega = f(x_1, \cdots, x_m) dx^1 \wedge \cdots \wedge dx^m,$$

where $f : \mathbb{R}^m \to \mathbb{R}$ is a smooth function. For a given (bounded) domain of integration D we define

$$\begin{aligned} \int_D \omega &:= \int_D f(x_1, \cdots, x_m) dx_1 \cdots dx_m = \int_D f d\mu \\ &= \int_D \omega_x(e_1, \cdots, e_m) d\mu. \end{aligned}$$

An m-form ω is compactly supported if $\mathrm{supp}(\omega) = \mathrm{cl}\{x \in \mathbb{R}^m \ : \ \omega(x) \neq 0\}$ is a compact set. The set of compactly supported m-forms on $\mathbb{R}^m$ is denoted by $\Gamma^m_c(\mathbb{R}^m)$, and is a linear subspace of $\Gamma^m(\mathbb{R}^m)$. Similarly, for any open set $U \subset \mathbb{R}^m$ we can define $\omega \in \Gamma^m_c(U)$. Clearly, $\Gamma^m_c(U) \subset \Gamma^m_c(\mathbb{R}^m)$, and can be viewed as a linear subspace via zero extension to $\mathbb{R}^m$. For any open set $U \subset \mathbb{R}^m$ there exists a domain of integration D such that $U \supset D \supset \mathrm{supp}(\omega)$ (see Exercises).

[31] Boundedness is needed because the rectangles are allowed to be open.

Definition 12.2. Let $U \subset \mathbb{R}^m$ be open and $\omega \in \Gamma_c^m(U)$, and let D be a domain of integration D such that $U \supset D \supset \mathrm{supp}(\omega)$. We define the integral

$$\int_U \omega := \int_D \omega.$$

If $U \subset \mathbb{H}^m$ open, then

$$\int_U \omega := \int_{D\cap\mathbb{H}^m} \omega.$$

The next theorem is the first step towards defining integrals on m-dimensional manifolds M.

Theorem 12.3. *Let $U,V \subset \mathbb{R}^m$ be open sets, $f : U \to V$ an orientation preserving diffeomorphism, and let $\omega \in \Gamma_c^m(V)$. Then,*

$$\int_V \omega = \int_U f^*\omega.$$

If f is orientation reversing, then $\int_U \omega = -\int_V f^*\omega$.

Proof: Assume that f is an orientation preserving diffeomorphism from U to V. Let E be a domain a domain of integration for ω, then $D = f^{-1}(E)$ is a domain of integration for $f^*\omega$. We now prove the theorem for the domains D and E. We use coordinates $\{x_i\}$ and $\{y_i\}$ on D and E respectively. We start with $\omega = g(y_1,\cdots,y_m)dy^1\wedge\cdots\wedge dy^m$. Using the change of variables formula for integrals and the pullback formula in (11) we obtain

$$\begin{aligned}
\int_E \omega &= \int_E g(y)dy_1\cdots dy_m \text{ (Definition)}\\
&= \int_D (g\circ f)(x)\det(J\tilde{f}|_x)dx^1\cdots dx^m\\
&= \int_D (g\circ f)(x)\det(J\tilde{f}|_x)dx^1\wedge\cdots\wedge dx^m\\
&= \int_D f^*\omega \text{ (Definition)}.
\end{aligned}$$

One has to introduce a $-$ sign in the orientation reversing case. ■

13. Partitions of unity

We start with introduce the notion partition of unity for smooth manifolds. We should point out that this definition can be used for arbitrary topological spaces.

Definition 13.1. Let M be smooth m-manifold with atlas $\mathcal{A} = \{(\varphi_i, U_i)\}_{i\in I}$. A ***partition of unity subordinate to*** $\mathcal{A}$ is a collection for smooth functions $\{\lambda_i : M \to \mathbb{R}\}_{i\in I}$ satisfying the following properties:

(i) $0 \le \lambda_i(p) \le 1$ for all $p \in M$ and for all $i \in I$;
(ii) $\mathrm{supp}(\lambda_i) \subset U_i$;
(iii) the set of supports $\{\mathrm{supp}(\lambda_i)\}_{i\in I}$ is locally finite;
(iv) $\sum_{i\in I} \lambda_i(p) = 1$ for all $p \in M$.

Condition (iii) says that every $p \in M$ has a neighborhood $U \ni p$ such that only finitely many λ_i's are nonzero in U. As a consequence of this the sum in Condition (iv) is always finite.

Theorem 13.2. *For any smooth m-dimensional manifold M with atlas $\mathcal{A}$ there exists a partition of unity $\{\lambda_i\}$ subordinate to $\mathcal{A}$.*

In order to prove this theorem we start with a series of auxiliary results and notions.

Lemma 13.3. *There exists a smooth function $h : \mathbb{R}^m \to \mathbb{R}$ such that $0 \le h \le 1$ on $\mathbb{R}^m$, and $h|_{B_1(0)} \equiv 1$ and $\mathrm{supp}(h) \subset B_2(0)$.*

Proof: Define the function $f_1 : \mathbb{R} \to \mathbb{R}$ as follows

$$f_1(t) = \begin{cases} e^{-1/t}, & t > 0, \\ 0, & t \le 0. \end{cases}$$

One can easily prove that f_1 is a C^∞-function on $\mathbb{R}$. If we set

$$f_2(t) = \frac{f_1(2-t)}{f_1(2-t) + f_1(t-1)}.$$

This function has the property that $f_2(t) \equiv 1$ for $t \le 1$, $0 < f_2(t) < 1$ for $1 < t < 2$, and $\mathrm{f}_2(t) \equiv 0$ for $t \ge 2$. Moreover, f_2 is smooth. To construct f we simply write $f(x) = f_2(|x|)$ for $x \in \mathbb{R}^m \backslash \{0\}$. Clearly, f is smooth on $\mathbb{R}^m \backslash \{0\}$, and since $f|_{B_1(0)} \equiv 1$ it follows that f is smooth on $\mathbb{R}^m$. ∎

An atlas $\mathcal{A}$ gives an open covering for M. The set U_i in the atlas need not be compact, nor locally finite. We say that a covering $\mathcal{U} = \{U_i\}$ of M is ***locally finite*** if every point $p \in M$ has a neighborhood that intersects only finitely $U_i \in \mathcal{U}$. If there exists another covering $\mathcal{V} = \{V_j\}$ such that every $V_j \in \mathcal{V}$ is contained in some $V_j \subset U_i \in \mathcal{U}$, then $\mathcal{V}$ is called a ***refinement of*** $\mathcal{U}$. A topological space X for which each open covering admits a locally finite refinement is called ***paracompact***.

Lemma 13.6. *Any topological manifold M allows a countable, locally finite covering by precompact open sets.*

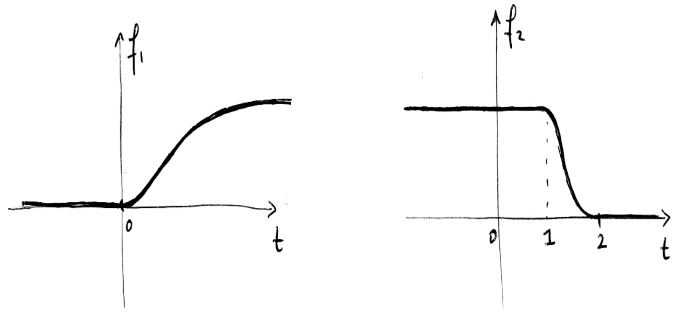

FIGURE 33. The functions f_1 and f_2.

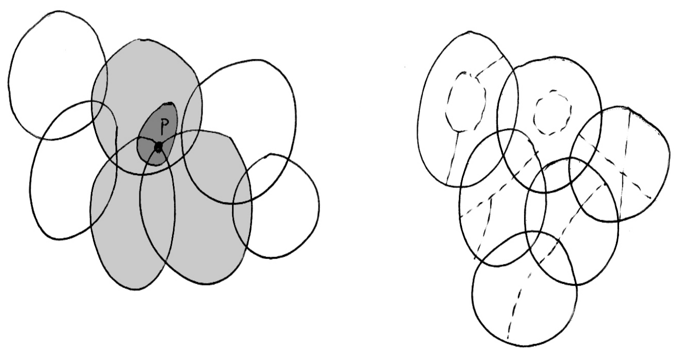

FIGURE 34. A locally finite covering [left], and a refinement [right].

Proof: We start with a countable covering of open balls $\mathcal{B} = \{B_i\}$. We now construct a covering $\mathcal{U}$ that satisfies

(i) all sets $U_i \in \mathcal{U}$ are precompact open sets in M;
(ii) $\overline{U_{i-1}} \subset U_i$, $i > 1$;
(iii) $B_i \subset U_i$.

We build the covering $\mathcal{U}$ from $\mathcal{B}$ using an inductive process. Let $U_1 = B_1$, and assume $U_1, \cdots, U_k$ have been constructed satisfying (i)-(iii). Then

$$\overline{U_k} \subset B_{i_1} \cup \cdots \cup B_{i_k},$$

where $B_{i_1} = B_1$. Now set

$$U_{k+1} = B_{i_1} \cup \cdots \cup B_{i_k}.$$

Choose i_k large enough so that $i_k \geq k+1$ and thus $B_{k+1} \subset U_{k+1}$.

From the covering $\mathcal{U}$ we can now construct a locally finite covering $\mathcal{V}$ by setting $V_i = U_i \backslash \overline{U_{i-2}}$, $i > 1$. ∎

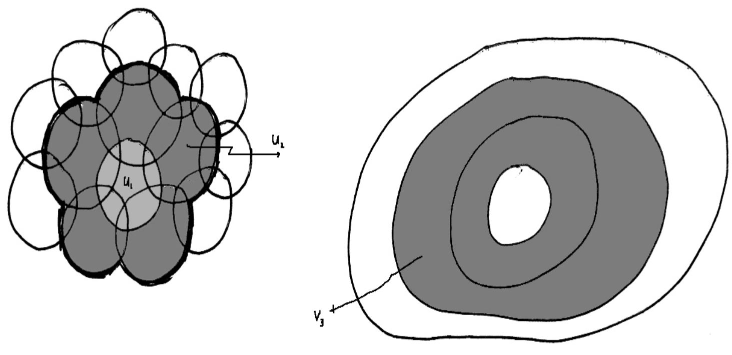

FIGURE 35. Constructing a nested covering $\mathcal{U}$ from a covering with balls $\mathcal{B}$ [left], and a locally finite covering obtained from the previous covering [right].

Next we seek a special locally finite refinement $\{W_j\} = \mathcal{W}$ of $\mathcal{V}$ which has special properties with respect to coordinate charts of M;

(i) $\mathcal{W}$ is a countable and locally finite;
(ii) each $W_j \subset \mathcal{W}$ is in the domain of some smooth coordinate map $\varphi_j :_j \to \mathbb{R}^m$ for M;
(iii) the collection $\mathcal{Z} = \{Z_j\}$, $Z_j = \varphi_j^{-1}(B_1(0))$ covers M.

The covering $\mathcal{W}$ is called ***regular***.

Lemma 13.8. *For any open covering $\mathcal{U}$ for a smooth manifold M there exists a regular refinement. In particular M is paracompact.*

Proof: Let $\mathcal{V}$ be a countable, locally finite covering as described in the previous lemma. Since $\mathcal{V}$ is locally finite we can find a neighborhood W_p for each $p \in M$ that intersects only finitely many set $V_j \in \mathcal{V}$. We want to choose W_p in a smart way. First we replace W_p by $W_p \cap \{V_j \ : \ p \in V_j\}$. Since $p \in U_i$ for some i we then replace W_p by $W_p \cap V_i$. Finally, we replace W_p by a small coordinate ball $B_r(p)$ so that W_p is in the domain of some coordinate map φ_p. This provides coordinate charts (W_p, φ_p). Now define $Z_p = \varphi_p^{-1}(B_1(0))$.

For every k, $\{Z_p \ : \ p \in \overline{V_k}\}$ is an open covering of $\overline{V_k}$, which has a finite subcovering, say $Z_k^1, \cdots, Z_k^{m_k}$. The sets Z_k^i are sets of the form Z_{p_i} for some $p_i \in \overline{V_k}$. The associated coordinate charts are $(W_k^1, \varphi_k^1), \cdots, (W_k^{m_k}, \varphi_k^{m_k})$. Each W_k^i is obtained via the construction above for $p_i \in \overline{V_k}$. Clearly, $\{W_k^i\}$ is a countable open covering of M that refines $\mathcal{U}$ and which, by construction, satisfies (ii) and (iii) in the definition of regular refinement. It is clear from compactness that $\{W_k^i\}$ is locally finite. Let

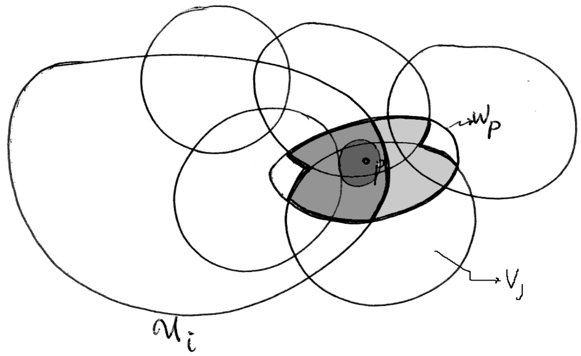

FIGURE 36. The different stages of constructing the sets W_p colored with different shades of grey.

us relabel the sets $\{W_k^i\}$ and denote this covering by $\mathcal{W} = \{W_i\}$. As a consequence M is paracompact. ■

Proof of Theorem 13.2: From Lemma 13.8 there exists a regular refinement $\mathcal{W}$ for $\mathcal{A}$. By construction we have $Z_i = \varphi_i^{-1}(B_1(0))$, and we define

$$\widehat{Z}_i := \varphi_i^{-1}(B_2(0)).$$

Using Lemma 13.3 we now define the functions $\mu_{:}M \to \mathbb{R}$;

$$\mu_i = \begin{cases} f_2 \circ \varphi_i & \text{on } W_i \\ 0 & \text{on } M \backslash \overline{V}_i. \end{cases}$$

These functions are smooth and $\mathrm{supp}(\mu_i) \subset W_i$. The quotients

$$\widehat{\lambda} = \frac{\mu_j(p)}{\sum_i \mu_i(p)},$$

are, due to the local finiteness of W_i and the fact that the denominator is positive for each $p \in M$, smooth functions on M. Moreover, $0 \le \widehat{\lambda}_j \le 1$, and $\sum_j \widehat{\lambda}_j \equiv 1$.

Since $\mathcal{W}$ is a refinement of $\mathcal{A}$ we can choose an index k_j such that $W_j \subset U_{k_j}$. Therefore we can group together some of the function $\widehat{\lambda}_j$:

$$\lambda_i = \sum_{j\,:\,k_j = i} \widehat{\lambda}_j,$$

which give us the desired partition functions with $0 \le \lambda_i \le 1$, $\sum_i \lambda_i \equiv 1$, and $\mathrm{supp}(\lambda_i) \subset U_i$. ■

Some interesting byproducts of the above theorem using partitions of unity are. In all these case M is assumed to be an smooth m-dimensional manifold.

Theorem 13.10. [32] *For any close subset $A \subset M$ and any open set $U \supset A$, there exists a smooth function $f : M \to \mathbb{R}$ such that*

(i) $0 \le f \le 1$ *on* M;
(ii) $f^{-1}(1) = A$;
(iii) $\mathrm{supp}(f) \subset U$.

Such a function f is called a ***bump function for*** A ***supported in*** U.

By considering functions $g = c(1-f) \ge 0$ we obtain smooth functions for which $g^{-1}(0)$ can be an arbitrary closed subset of M.

Theorem 13.11. [33] *Let $A \subset M$ be a closed subset, and $f : A \to \mathbb{R}^k$ be a smooth mapping.*[34] *Then for any open subset $U \subset M$ containing A there exists a smooth mapping $f^\dagger : M \to \mathbb{R}^k$ such that $f^\dagger|_A = f$, and* $\mathrm{supp}(f^\dagger) \subset U$.

14. Integration on of m-forms on m-dimensional manifolds.

In order to introduce the integral of an m-form on M we start with the case of forms supported in a single chart. In what follows we assume that M is an oriented manifold with an oriented atlas $\mathcal{A}$, i.e. a consistent choice of smooth charts such that the transitions mappings are orientation preserving.

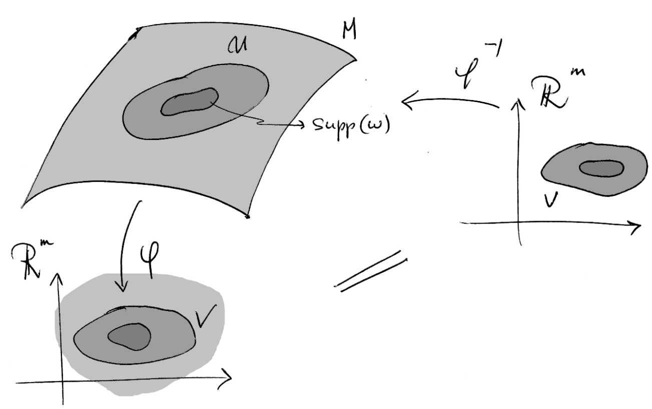

FIGURE 37. An m-form supported in a single chart.

[32] See Lee, Prop. 2.26.

[33] See Lee, Prop. 2.26.

[34] A mapping from a closed subset $A \subset M$ is said to be smooth if for every point $p \in A$ there exists an open neighborhood $W \subset M$ of p, and a smooth mapping $f^\dagger : W \to \mathbb{R}^k$ such that $f^\dagger|_{W \cap A} = f$.

Let $\omega \in \Gamma_c^m(M)$, with $\mathrm{supp}(\omega) \subset U$ for some chart U in $\mathcal{A}$. Now define the integral of ω over M as follows:

$$\int_M \omega := \int_{\varphi(U)} (\varphi^{-1})^* \omega,$$

where the pullback form $(\varphi^{-1})^*\omega$ is compactly supported in $V = \varphi(U)$ (φ is a diffeomorphism). The integral over V of the pullback form $(\varphi^{-1})^*\omega$ is defined in Definition 12.2. It remains to show that the integral $\int_M \omega$ does not depend on the particular chart. Consider a different chart U', possibly in a different oriented atlas

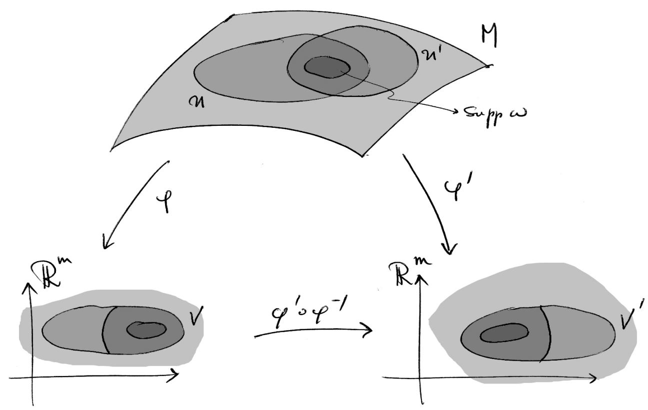

FIGURE 38. Different charts U and U' containing $\mathrm{supp}(\omega)$, i.e. $\mathrm{supp}(\omega) \subset U \cap U'$.

$\mathcal{A}'$ for M (same orientation), then

$$\begin{aligned}
\int_{V'} (\varphi'^{-1})^*\omega &= \int_{\varphi'(U\cap U')} (\varphi'^{-1})^*\omega = \int_{\varphi(U\cap U')} (\varphi' \circ \varphi^{-1})^* (\varphi'^{-1})^*\omega \\
&= \int_{\varphi(U\cap U')} (\varphi^{-1})^* (\varphi')^* (\varphi'^{-1})^*\omega = \int_{\varphi(U\cap U')} (\varphi^{-1})^*\omega \\
&= \int_V (\varphi^{-1})^*\omega,
\end{aligned}$$

which show that the definition is independent of the chosen chart. We crucially do use the fact that $\mathcal{A} \cup \mathcal{A}'$ is an oriented atlas for M. To be more precise the mappings $\varphi' \circ \varphi^{-1}$ are orientation preserving.

By choosing a partition of unity as described in the previous section we can now define the integral over M of arbitrary m-forms $\omega \in \Gamma_c^m(M)$.

Definition 14.3. *Let* $\omega \in \Gamma_c^m(M)$ *and let* $\mathcal{A}_I = \{(U_i, \varphi_i\}_{i\in I} \subset \mathcal{A}$ *be a finite subcovering of* $\mathrm{supp}(\omega)$ *coming from an oriented atlas* $\mathcal{A}$ *for* M*. Let* $\{\lambda_i\}$ *be a partition of unity subordinate to* $\mathcal{A}_I$*. Then the* ***integral of*** ω ***over*** M *is defined as*

$$\int_M \omega := \sum_i \int_M \lambda_i \omega,$$

where the integrals $\int_M \lambda_i\omega$ *are integrals of form that have support in single charts as defined above.*

We need to show now that the integral is indeed well-defined, i.e. the sum is finite and independent of the atlas and partition of unity. Since $\mathrm{supp}(\omega)$ is compact $\mathcal{A}_I$ exists by default, and thus the sum is finite.

Lemma 14.4. *The above definition is independent of the chosen partition of unity and covering* $\mathcal{A}_I$*.*

Proof: Let $\mathcal{A}'_J \subset \mathcal{A}'$ be another finite covering of $\mathrm{supp}(\omega)$, where $\mathcal{A}'$ is a com-

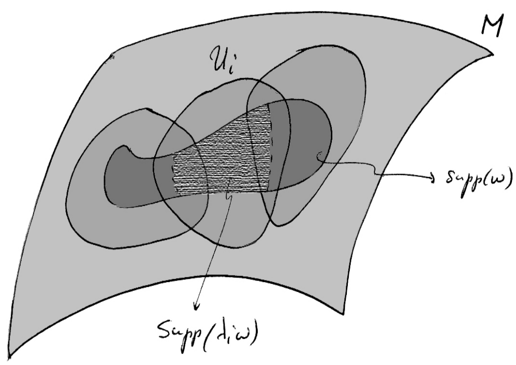

FIGURE 39. Using a partition of unity we can construct m-forms which are all supported in one, but possibly different charts U_i.

patible oriented atlas for M, i.e. $\mathcal{A}\cup\mathcal{A}'$ is a oriented atlas. Let $\{\lambda'_j\}$ be a partition of unity subordinate to $\mathcal{A}'_J$. We have

$$\int_M \lambda_i\omega = \int_M \Big(\sum_j \lambda'_j\Big)\lambda_i\omega = \sum_j \int_M \lambda'_j\lambda_i\omega.$$

By summing over i we obtain $\sum_i \int_M \lambda_i\omega = \sum_{i,j}\int_M \lambda'_j\lambda_i\omega$. Each term $\lambda'_j\lambda_i\omega$ is supported in some U_i and by previous independent of the coordinate mappings. Similarly, if we interchange the i's and j's, we obtain that $\sum_j \int_M \lambda'_j\omega = \sum_{i,j}\int_M \lambda'_j\lambda_i\omega$, which proves the lemma. ∎

◀ **14.6** *Remark.* If M is a compact manifold that for any $\omega \in \Gamma^m(M)$ it holds that $\mathrm{supp}(\omega) \subset M$ is a compact set, and therefore $\Gamma^m_c(M) = \Gamma^m(M)$ in the case of compact manifolds. ▶

So far we considered integral of m-forms over M. One can of course integrate n-forms on M over n-dimensional immersed or embedded submanifolds $N \subset M$. Given $\omega \in \Gamma^n(M)$ for which the restriction to N, $\omega|_N$ has compact support in N, then

$$\int_N \omega := \int_N \omega|_N.$$

As a matter of fact we have $i : N \to M$, and $\omega|_N = i^*\omega$, so that $\int_N \omega := \int_N \omega|_N = \int_N i^*\omega$. If M is compact with boundary ∂M, then $\int_{\partial M} \omega$ is well-defined for any $(m-1)$-form $\omega \in \Gamma^{m-1}(M)$.

Theorem 14.7. *Let $\omega \in \Gamma^m_c(M)$, and $f : N \to M$ is a diffeomorphism. Then*

$$\int_M \omega = \int_N f^*\omega.$$

Proof: By the definition of integral above we need to prove the above statement for the terms

$$\int_M \lambda_i \omega = \int_N f^*\lambda_i \omega.$$

Therefore it suffices to prove the theorem for forms ω whose support is in a single chart U.

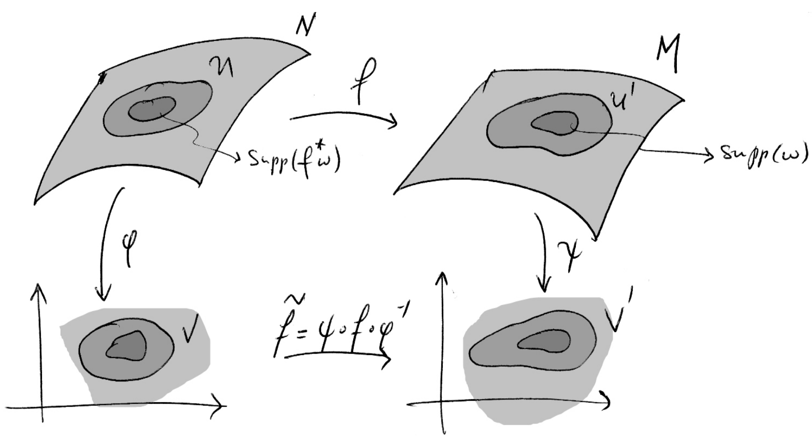

FIGURE 40. The pullback of an m-form.

$$\begin{aligned}\int_U f^*\omega &= \int_{\varphi(U)} (\varphi^{-1})^* f^*\omega = \int_{\varphi(U)} (\varphi^{-1})^* f^*\psi^*(\psi^{-1})^*\omega \\ &= \int_{\psi(U')} (\psi^{-1})^*\omega = \int_{U'} \omega.\end{aligned}$$

By applying this to $\lambda_i\omega$ and summing we obtain the desired result. ■

For sake of completeness we summarize the most important properties of the integral.

Theorem 14.9. [35] *Let N,M be oriented manifolds (with or without boundary) of dimension m , and $\omega,\eta \in \Gamma^m_c(M)$, are m-forms. Then*

(i) $\int_M a\omega + b\eta = a\int_M \omega + b\int_M \eta$;

(ii) *if $-\mathcal{O}$ is the opposite orientation to $\mathcal{O}$, then*

$$\int_{M,-\mathcal{O}} \omega = -\int_{M,\mathcal{O}} \omega;$$

(iii) *if ω is an orientation form, then $\int_M \omega > 0$;*

(iv) *if $f : N \to M$ is a diffeomorphism, then*

$$\int_M \omega = \int_N f^*\omega.$$

For practical situations the definition of the integral above is not convenient since constructing partitions of unity is hard in practice. If the support of a differential form ω can be parametrized be appropriate parametrizations, then the integral can be easily computed from this. Let $D_i \subset \mathbb{R}^m$ — finite set of indices i — be compact domains of integration, and $g_i : D_i \to M$ are smooth mappings satisfying

(i) $E_i = g_i(D_i)$, and $g_i : \mathrm{int}\,(D_i) \to \mathrm{int}\,(E_i)$ are orientation preserving diffeomorphisms;

(ii) $E_i \cap E_j$ intersect only along their boundaries, for all $i \neq j$.

Theorem 14.10. [36] *Let $\{(g_i,D_i)\}$ be a finite set of parametrizations as defined above. Then for any $\omega \subset \Gamma^m_c(M)$ such that $\mathrm{supp}(\omega) \subset \cup_i E_i$ it holds that*

$$\int_M \omega = \sum_i \int_{D_i} g_i^*\omega.$$

Proof: As before it suffices the prove the above theorem for a single chart U, i.e. $\mathrm{supp}(\omega) \subset U$. One can choose U to have a boundary ∂U so that $\varphi(\partial U)$ has measure zero, and φ maps $\mathrm{cl}(U)$ to a compact domain of integration $K \subset \mathbb{H}^m$. Now set

$$A_i = \mathrm{cl}(U) \cap E_i, \quad B_i = g_i^{-1}(A_i), \quad C_i = \varphi_i(A_i).$$

We have

$$\int_{C_i} (\varphi^{-1})^*\omega = \int_{B_i} (\varphi \circ g_i)^*(\varphi^{-1})^*\omega = \int_{B_i} g_i^*\omega.$$

[35] See Lee Prop. 14.6.

[36] See Lee Prop. 14.7.

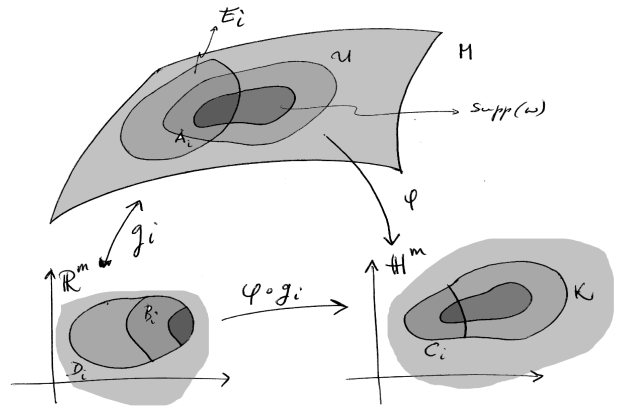

FIGURE 41. Carving up supp(ω) via domains of integration for parametrizations g_i for M.

Since the interiors of the sets C_i (and thus A_i) are disjoint it holds that

$$\begin{aligned}\int_M \omega &= \int_K (\varphi^{-1})^*\omega = \sum_i \int_{C_i} (\varphi^{-1})^*\omega \\ &= \sum_i \int_{B_i} g_i^*\omega = \sum_i \int_{D_i} g_i^*\omega,\end{aligned}$$

which proves the theorem. ■

◀ **14.12** *Remark.* From the previous considerations computing $\int_M \omega$ boils down to computing $(\varphi^{-1})^*\omega$, or $g^*\omega$ for appropriate parametrizations, and summing the various contrubutions. Recall from Section 12 that in order to integrate one needs to evaluate $(\varphi^{-1})^*\omega_x(e_1,\cdots,e_m)$, which is given by the formula

$$\big((\varphi^{-1})^*\omega\big)_x(e_1,\cdots,e_m) = \omega_{\varphi^{-1}(x)}(\varphi_*^{-1}(e_1),\cdots,\varphi_*^{-1}(e_m)).$$

For a single chart contribution this then yields the formula

$$\begin{aligned}\int_U \omega &= \int_{\varphi(U)} \omega_{\varphi^{-1}(x)}(\varphi_*^{-1}(e_1),\cdots,\varphi_*^{-1}(e_m))d\mu \\ &= \int_{\varphi(U)} \omega_{\varphi^{-1}(x)}\Big(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_1}\Big)d\mu\end{aligned}$$

or in the case of a parametrization $g : D \to M$:

$$\int_{g(D)} \omega = \int_D \omega_{g(x)}(g_*(e_1),\cdots,g_*(e_m))d\mu.$$

These expression are useful for computing integrals. ▶

◀ **14.13** *Example.* Consider the 2-sphere parametrized by the mapping $g : \mathbb{R}^2 \to S^2 \subset \mathbb{R}^3$ given as

$$g(\varphi,\theta) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \sin(\varphi)\cos(\theta) \\ \sin(\varphi)\sin(\theta) \\ \cos(\varphi) \end{pmatrix}.$$

This mapping can be viewed as a covering map. From this expression we derive various charts for S^2. Given the 2-form $\omega = zdx \wedge dz$ let us compute the pullback form $g^*\omega$ on $\mathbb{R}^2$. We have that

$$\begin{aligned} g_*(e_1) &= \cos(\varphi)\cos(\theta)\frac{\partial}{\partial x} + \cos(\varphi)\sin(\theta)\frac{\partial}{\partial y} - \sin(\varphi)\frac{\partial}{\partial z}, \\ g_*(e_2) &= -\sin(\varphi)\sin(\theta)\frac{\partial}{\partial x} + \sin(\varphi)\cos(\theta)\frac{\partial}{\partial y}, \end{aligned}$$

and therefore

$$g^*\omega(e_1,e_2) = \omega_{g(x)}(g_*(e_1), g_*(e_2)) = -\cos(\varphi)\sin^2(\varphi)\sin(\theta),$$

and thus

$$g^*\omega = -\cos(\varphi)\sin^2(\varphi)\sin(\theta)d\varphi \wedge d\theta.$$

The latter gives that $\int_{S^2}\omega = -\int_0^{2\pi}\int_0^{\pi}\cos(\varphi)\sin^2(\varphi)\sin(\theta)d\varphi d\theta = 0$, which shows that ω is not a volume form on S^2.

If we perform the same calculations for $\omega = xdy\wedge dz + ydz\wedge dx + zdx\wedge dy$, then

$$\int_{S^2}\omega = \int_0^{2\pi}\int_0^{\pi}\sin(\varphi)d\varphi d\theta = 4\pi.$$

The pullback form $g^*\omega = \sin(\varphi)d\varphi\wedge d\theta$, which shows that ω is a volume form on S^2. ▶

15. The exterior derivative

The ***exterior derivative*** d of a differential form is an operation that maps an k-form to a $(k+1)$-form. Write a k-form on $\mathbb{R}^m$ in the following notation

$$\omega = \omega_{i_1\cdots i_k}dx^{i_1}\wedge\cdots\wedge dx^{i_k} = \omega_I dx^I, \qquad I = (i_1\cdots i_k),$$

then we define

$$d\omega = d\omega_I \wedge dx^I. \tag{14}$$

Written out in all its differentials this reads

$$d\omega = d\Big(\omega_{i_1\cdots i_k}dx^{i_1}\wedge\cdots\wedge dx^{i_k}\Big) = \frac{\partial\omega_I}{\partial x_i}dx^i\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}.$$

Of course the Einstein summation convention is again applied here. This is the definition that holds for all practical purposes, but it is a local definition. We need to prove that d extends to differential forms on manifolds.

◀ **15.1** *Example.* Consider $\omega_0 = f(x,y)$, a 0-form on $\mathbb{R}^2$, then

$$d\omega_0 = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

For a 1-form $\omega_1 = f_1(x,y)dx + f_2(x,y)dy$ we obtain

$$\begin{aligned} d\omega_1 &= \frac{\partial f_1}{\partial x}dx\wedge dx + \frac{\partial f_1}{\partial y}dy\wedge dx + \frac{\partial f_2}{\partial x}dx\wedge dy + \frac{\partial f_2}{\partial y}dy\wedge dy \\ &= \frac{\partial f_1}{\partial y}dy\wedge dx + \frac{\partial f_2}{\partial x}dx\wedge dy = \Big(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\Big)dx\wedge dy. \end{aligned}$$

Finally, for a 2-form $\omega_2 = g(x,y)dx\wedge dy$ the d-operation gives

$$d\omega_2 = \frac{\partial g}{\partial x}dx\wedge dx\wedge dy + \frac{\partial g}{\partial y}dy\wedge dx\wedge dy = 0.$$

The latter shows that d applied to a top-form always gives 0. ▶

◀ **15.2** *Example.* In the previous example $d\omega_0$ is a 1-form, and $d\omega_1$ is a 2-form. Let us now apply the d operation to these forms:

$$d(d\omega_0) = \Big(\frac{\partial^2 f}{\partial x\partial y} - \frac{\partial^2 f}{\partial y\partial x}\Big)dx\wedge dy = 0,$$

and

$$d(d\omega_1) = 0,$$

since d acting on a 2-form always gives 0. These calculations seem to suggest that in general $d\circ d = 0$. ▶

As our examples indicate $d^2\omega = 0$. One can also have forms ω for which $d\omega = 0$, but $\omega \neq d\sigma$. We say that ω is a ***closed form***, and when $\omega = d\sigma$, then ω is called an ***exact form***. Clearly, closed forms form a possibly larger class than exact forms. In the next chapter on De Rham cohomology we are going to come back to this phenomenon in detail. On $\mathbb{R}^m$ one will not be able to find closed forms that are not exact, i.e. on $\mathbb{R}^m$ all closed forms are exact. However, if we consider different manifolds we can have examples where this is not the case.

◀ **15.3** *Example.* Let $M = \mathbb{R}^2\backslash\{(0,0)\}$, and consider the 1-form

$$\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy,$$

which clearly is a smooth 1-form on M. Notice that ω does not extend to a 1-form on $\mathbb{R}^2$! It holds that $d\omega = 0$, and thus ω is a closed form on M. Let $\gamma\colon [0,2\pi) \to M$, $t \mapsto (\cos(t), \sin(t))$ be an embedding of S^1 into M, then

$$\int_\gamma \omega = \int_0^{2\pi} \gamma^*\omega = \int_0^{2\pi} dt = 2\pi.$$

Assume that ω is an exact 1-form on M, then $\omega = df$, for some smooth function $f : M \to \mathbb{R}$. In Section 5 we have showed that

$$\int_\gamma \omega = \int_\gamma df = \int_0^{2\pi} \gamma^* df = (f\circ\gamma)'(t)dt = f(1,0) - f(1,0) = 0,$$

which contradicts the fact that $\int_\gamma \omega = \int_\gamma df = 2\pi$. This proves that ω is not exact. ▶

For the exterior derivative in general we have the following theorem.

Theorem 15.4. [37] *Let M be a smooth m-dimensional manifold. Then for all $k \geq 0$ there exist unique linear operations $d : \Gamma^k(M) \to \Gamma^{k+1}(M)$ such that;*

(i) *for any 0-form $\omega = f$, $f \in C^\infty(M)$ it holds that*

$$d\omega(X) = df(X) = Xf, \qquad X \in \mathcal{F}(M);$$

(ii) *if $\omega \in \Gamma^k(M)$ and $\eta \in \Gamma^\ell(M)$, then*

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta;$$

(iii) $d \circ d = 0$;

(iv) *if $\omega \in \Gamma^m(M)$, then $d\omega = 0$.*

*This operation d is called the **exterior derivative** on differential forms, and is a unique anti-derivation (of degree 1) on $\Gamma(M)$ with $d^2 = 0$.*

Proof: Let us start by showing the existence of d on a chart $U \subset M$. We have local coordinates $x = \varphi(p)$, $p \in U$, and we define

$$d_U : \Gamma^k(U) \to \Gamma^{k+1}(U),$$

via (14). Let us write d instead of d_U for notational convenience. We have that $d(fdx^I) = df \wedge dx^I$. Due to the linearity of d it holds that

$$\begin{aligned} d(fdx^I \wedge gdx^J) &= d(fgdx^I \wedge dx^J) = d(fg)dx^K \\ &= gdfdx^K + fdgdx^K \\ &= (df \wedge dx^I) \wedge (gdx^J) + fdg \wedge dx^I \wedge dx^J \\ &= (df \wedge dx^I) \wedge (gdx^J) + (-1)^k fdx^I \wedge dg \wedge dx^J \\ &= d(fdx^I) \wedge (gdx^J) + (-1)^k (fdx^I) \wedge d(gdx^J), \end{aligned}$$

[37] See Lee Thm. 12.14

which proves (ii). As for (iii) we argue as follows. In the case of a 0-form we have that

$$\begin{aligned} d(df) &= d\Big(\frac{\partial f}{\partial x_i}dx^i = \frac{\partial^2 f}{\partial x_j \partial x_i}\Big)dx^j \wedge dx^i \\ &= \sum_{i<j}\Big(\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_i}\Big)dx^i \wedge dx^j = 0. \end{aligned}$$

Given a k-form $\omega = \omega_I dx^I$, then, using (ii), we have

$$\begin{aligned} d(d\omega) &= d\Big(d\omega_I \wedge dx^I\Big) \\ &= d(d\omega_I) \wedge dx^I + (-1)^{k+1} d\omega^I \wedge d(dx^I) = 0, \end{aligned}$$

since ω_I is a 0-form, and $d(dx^I) = (-1)^j dx^{i_1} \wedge \cdots \wedge d(dx^{i_j}) \wedge \cdots \wedge dx^{i_k} = 0$. The latter follows from $d(dx^{i_j}) = 0$. The latter also implies (iv), finishing the existence proof in the one chart case. The operation $d = d_U$ is well-defined, satisfying (i)-(iv), for any chart (U, φ).

The operation d_U is unique, for if there exists yet another exterior derivative $\widetilde{d}_U$, which satisfies (i)-(iv), then for $\omega = \omega_I dx^I$,

$$\widetilde{d}\omega = \widetilde{d}\omega_I \wedge dx^I + \omega^I \widetilde{d}(dx^I),$$

where we used (ii). From (ii) it also follows that $\widetilde{d}(dx^I) = (-1)^j dx^{i_1} \wedge \cdots \wedge \widetilde{d}(dx^{i_j}) \wedge \cdots \wedge dx^{i_k} = 0$. By (i) $d(\varphi(p)_{i_j}) = dx^{i_j}$, and thus by (iv) $\widetilde{d}(dx^{i_j}) = \widetilde{d} \circ \widetilde{d}(\varphi(p)_{i_j}) = 0$, which proves the latter. From (i) it also follows that $\widetilde{d}\omega_I = d\omega_I$, and therefore

$$\widetilde{d}\omega = \widetilde{d}\omega_I \wedge dx^I + \omega^I \widetilde{d}(dx^I) = d\omega_I \wedge dx^I = d\omega,$$

which proves the uniqueness of d_U.

Before giving the defining d for $\omega \in \Gamma^k(M)$ we should point out that d_U trivially satisfies (i)-(iii) of Theorem 15.5 (use (14). Since we have a unique operation d_U for every chart U, we define for $p \in U$

$$(d\omega)_p = (d_U \omega|_U)_p,$$

as the exterior derivative for forms on M. If there is a second chart $U' \ni p$, then by uniqueness it holds that on

$$d_U(\omega|_{U \cap U'}) = d_{U \cap U'}(\omega|_{U \cap U'}) = d_{U'}(\omega|_{U \cap U'}),$$

which shows that the above definition is independent of the chosen chart.

The last step is to show that d is also uniquely defined on $\Gamma^k(M)$. Let $p \in U$, a coordinate chart, and consider the restriction

$$\omega|_U = \omega_I dx^I,$$

where $\omega_I \in C^\infty(U)$. Let $W \subset U$ be an open set containing p, with the additional property that $\mathrm{cl}(W) \subset U$ is compact. By Theorem 13.10 we can find a bump-function $g \in C^\infty(M)$ such that $g|_W = 1$, and $\mathrm{supp}(g) \subset U$. Now define

$$\widetilde{\omega} = g\omega_I d(gx_{i_1}) \wedge \cdots \wedge d(gx_{i_k}).$$

Using (i) we have that $d(gx_i)|_W = dx^i$, and therefore $\tilde{\omega}|_W = \omega|_W$. Set $\eta = \widetilde{\omega} - \omega$, then $\eta|_W = 0$. Let $p \in W$ and $h \in C^\infty(M)$ satisfying $h(p) = 1$, and $\mathrm{supp}(h) \subset W$. Thus, $h\omega \equiv 0$ on M, and

$$0 = d(h\omega) = dh \wedge \omega + hd\omega.$$

This implies that $(d\omega)_p = -(df \wedge \omega)_p = 0$, which proves that $(d\widetilde{\omega})|_W = (d\omega)|_W$.

If we use (ii)-(iii) then

$$\begin{aligned} d\widetilde{\omega} &= d\big(g\omega_I d(gx_{i_1}) \wedge \cdots \wedge d(gx_{i_k})\big) \\ &= d(g\omega_I) \wedge d(gx_{i_1}) \wedge \cdots \wedge d(gx_{i_k}) + g\omega_I d\big(d(gx_{i_1}) \wedge \cdots \wedge d(gx_{i_k})\big) \\ &= d(g\omega_I) \wedge d(gx_{i_1}) \wedge \cdots \wedge d(gx_{i_k}). \end{aligned}$$

It now follows that since $g|_W = 1$, and since $(d\widetilde{\omega})|_W = (d\omega)|_W$, that

$$(d\omega)|_W = \frac{\partial \omega_I}{\partial x_i} dx^i \wedge dx^I,$$

which is exactly (14). We have only used properties (i)-(iii) the derive this expression, and since p is arbitrary it follows that $d : \Gamma^k(M) \to \Gamma^{k+1}(M)$ is uniquely defined. ■

The exterior derivative has other important properties with respect to restrictions and pullbacks that we now list here.

Theorem 15.5. *Let M be a smooth m-dimensional manifold, and let $\omega \in \Gamma^k(M)$, $k \geq 0$. Then*

(i) *in each chart (U, φ) for M, $d\omega$ in local coordinates is given by (14);*
(ii) *if $\omega = \omega'$ on some open set $U \subset M$, then also $d\omega = d\omega'$ on U;*
(iii) *if $U \subset M$ is open, then $d(\omega|_U) = (d\omega)|_U$;*
(iv) *if $f : N \to M$ is a smooth mapping, then*

$$f^*(d\omega) = d(f^*\omega),$$

i.e. $f^* : \Gamma^k(M) \to \Gamma^k(N)$, and d commute as operations.

Proof: Let us restrict ourselves here to proof of (iv). It suffices to prove (iv) in a chart U, and $\omega = \omega_I dx^I$. Let us first compute $f^*(d\omega)$:

$$\begin{aligned} f^*(d\omega) &= f^*\big(d\omega_I \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}\big) \\ &= d(\omega_I \circ f) \wedge d(\varphi \circ f)_{i_1} \wedge \cdots \wedge d(\varphi \circ f)_{i_k}. \end{aligned}$$

Similarly,

$$\begin{aligned} d(f^*\omega) &= d\big((\omega_I\circ f)\wedge d(\varphi\circ f)_{i_1}\wedge\cdots\wedge d(\varphi\circ f)_{i_k}\big)\\ &= d(\omega_I\circ f)\wedge d(\varphi\circ f)_{i_1}\wedge\cdots\wedge d(\varphi\circ f)_{i_k}, \end{aligned}$$

which proves the theorem. ∎

16. **Stokes' Theorem**

The last section of this chapter deals with a generalization of the Fundamental Theorem of Integration: Stokes' Theorem. The Stokes' Theorem allows us to compute $\int_M d\omega$ in terms of a boundary integral for ω. In a way the $(m-1)$-form act as the 'primitive', 'anti-derivative' of the $d\omega$. Therefore if we are interested in $\int_M \sigma$ using Stokes' Theorem we need to first 'integrate' σ, i.e. write $\sigma = d\omega$. This is not always possible as we saw in the previous section.

Stokes' Theorem can be now be phrased as follows.

Theorem 16.1. *Let M be a smooth m-dimensional manifold with or without boundary ∂M, and $\omega\in\Gamma_c^{m-1}(M)$. Then,*

$$\int_M d\omega = \int_{\partial M} j^*\omega, \tag{15}$$

where $j:\partial M\hookrightarrow M$ is the natural embedding of the boundary ∂M into M.

In order to prove this theorem we start with the following lemma.

Lemma 16.2. *Let $\omega\in\Gamma_c^{m-1}(\mathbb{H}^m)$ be given by*

$$\omega = \omega_i dx^1\wedge\cdots\wedge\widehat{dx^i}\wedge\cdots\wedge dx^m.$$

Then,

(i) $\int_{\mathbb{H}^M} d\omega = 0$, *if* $\mathrm{supp}(\omega)\subset\mathrm{int}\,(\mathbb{H}^m)$*;*

(ii) $\int_{\mathbb{H}^m} d\omega = \int_{\partial\mathbb{H}^m} j^*\omega$, *if* $\mathrm{supp}(\omega)\cap\partial\mathbb{H}^m\neq\varnothing$,

where $j:\partial\mathbb{H}^m\hookrightarrow\mathbb{H}^m$ is the canonical inclusion.

Proof: We may assume without loss of generality that $\mathrm{supp}(\omega)\subset[0,R]\times\cdots\times[0,R]=I_R^m$. Now,

$$\begin{aligned} \int_{\mathbb{H}^m} d\omega &= (-1)^{i+1}\int_{I_R^m}\frac{\partial\omega_i}{\partial x_i}dx_1\cdots dx_m\\ &= (-1)^{i+1}\int_{I_R^{m-1}}\Big(\omega_i|_{x_i=R}-\omega_i|_{x_i=0}\Big)dx_1\cdots\widehat{dx_i}\cdots dx_m. \end{aligned}$$

If $\mathrm{supp}(\omega)\subset\mathrm{int}\,I_R^m$, then $\omega_i|_{x_i=R}-\omega_i|_{x_i=0}=0$ for all i, and

$$\int_{\mathbb{H}^m} d\omega = 0.$$

If $\mathrm{supp}(\omega)\cap\partial\mathbb{H}^m \neq \varnothing$, then $\omega_i|_{x_i=R} - \omega_i|_{x_i=0} = 0$ for all $i \le m-1$, and $\omega_m|_{x_m=R} = 0$. Therefore,

$$\begin{aligned}
\int_{\mathbb{H}^m} d\omega &= (-1)^{i+1}\int_{I_R^m} \frac{\partial\omega_i}{\partial x_i} dx_1\cdots dx_m \\
&= (-1)^{i+1}\int_{I_R^{m-1}} \Big(\omega_i|_{x_i=R} - \omega_i|_{x_i=0}\Big) dx_1\cdots \widehat{dx_i}\cdots dx_m \\
&= (-1)^{m+1}\int_{I_R^{m-1}} \Big(-\omega_i|_{x_m=0}\Big) dx_1\cdots dx_{m-1} \\
&= (-1)^{m}\int_{I_R^{m-1}} \Big(\omega_i|_{x_m=0}\Big) dx_1\cdots dx_{m-1}.
\end{aligned}$$

The mapping $j:\partial\mathbb{H}^m \to \mathbb{H}^m$, which, in coordinates, is given by $\tilde{j} = j\circ\varphi : (x_1,\cdots,x_{m-1}) \mapsto (x_1,\cdots,x_{m-1},0)$, where $\varphi(x_1,\cdots,x_{m-1},0) = (x_1,\cdots,x_{m-1})$. The latter mapping is orientation preserving if m is even and orientation reversing if m is odd. Under the mapping j we have that

$$j_*(e_i) = \mathbf{e}_i, \qquad i = 1,\cdots,m-1,$$

where the e_i's on the left hand side are the unit vectors in $\mathbb{R}^{m-1}$, and the bold face $\mathbf{e}_k$ are the unit vectors in $\mathbb{R}^m$. We have that the induced orientation for $\partial\mathbb{H}^m$ is obtained by the rotation $\mathbf{e}_1 \to -\mathbf{e}_m$, $\mathbf{e}_m \to \mathbf{e}_1$, and therefore

$$\partial\mathcal{O} = [\mathbf{e}_2,\cdots,\mathbf{e}_{m-1},\mathbf{e}_1].$$

Under φ this corresponds with the orientation $[e_2,\cdots,e_{m-1},e_1]$ of $\mathbb{R}^{m-1}$, which is indeed the standard orientation for m even, and the opposite orientation for m odd. The pullback form on $\partial\mathbb{H}^m$, using the induced orientation on $\partial\mathbb{H}^m$, is given by

$$\begin{aligned}
&(j^*\omega)_{(x_1,\cdots,x_{m-1})}(e_2,\cdots,e_{m-1},e_1) \\
&\quad = \omega_{(x_1,\cdots,x_{m-1},0)}\big(j_*(e_2),\cdots,j_*(e_{m-2}),j_*(e_1)\big) \\
&\quad = (-1)^m\omega_m(x_1,\cdots,x_{m-1},0),
\end{aligned}$$

where we used the fact that

$$\begin{aligned}
&dx_1\wedge\cdots\wedge dx_{m-1}\big(j_*(e_2),\cdots,j_*(e_{m-1}),j_*(e_1))\big) \\
&\quad = dx_1\wedge\cdots\wedge dx_{m-1}(\mathbf{e}_2,\cdots,\mathbf{e}_{m-1},\mathbf{e}_1) \\
&\quad = (-1)^m dx_2\wedge\cdots\wedge dx_{m-1}\wedge dx_1(\mathbf{e}_2,\cdots,\mathbf{e}_{m-1},\mathbf{e}_1) = (-1)^m.
\end{aligned}$$

Combining this with the integral over $\mathbb{H}^m$ we finally obtain

$$\int_{\mathbb{H}^m} d\omega = \int_{\partial\mathbb{H}^m} j^*\omega,$$

which completes the proof. ∎

Proof of Theorem 16.1: Let us start with the case that $\mathrm{supp}(\omega) \subset U$, where (U,φ) is an oriented chart for M. Then by the definition of the integral we have that

$$\int_M d\omega = \int_{\varphi(U)} (\varphi^{-1})^*(d\omega) = \int_{\mathbb{H}^m} (\varphi^{-1})^*(d\omega) = \int_{\mathbb{H}^m} d\Big((\varphi^{-1})^*\omega\Big),$$

where the latter equality follows from Theorem 15.5. It follows from Lemma 16.2 that if $\mathrm{supp}(\omega) \subset \mathrm{int}\,(\mathbb{H}^m)$, then the latter integral is zero and thus $\int_M \omega = 0$. Also, using Lemma 16.2, it follows that if $\mathrm{supp}(\omega) \cap \partial\mathbb{H}^m \neq \varnothing$, then

$$\begin{aligned}\int_M d\omega &= \int_{\mathbb{H}^m} d\Big((\varphi^{-1})^*\omega\Big) = \int_{\partial\mathbb{H}^m} j'^*(\varphi^{-1})^*\omega \\ &= \int_{\partial\mathbb{H}^m} (\varphi^{-1}\circ j')^*\omega = \int_{\partial M} j^*\omega,\end{aligned}$$

where $j' : \partial\mathbb{H}^m \to \mathbb{H}^m$ is the canonical inclusion as used Lemma 16.2, and $j = \varphi^{-1}\circ j' : \partial M \to M$ is the inclusion of the boundary of M into M.

Now consider the general case. As before we choose a finite covering $\mathcal{A}_I \subset \mathcal{A}$ of $\mathrm{supp}(\omega)$ and an associated partition of unity $\{\lambda_i\}$ subordinate to $\mathcal{A}_I$. Consider the forms $\lambda_i\omega$, and using the first part we obtain

$$\begin{aligned}\int_{\partial M} j^*\omega &= \sum_i \int_{\partial M} j^*\lambda_i\omega = \sum_i \int_M d(\lambda_i\omega) \\ &= \sum_i \int_M \Big(d\lambda_i \wedge \omega + \lambda_i d\omega\Big) \\ &= \int_M d\Big(\sum_i \lambda_i\Big)\wedge\omega + \int_M \Big(\sum_i \lambda_i\Big) d\omega \\ &= \int_M d\omega,\end{aligned}$$

which proves the theorem. ■

◀ **16.3** *Remark.* If M is a closed (compact, no boundary), oriented manifold manifold, then by Stokes' Theorem for any $\omega \in \Gamma^{m-1}(M)$,

$$\int_M d\omega = 0,$$

since $\partial M = \varnothing$. As a consequence volume form σ cannot be exact. Indeed, $\int_M \sigma > 0$, and thus if σ were exact, then $\sigma = d\omega$, which implies that $\int_M \sigma = \int_M d\omega = 0$, a contradiction. ▶

Let us start with the obvious examples of Stokes' Theorem in $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$.

◀ **16.4** *Example.* Let $M = [a,b] \subset \mathbb{R}$, and consider the 0-form $\omega = f$, then, since M is compact, ω is compactly supported 0-form. We have that $d\omega = f'(x)dx$, and

$$\int_a^b f'(x)dx = \int_{\{a,b\}} f = f(b) - f(a),$$

which is the fundamental theorem of integration. Since on M any 1-form is exact the theorem holds for 1-forms. ▶

◀ **16.5** *Example.* Let $M = \gamma \subset \mathbb{R}^2$ be a curve parametrized given by $\gamma\colon [a,b] \to \mathbb{R}^2$. Consider a 0-form $\omega = f$, then

$$\int_\gamma f_x(x,y)dx + f_y(x,y)dy = \int_{\{\gamma(a),\gamma(b)\}} f = f(\gamma(b)) - f(\gamma(a)).$$

Now let $M = \Omega \subset \mathbb{R}^2$, a closed subset of $\mathbb{R}^2$ with a smooth boundary $\partial\Omega$, and consider a 1-form $\omega = f(x,y)dx + g(x,y)dy$. Then, $d\omega = \big(g_x - f_y\big)dx \wedge dy$, and Stokes' Theorem yields

$$\int_\Omega \Big(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\Big)dxdy = \int_{\partial\Omega} fdx + gdy,$$

also known as Green's Theorem in $\mathbb{R}^2$. ▶

◀ **16.6** *Example.* In the case of a curve $M = \gamma \subset \mathbb{R}^3$ we obtain the line-integral for 0-forms as before:

$$\int_\gamma f_x(x,y,z)dx + f_y(x,y,z)dy + f_z(x,y,z) = f(\gamma(b)) - f(\gamma(a)).$$

If we write the vector field

$$F = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial z},$$

then

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial x}\frac{\partial}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial}{\partial z},$$

and the above expression can be rewritten as

$$\int_\gamma \nabla f \cdot ds = f|_{\partial\gamma}.$$

Next, let $M = S \subset \mathbb{R}^3$ be an embedded or immersed hypersurface, and let $\omega = fdx + gdy + hdz$ be a 1-form. Then,

$$d\omega = \Big(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\Big)dy \wedge dz + \Big(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\Big)dz \wedge dx + \Big(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\Big)dx \wedge dy.$$

Write the vector fields

$$\text{curl } F = \nabla \times F = \Big(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\Big)\frac{\partial}{\partial x} + \Big(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\Big)\frac{\partial}{\partial y} + \Big(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\Big)\frac{\partial}{\partial z}.$$

Furthermore set

$$ds = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}, \qquad dS = \begin{pmatrix} dydz \\ dzdx \\ dxdy \end{pmatrix},$$

then from Stokes' Theorem we can write the following surface and line integrals:

$$\int_S \nabla \times F \cdot dS = \int_{\partial S} F \cdot ds,$$

which is usually referred to as the classical Stokes' Theorem in $\mathbb{R}^3$. The version in Theorem 16.1 is the general Stokes' Theorem. Finally let $M = \Omega$ a closed subset of $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$, and consider a a 2-form $\omega = f dy \wedge dz + g dz \wedge dx + h dx \wedge dy$. Then

$$d\omega = \Big(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\Big) dx \wedge dy \wedge dz.$$

Write

$$\text{div } F = \nabla \cdot F = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}, \quad dV = dxdydz,$$

then from Stokes' Theorem we obtain

$$\int_\Omega \nabla \cdot F dV = \int_{\partial\Omega} F \cdot dS,$$

which is referred to as the Gauss Divergence Theorem. ▶

V. De Rham cohomology

17. Definition of De Rham cohomology

In the previous chapters we introduced and integrated m-forms over manifolds M. We recall that k-form $\omega \in \Gamma^k(M)$ is closed if $d\omega = 0$, and a k-form $\omega \in \Gamma^k(M)$ is exact if there exists a $(k-1)$-form $\sigma \in \Gamma^{k-1}(M)$ such that $\omega = d\sigma$. Since $d^2 = 0$, exact forms are closed. We define

$$\begin{aligned} Z^k(M) &= \left\{\omega \in \Gamma^k(M) \,:\, d\omega = 0\right\} = \mathrm{Ker}(d), \\ B^k(M) &= \left\{\omega \in \Gamma^k(M) \,:\, \exists\, \sigma \in \Gamma^{k-1}(M) \,\ni\, \omega = d\sigma\right\} = \mathrm{Im}(d), \end{aligned}$$

and in particular

$$B^k(M) \subset Z^k(M).$$

The sets Z^k and B^k are real vector spaces, with B^k a vector subspace of Z^k. This leads to the following definition.

Definition 17.1. Let M be a smooth m-dimensional manifold then the ***de Rham cohomology groups*** are defined as

$$H^k_{dR}(M) := Z^k(M)/B^k(M), \qquad k = 0, \cdots m, \tag{16}$$

where $B^0(M) := 0$.

It is immediate from this definition that $Z^0(M)$ are smooth functions on M that are constant on each connected component of M. Therefore, when M is connected, then $H^0_{\mathrm{dR}}(M) \cong \mathbb{R}$. Since $\Gamma^k(M) = \{0\}$, for $k > m = \dim\, M$, we have that $\mathrm{H}^k_{dR}(M) = 0$ for all $k > m$. For $k < 0$, we set $H^k_{dR}(M) = 0$.

◀ **17.2** *Remark.* The de Rham groups defined above are in fact real vector spaces, and thus groups under addition in particular. The reason we refer to de Rham cohomology groups instead of de Rham vector spaces is because (co)homology theories produce abelian groups. ▶

An equivalence class $[\omega] \in H^k_{\mathrm{dR}}(M)$ is called a ***cohomology class***, and two form $\omega, \omega' \in Z^k(M)$ are ***cohomologous*** if $[\omega] = [\omega']$. This means in particular that ω and ω' differ by an exact form, i.e.

$$\omega' = \omega + d\sigma.$$

Now let us consider a smooth mapping $f : N \to M$, then we have that the pull-back f^* acts as follows: $f^* : \Gamma^k(M) \to \Gamma^k(N)$. From Theorem 15.5 it follows that $d \circ f^* = f^* \circ d$ and therefore f^* descends to homomorphism in cohomology. This can be seen as follows:

$$df^*\omega = f^* d\omega = 0, \quad \text{and} \quad f^* d\sigma = d(f^*\sigma),$$

and therefore the closed forms $Z^k(M)$ get mapped to $Z^k(N)$, and the exact form $B^k(M)$ get mapped to $B^k(N)$. Now define

$$f^*[\omega] = [f^*\omega],$$

which is well-defined by

$$f^*\omega' = f^*\omega + f^* d\sigma = f^*\omega + d(f^*\sigma)$$

which proves that $[f^*\omega'] = [f^*\omega]$, whenever $[\omega'] = [\omega]$. Summarizing, f^* maps cohomology classes in $H^k_{\rm dR}(M)$ to classes in $H^k_{\rm dR}(N)$:

$$f^* : H^k_{\rm dR}(M) \to H^k_{\rm dR}(N),$$

Theorem 17.3. *Let $f : N \to M$, and $g : M \to K$, then*

$$g^* \circ f^* = (f \circ g)^* : \ H^k_{\rm dR}(K) \to H^k_{\rm dR}(N),$$

Moreover, id^* is the identity map on cohomology.

Proof: Since $g^* \circ f^* = (f \circ g)^*$ the proof follows immediately. ∎

As a direct consequence of this theorem we obtain the invariance of de Rham cohomology under diffeomorphisms.

Theorem 17.4. *If $f : N \to M$ is a diffeomorphism, then $H^k_{\rm dR}(M) \cong H^k_{\rm dR}(N)$.*

Proof: We have that $\mathrm{id} = f \circ f^{-1} = f^{-1} \circ f$, and by the previous theorem

$$\mathrm{id}^* = f^* \circ (f^{-1})^* = (f^{-1})^* \circ f^*,$$

and thus f^* is an isomorphism. ∎

18. Homotopy invariance of cohomology

We will prove now that the de Rham cohomology of a smooth manifold M is even invariant under homeomorphisms. As a matter of fact we prove that the de Rham cohomology is invariant under homotopies of manifolds.

Definition 18.1. Two smooth mappings $f, g : N \to M$ are said to be ***homotopic*** if there exists a continuous map $H : N \times [0,1] \to M$ such that

$$\begin{aligned} H(p,0) &= f(p) \\ H(p,1) &= g(p), \end{aligned}$$

for all $p \in N$. Such a mapping is called a ***homotopy*** from/between f to/and g. If in addition H is smooth then f and g are said to be ***smoothly homotopic***, and H is called a ***smooth homotopy***.

Using the notion of smooth homotopies we will prove the following crucial property of cohomology:

Theorem 18.2. *Let $f, g : N \to M$ be two smoothly homotopic maps. Then for $k \geq 0$ it holds for $f^*, g^* : H^k_{\mathrm{dR}}(M) \to H^k_{\mathrm{dR}}(N)$, that*

$$f^* = g^*.$$

◀ **18.3** *Remark.* It can be proved in fact that the above results holds for two homotopic (smooth) maps f and g. This is achieved by constructing a smooth homotopy from a homotopy between maps. ▶

Proof of Theorem 18.2: A map $\mathbf{h} : \Gamma^k(M) \to \Gamma^{k-1}(N)$ is called a ***homotopy map*** between f^* and g^* if

$$d\mathbf{h}(\omega) + \mathbf{h}(d\omega) = g^*\omega - f^*\omega, \quad \omega \in \Gamma^k(M). \tag{17}$$

Now consider the embedding $i_t : N \to N \times I$, and the trivial homotopy between i_0 and i_1 (just the identity map). Let $\omega \in \Gamma^k(N \times I)$, and define the mapping

$$\mathbf{h}(\omega) = \int_0^1 i_{\frac{\partial}{\partial t}} \omega dt,$$

which is a map from $\Gamma^k(N \times I) \to \Gamma^{k-1}(N)$. Choose coordinates so that either

$$\omega = \omega_I(x,t) dx^I, \quad \text{or} \quad \omega = \omega_{I'}(x,t) dt \wedge d^x I'.$$

In the first case we have that $i_{\frac{\partial}{\partial t}} \omega = 0$ and therefore $d\mathbf{h}(\omega) = 0$. On the other hand

$$\begin{aligned} \mathbf{h}(d\omega) &= \mathbf{h}\Big(\frac{\partial \omega_I}{\partial t} dt \wedge dx^I + \frac{\partial \omega_I}{\partial x_i} dx^i \wedge dx^I\Big) \\ &= \Big(\int_0^1 \frac{\partial \omega_I}{\partial t} dt\Big) dx^I \\ &= \big(\omega_I(x,1) - \omega_I(x,0)\big) dx^I = i_1^*\omega - i_0^*\omega, \end{aligned}$$

which prove (17) for i_0^* and i_1^*, i.e.

$$d\mathbf{h}(\omega) + \mathbf{h}(d\omega) = i_1^*\omega - i_0^*\omega.$$

In the second case we have

$$\begin{aligned}
\mathbf{h}(d\omega) &= \mathbf{h}\Big(\frac{\partial\omega_I}{\partial x_i}dx^i\wedge dt\wedge dx^{I'}\Big)\\
&= \int_0^1 \frac{\partial\omega_I}{\partial dx_i} i_{\frac{\partial}{\partial x_i}}\big(dx^i\wedge dt\wedge dx^{I'}\big)dt\\
&= -\Big(\int_0^1 \frac{\partial\omega_I}{\partial x_i}dt\Big)dx^i\wedge dx^{I'}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
d\mathbf{h}(\omega) &= d\Big(\Big(\int_0^1 \omega_{I'}(x,t)dt\Big)dx^{I'}\Big)\\
&= \frac{\partial}{\partial x_i}\Big(\int_0^1 \omega_{I'}(x,t)dt\Big)dx^i\wedge dx^{I'}\\
&= \Big(\int_0^1 \frac{\partial\omega_I}{\partial x_i}dt\Big)dx^i\wedge dx^{I'}\\
&= -\mathbf{h}(d\omega).
\end{aligned}$$

This gives the relation that

$$d\mathbf{h}(\omega)+\mathbf{h}(d\omega)=0,$$

and since $i_1^*\omega = i_0^*\omega = 0$ in this case, this then completes the argument in both cases, and $\mathbf{h}$ as defined above is a homotopy map between i_0^* and i_1^*.

By assumption we have a smooth homotopy $H : N\times[0,1]\to M$ bewteen f and g, with $f = H\circ i_0$, and $g = H\circ i_1$. Consider the composition $\widetilde{\mathbf{h}} = \mathbf{h}\circ H^*$. Using the relations above we obtain

$$\begin{aligned}
\widetilde{\mathbf{h}}(d\omega)+d\widetilde{\mathbf{h}}(\omega) &= \mathbf{h}(H^*d\omega)+d\mathbf{h}(H^*\omega)\\
&= \mathbf{h}(d(H^*\omega))+d\mathbf{h}(H^*\omega)\\
&= i_1^*H^*\omega - i_0^*H^*\omega\\
&= (H\circ i_1)^*\omega-(H\circ i_0)^*\omega\\
&= g^*\omega - f^*\omega.
\end{aligned}$$

If we assume that ω is closed then

$$g^*\omega - f^*\omega = d\mathbf{h}(H^*\omega),$$

and thus

$$0=[d\mathbf{h}(H^*\omega)] = [g^*\omega - f^*\omega] = g^*[\omega]-f^*[\omega],$$

which proves the theorem. ■

◀ **18.4** *Remark.* Using the same ideas as for the Whitney embedding theorem one can prove, using approximation by smooth maps, that Theorem 18.2 holds true for continuous homotopies between smooth maps. ▶

Definition 18.5. Two manifolds N and M are said to be ***homotopy equivalent***, if there exist smooth maps $f : N \to M$, and $g : M \to N$ such that

$$g \circ f \cong \mathrm{id}_N, \qquad f \circ g \cong \mathrm{id}_M \quad \text{(homotopic maps)}.$$

We write $N \sim M$., The maps f and g are ***homotopy equivalences*** are each other ***homotopy inverses*** . If the homotopies involved are smooth we say that N and M ***smoothly homotopy equivalent***.

◀ **18.6** *Example.* Let $N = S^1$, the standard circle in $\mathbb{R}^2$, and $M = \mathbb{R}^2 \backslash \{(0,0)\}$. We have that $N \sim M$ by considering the maps

$$f = i : S^1 \hookrightarrow \mathbb{R}^2 \backslash \{(0,0)\}, \qquad g = \mathrm{id}/|\cdot|.$$

Clearly, $(g \circ f)(p) = p$, and $(f \circ g)(p) = p/|p|$, and the latter is homotopic to the identity via $H(p,t) = tp + (1-t)p/|p|$. ▶

Theorem 18.7. *Let N and M be smoothly homotopically equivalent manifold, $N \sim M$, then*

$$H^k_{\mathrm{dR}}(N) \cong H^k_{\mathrm{dR}}(M),$$

and the homotopy equivalences f g between N and M, and M and N respectively are isomorphisms.

As before this theorem remains valid for continuous homotopy equivalences of manifolds.

VI. Exercises

A number of the exercises given here are taken from the Lecture notes by J. Bochnak.

Manifolds

Topological Manifolds

☞ **1** Given the function $g:\mathbb{R}\to\mathbb{R}^2$, $g(t)=(\cos(t),\sin(t))$. Show that $f(\mathbb{R})$ is a manifold.

☞ **2** Given the set $\mathbb{T}^2=\{p=(p_1,p_2,p_3)\mid 16(p_1^2+p_2^2)=(p_1^2+p_2^2+p_3^2+3)^2\}\subset\mathbb{R}^3$, called the ***2-torus***.

(***i***) Consider the product manifold $S^1\times S^1=\{q=(q_1,q_2,q_3,q_4)\mid q_1^2+q_2^2=1,\ q_3^2+q_4^2=1\}$, and the mapping $f:S^1\times S^1\to\mathbb{T}^2$, given by

$$f(q)=\big(q_1(2+q_3),q_2(2+q_3),q_4\big).$$

Show that f is onto and $f^{-1}(p)=\big(\frac{p_1}{r},\frac{p_2}{r},r-2,p_3\big)$, where $r=\frac{|p|^2+3}{4}$.

(***ii***) Show that f is a homeomorphism between $S^1\times S^1$ and $\mathbb{T}^2$.

☞ **3** (***i***) Find an atlas for $\mathbb{T}^2$ using the mapping f in 2.

(***ii***) Give a parametrization for $\mathbb{T}^2$.

4 Show that

$$A_{4,4}=\{(p_1,p_2)\in\mathbb{R}^2\mid p_1^4+p_2^4=1\},$$

is a manifold and $A_{4,4}\cong S^1$ (homeomorphic).

☞ **5** (***i***) Show that an open subset $U\subset M$ of a manifold M is again a manifold.

(***ii***) Let N and M be manifolds. Show that their cartesian production $N\times M$ is also a manifold.

☞ **6** Show that

(***i***) $\{A \in M_{2,2}(\mathbb{R}) \mid \det(A) = 1\}$ is a manifold.

(***ii***) $Gl(n,\mathbb{R}) = \{A \in M_{n,n}(\mathbb{R}) \mid \det(A) \neq 0\}$ is a manifold.

(***iii***) Determine the dimensions of the manifolds in (a) and (b).

7 Construct a simple counterexample of a set that is not a manifold.

8 Show that in Definition 1.1 an open set $U' \subset \mathbb{R}^n$ can be replaced by an open disc $D^n \subset \mathbb{R}^n$.

9 Show that $P\mathbb{R}^n$ is a Hausdorff space and is compact.

10 Define the Grassmann manifold $G^k\mathbb{R}^n$ as the set of all k-dimensional linear subspaces in $\mathbb{R}^n$. Show that $G^k\mathbb{R}^n$ is a manifold.

11 Consider X to be the parallel lines $\big(\mathbb{R} \times \{0\}\big) \cup \big(\mathbb{R} \times \{1\}\big)$. Define the equivalence relation $(x,0) \sim (x,1)$ for all $x \neq 0$. Show that $M = X/\sim$ is a topological space that satisfies (ii) and (iii) of Definition 1.1.

☞ **12** Let M be an uncountable union of copies of $\mathbb{R}$. Show that M is a topological space that satisfies (i) and (ii) of Definition 1.1.

13 Let $M = \{(x,y) \in \mathbb{R}^2 \ : \ xy = 0\}$. Show that M is a topological space that satisfies (i) and (iii) of Definition 1.1.

Differentiable manifolds

☞ **14** Show that cartesian products of differentiable manifolds are again differentaible manifolds.

☞ **15** Show that $P\mathbb{R}^n$ is a smooth manifold.

16 Prove that $P\mathbb{R}$ is diffeomorphic to the standard unit circle in $\mathbb{R}^2$ by constructing a diffeomorphism.

☞ **17** Which of the atlases for S^1 given in Example 2, Sect. 1, are compatible. If not compatible, are they diffeomorphic?

☞ **18** Show that the standard torus $\mathbb{T}^2 \subset \mathbb{R}^3$ is diffeomorphic to $S^1 \times S^1$, where $S^1 \subset \mathbb{R}^2$ is the standard circle in $\mathbb{R}^2$.

19 Prove that the taking the union of C^∞-atlases defines an equivalence relation (Hint: use Figure 7 in Section 2).

☞ **20** Show that diffeomorphic defines an equivalence relation.

☞ **21** Prove Theorem 2.15.

Immersion, submersion and embeddings

☞ **22** Show that the torus the map f defined in Exer. 2 of 1.1 yields a smooth embedding of the torus $\mathbb{T}^2$ in $\mathbb{R}^3$.

23 Let k, m, n be positive integers. Show that the set

$$A_{k,m,n} := \{(x,y,z) \in \mathbb{R}^3 \ : \ x^k + y^m + z^n = 1\},$$

is a smooth embedded submanifold in $\mathbb{R}^3$, and is diffeomorphic to S^2 when these numbers are even.

24 Prove Lemma 3.18.

☞ **25** Let $f : \mathbb{R}^{n+1}\backslash\{0\} \to \mathbb{R}^{k+1}\backslash\{0\}$ be a smooth mapping such that $f(\lambda x) = \lambda^d f(x)$, $d \in \mathbb{N}$, for all $\lambda \in \mathbb{R}\backslash\{0\}$. This is called a homogeneous mapping of degree d. Show that $\widetilde{f} : P\mathbb{R}^n \to P\mathbb{R}^k$, defined by $\widetilde{f}([x]) = [f(x)]$, is a smooth mapping.

☞ **26** Show that the open ball $B^n \subset \mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$.

☞ **27** (Lee) Consider the mapping $f(x,y,s) = (x^2 + y, x^2 + y^2 + s^2 + y)$ from $\mathbb{R}^3$ to $\mathbb{R}^2$. Show that $q = (0,1)$ is a regular value, and $f^{-1}(q)$ is diffeomorphic to S^1 (standard).

28 Prove Theorem 3.24 (Hint: prove the steps indicated above Theorem 3.24).

29 Prove Theorem 3.26.

30 Let N, M be two smooth manifolds of the same dimension, and $f : N \to M$ is a smooth mapping. Show (using the Inverse Function Theorem) that if f is a bijection then it is a diffeomorphism.

31 Prove Theorem 3.28.

☞ **32** Show that the map $f : S^n \to P\mathbb{R}^n$ defined by $f(x_1, \cdots, x_{n+1}) = [(x_1, \cdots, x_{n+1})]$ is smooth and everywhere of rank n (see Boothby).

☞ **33** (Lee) Let $a \in \mathbb{R}$, and $M_a = \{(x,y) \in \mathbb{R}^2 \; : \; y^2 = x(x-1)(x-a)\}$.

(***i***) For which values of a is M_a an embedded submanifold of $\mathbb{R}^2$?

(***ii***) For which values of a can M_a be given a topology and smooth structure so that M_a is an immersed submanifold?

☞ **34** Consider the map $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x_1,x_2) = x_1^3 + x_1x_2 + x_2^3$. Which level sets of f are smooth embedded submanifolds of $\mathbb{R}^2$.

35 Let M be a smooth m-dimensional manifold with boundary ∂M (see Lee, pg. 25). Show that ∂M is an $(m-1)$-dimensional manifold (no boundary), and that there exists a unique differentiable structure on ∂M such that $i : \partial M \hookrightarrow M$ is a (smooth) embedding.

☞ **36** Let $f : N \to M$ be a smooth mapping. Let $S = f^{-1}(q) \subset N$ is a smooth embedded submanifold of codimension m. Is q necessarily a regular value of f?

☞ **37** Use Theorem 3.27 to show that the 2-torus $\mathbb{T}^2$ is a smooth embedded manifold in $\mathbb{R}^4$.

38 Prove Theorem 3.37.

39 Show that an m-dimensional linear subspace in $\mathbb{R}^\ell$ is an m-dimensional manifold.

40 Prove that the annulus $A = \{(p_1,p_2) \in \mathbb{R}^2 \; : \; 1 \le p_1^2 + p_2^2 \le 4\}$ is a smooth manifold two dimensional manifold in $\mathbb{R}^2$ with boundary.

Tangent and cotangent spaces

Tangent spaces

☞ **41** Let $U \subset \mathbb{R}^k$, $y \in \mathbb{R}^m$, and $f : U \to \mathbb{R}^m$ a smooth function. By Theorem 3.28 $M = f^{-1}(y) = \{p \in \mathbb{R}^k \,:\, f(p) = y\}$ is a smooth embedded manifold in $\mathbb{R}^k$ if $\mathrm{rk}(Jf)|_p = m$ for all $p \in M$. Show that

$$T_pM \cong \{X_p \in \mathbb{R}^k \,:\, X_p \cdot \partial_{p_i} f = 0,\ i = 1, \cdots, m\} = \ker Jf|_p,$$

i.e. the tangent space is the kernel of the Jacobian. Using this identification from now on, prove that

$$M_p = p + T_pM \subset \mathbb{R}^k$$

is tangent to M at p.

☞ **42** Given the set $M = \{(p_1, p_2, p_3) \in \mathbb{R}^3 \,:\, p_1^3 + p_2^3 + p_3^3 - 3p_1p_2p_3 = 1\}$.

(***i***) Show that M is smooth embedded manifold in $\mathbb{R}^3$ of dimension 2.

(***ii***) Compute T_pM, at $p = (0, 0, 1)$.

☞ **43** Given the set $M = \{(p_1, p_2, p_3) \in \mathbb{R}^3 \,:\, p_1^2 - p_2^2 + p_1p_3 - 2p_2p_3 = 0,\ 2p_1 - p_2 + p_3 = 3\}$.

(***i***) Show that M is smooth embedded manifold in $\mathbb{R}^3$ of dimension 1.

(***ii***) Compute T_pM, at $p = (1, -1, 0)$.

44 Prove Lemma 4.5.

☞ **45** Express the following planar vector fields in polar coordinates:

(***i***) $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$;

(***ii***) $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$;

(***iii***) $X = (x^2 + y^2)\frac{\partial}{\partial x}$.

46 Find a vector field on S^2 that vanishes at exactly one point.

47 Let $S^2 \subset \mathbb{R}^2$ be the standard unit sphere and $f : S^2 \to S^2$ a smooth map defined as a θ-degree rotation around the polar axis. Compute $f_* : T_pS^2 \to T_{f(p)}S^2$ and give a matrix representation of f_* in terms of a canonical basis.

Cotangent spaces

☞ **48** (Lee) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(p_1, p_2, p_3) = |p|^2$, and $g : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$g(x_1, x_2) = \Big(\frac{2x_1}{|x|^2+1}, \frac{2x_2}{|x|^2+1}, \frac{|x|^2-1}{|x|^2+1}\Big).$$

Compute both $g^* df$, and $d(f \circ g)$ and compare the two answers.

☞ **49** (Lee) Compute df in coordinates, and determine the set points where df vanishes.

(***i***) $M = \{(p_1, p_2) \in \mathbb{R}^2 \;:\; p_2 > 0\}$, and $f(p) = \frac{p_1}{|p|^2}$ — standard coordinates in $\mathbb{R}^2$.

(***ii***) As the previous, but in polar coordinates.

(***iii***) $M = S^2 = \{p \in \mathbb{R}^3 \;:\; |p| = 1\}$, and $f(p) = p_3$ — stereographic coordinates.

(***iv***) $M = \mathbb{R}^n$, and $f(p) = |p|^2$ — standard coordinates in $\mathbb{R}^n$.

☞ **50** Express in polar coordinates

(***i***) $\theta = x^2 dx + (x+y)dy$;

(***ii***) $\theta = xdx + ydy + zdz$.

51 Prove Lemma 5.3.

☞ **52** (Lee) Let $f : N \to M$ (smooth), $\omega \in \Lambda^1(N)$, and $\gamma : [a,b] \to N$ is a smooth curve. Show that

$$\int_\gamma f^*\omega = \int_{f\circ\gamma} \omega.$$

☞ **53** Given the following 1-forms on $\mathbb{R}^3$:

$$\alpha = -\frac{4z}{(x^2+1)^2}dx + \frac{2y}{y^2+1}dy + \frac{2x}{x^2+1}dz,$$
$$\omega = -\frac{4xz}{(x^2+1)^2}dx + \frac{2y}{y^2+1}dy + \frac{2}{x^2+1}dz.$$

(***i***) Let $\gamma(t) = (t,t,t)$, $t \in [0,1]$, and compute $\int_\gamma \alpha$ and $\int_\gamma \omega$.

(***ii***) Let γ be a piecewise smooth curve going from $(0,0,0)$ to $(1,0,0)$ to $(1,1,0)$ to $(1,1,1)$, and compute the above integrals.

(***iii***) Which of the 1-forms α and ω is exact.

(***iv***) Compare the answers.

Vector bundles

☞ **54** Let $M \subset \mathbb{R}^\ell$ be a manifold in $\mathbb{R}^\ell$. Show that TM as defined in Section 6 is a smooth embedded submanifold of $\mathbb{R}^{2\ell}$.

55 Let $S^1 \subset \mathbb{R}^2$ be the standard unit circle. Show that TS^1 is diffeomorphic to $S^1 \times \mathbb{R}$.

56 Let $M \subset \mathbb{R}^\ell$ be a manifold in $\mathbb{R}^\ell$. Show that T^*M as defined in Section 6 is a smooth embedded submanifold of $\mathbb{R}^{2\ell}$.

57 Show that the open Möbius band is a smooth rank-1 vector bundle over S^1.

Tensors

Tensors and tensor products

☞ **58** Describe the standard inner product on $\mathbb{R}^n$ as a covariant 2-tensor.

☞ **59** Construct the determinant on $\mathbb{R}^2$ and $\mathbb{R}^3$ as covariant tensors.

60 Given the vectors $a = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$, and $b = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$. Compute $a^* \otimes b$ and $b^* \otimes a$.

☞ **61** Given finite dimensional vector spaces V and W, prove that $V \otimes W$ and $W \otimes V$ are isomorphic.

62 Given finite dimensional vector spaces U, V and W, prove that $(U \otimes V) \otimes W$ and $U \otimes (V \otimes W)$ are isomorphic.

☞ **63** Show that $V \otimes \mathbb{R} \simeq V \simeq \mathbb{R} \otimes V$.

Symmetric and alternating tensors

☞ **64** Show that for $T \in T^s(V)$ the tensor $\mathrm{Sym}\, T$ is symmetric.

65 Prove that a tensor $T \in T^s(V)$ is symmetric if and only if $T = \mathrm{Sym}\, T$.

66 Show that the algebra

$$\Sigma^*(V) = \bigoplus_{k=0}^{\infty} \Sigma^k(V),$$

is a commutative algebra.

☞ **67** Prove that for any $T \in T^s(V)$ the tensor $\mathrm{Alt}\, T$ is alternating.

68 Show that a tensor $T \in T^s(V)$ is alternating if and only if $T = \mathrm{Alt}\, T$.

69 Given the vectors $a = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$, and $c = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. Compute $a \wedge b \wedge c$, and compare this with $\det(a, b, c)$.

70 Given vectors $a_1, \cdots, a_n$ show that $a_1 \wedge \cdots \wedge a_n = \det(a_1, \cdots, a_n)$.

71 Prove Lemma 8.11.

Tensor bundles and tensor fields

72 Let $M \subset \mathbb{R}^\ell$ be an embedded m-dimensional manifold. Show that $T^r M$ is a smooth manifold in $\mathbb{R}^{\ell+\ell^r}$.

73 Similarly, show that $T_s M$ is a smooth manifold in $\mathbb{R}^{\ell+\ell^s}$, and $T^r_s M$ is a smooth manifold in $\mathbb{R}^{\ell+\ell^r+\ell^s}$.

74 Prove that the tensor bundles introduced in Section 7 are smooth manifolds.

☞ **75** One can consider symmetric tensors in T_pM as defined in Section 8. Define and describe $\Sigma^r(T_pM) \subset T^r(T_pM)$ and $\Sigma^r M \subset T^r M$.

☞ **76** Describe a smooth covariant 2-tensor field in $\Sigma^2 M \subset T^2 M$. How does this relate to an (indefinite) inner product on T_pM?

77 Prove Lemma 9.2.

☞ **78** Given the manifolds $N = M = \mathbb{R}^2$, and the smooth mapping

$$q = f(p) = (p_1^2 - p_2, p_1 + 2p_2^3),$$

acting from N to M. Consider the tensor spaces $T^2(T_pN) \cong T^2(T_qM) \cong T^2(\mathbb{R}^2)$ and compute the matrix for the pullback f^*.

☞ **79** In the above problem consider the 2-tensor field σ on M, given by

$$\sigma = dy^1 \otimes dy^2 + q_1 dy^2 \otimes dy^2.$$

(***i***) Show that $\sigma \in \mathcal{F}^2(M)$.

(***ii***) Compute the pulback $f^*\sigma$ and show that $f^*\sigma \in \mathcal{F}^2(N)$.

(***iii***) Compute $f^*\sigma(X,Y)$, where $X, Y \in \mathcal{F}(N)$.

Differential forms

☞ **80** Given the differential form $\sigma = dx^1 \wedge d^{x2} - dx^2 \wedge dx^3$ on $\mathbb{R}^3$. For each of the following vector fields X compute $i_X\sigma$:

(***i***) $X = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}$;

(***ii***) $X = x_1 \frac{\partial}{\partial x_1} - x_2^2 \frac{\partial}{\partial x_3}$;

(***iii***) $X = x_1 x_2 \frac{\partial}{\partial x_1} - \sin(x_3) \frac{\partial}{\partial x_2}$;

☞ **81** Given the mapping

$$f(p) = (\sin(p_1 + p_2), p_1^2 - p_2),$$

acting from $\mathbb{R}^2$ to $\mathbb{R}^2$ and the 2-form $\sigma = p_1^2 dx^1 \wedge dx^2$. Compute the pullback form $f^*\sigma$.

☞ **82** Prove Lemma 10.2.

☞ **83** Derive Formula (10).

Orientations

☞ **84** Show that S^n is orientable and give an orientation.

85 Show that the standard n-torus $\mathbb{T}^n$ is orientable and find an orientation.

☞ **86** Prove that the Klein bottle and the projective space $P\mathbb{R}^2$ are non-orientable.

☞ **87** Give an orientation for the projective space $P\mathbb{R}^3$.

88 Prove that the projective spaces $P\mathbb{R}^n$ are orientable for $n = 2k+1$, $k \geq 0$.

89 Show that the projective spaces $P\mathbb{R}^n$ are non-orientable for $n = 2k$, $k \geq 1$.

Integration on manifolds

Integrating m-forms on $\mathbb{R}^m$

☞ **90** Let $U \subset \mathbb{R}^m$ be open and let $K \subset U$ be compact. Show that there exists a domain of integration D such that $K \subset D \subset \mathbb{R}^m$.

91 Show that Definition 12.2 is well-posed.

Partitions of unity

92 Show that the function f_1 defined in Lemma 13.3 is smooth.

☞ **93** If $\mathcal{U}$ is open covering of M for which each set U_i intersects only finitely many other sets in $\mathcal{U}$, prove that $\mathcal{U}$ is locally finite.

94 Give an example of uncountable open covering $\mathcal{U}$ of the interval $[0,1] \subset \mathbb{R}$, and a countable refinement $\mathcal{V}$ of $\mathcal{U}$.

Integration of m-forms on m-dimensional manifolds

☞ **95** Let $S^2 = \partial B^3 \subset \mathbb{R}^3$ oriented via the standard orientation of $\mathbb{R}^3$, and consider the 2-form

$$\omega = x dy \wedge dz + y dz \wedge dx + z dx \wedge dy.$$

Given the parametrization

$$F(\varphi, \theta) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \sin(\varphi)\cos(\theta) \\ \sin(\varphi)\sin(\theta) \\ \cos(\varphi) \end{pmatrix},$$

for S^2, compute $\int_{S^2} \omega$.

☞ **96** Given the 2-form $\omega = x dy \wedge dz + z dy \wedge dx$, show that

$$\int_{S^2} \omega = 0.$$

☞ **97** Consider the circle $S^1 \subset \mathbb{R}^2$ parametrized by

$$F(\theta) = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos(\theta) \\ r\sin(\theta) \end{pmatrix}.$$

Compute the integral over S^1 of the 1-form $\omega = xdy - ydx$.

98 If in the previous problem we consider the 1-form

$$\omega = \frac{x}{\sqrt{x^2+y^2}}dy - \frac{y}{\sqrt{x^2+y^2}}dx.$$

Show that $\int_{S^1} \omega$ represents the induced Euclidian length of S^1.

☞ **99** Consider the embedded torus

$$\mathbb{T}^2 = \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 \ : \ x_1^2+x_2^2 = 1, \quad x_3^2+x_4^2 = 1\}.$$

Compute the integral over $\mathbb{T}^2$ of the 2-form

$$\omega = x_1^2 dx_1 \wedge dx_4 + x_2 dx_3 \wedge dx_1.$$

☞ **100** Consider the following 3-manifold M parametrized by $g : [0,1]^3 \to \mathbb{R}^4$,

$$\begin{pmatrix} r \\ s \\ t \end{pmatrix} \mapsto \begin{pmatrix} r \\ s \\ t \\ (2r-t)^2 \end{pmatrix}.$$

Compute

$$\int_M x_2 dx_2 \wedge dx_3 \wedge dx_4 + 2x_1x_3 dx_1 \wedge dx_2 \wedge dx_3.$$

The exterior derivative

☞ **101** Let $(x,y,z) \in R^3$. Compute the exterior derivative $d\omega$, where ω is given as:

(***i***) $\omega = e^{xyz}$;

(***ii***) $\omega = x^2 + z\sin(y)$;

(***iii***) $\omega = xdx + ydy$;

(***iv***) $\omega = dx + xdy + (z^2 - x)dz$;

(***v***) $\omega = xydx \wedge dz + zdx \wedge dy$;

(***vi***) $\omega = dx \wedge dz$.

☞ **102** Which of the following forms on $\mathbb{R}^3$ are closed?

(***i***) $\omega = x dx \wedge dy \wedge dz$;

(***ii***) $\omega = z dy \wedge dx + x dy \wedge dz$;

(***iii***) $\omega = x dx + y dy$;

(***iv***) $\omega = z dx \wedge dz$.

☞ **103** Verify which of the following forms ω on $\mathbb{R}^2$ are exact, and if so write $\omega = d\sigma$:

(***i***) $\omega = x dy - y dx$;

(***ii***) $\omega = x dy + y dx$;

(***iii***) $\omega = dx \wedge dy$;

(***iv***) $\omega = (x^2 + y^3) dx \wedge dy$;

☞ **104** Verify which of the following forms ω on $\mathbb{R}^3$ are exact, and if so write $\omega = d\sigma$:

(***i***) $\omega = x dx + y dy + z dz$;

(***ii***) $\omega = x^2 dx \wedge dy + z^3 dx \wedge dz$;

(***iii***) $\omega = x^2 y dx \wedge dy \wedge dz$.

105 Verify that on $\mathbb{R}^2$ and $\mathbb{R}^3$ all closed k-forms, $k \geq 1$, are exact.

106 Find a 2-form on $\mathbb{R}^3 \backslash \{(0,0,0)\}$ which is closed but not exact.

Stokes' Theorem

107 Let $\Omega \subset \mathbb{R}^3$ be a parametrized 3-manifold,i.e. a solid, or 3-dimensional domain. Show that the standard volume of Ω is given by

$$\mathrm{Vol}(M) = \frac{1}{3} \int_{\partial\Omega} x dy \wedge dz - y dx \wedge dz + z dx \wedge dy.$$

108 Let $\Omega \subset \mathbb{R}^n$ be an n-dimensional domain. Prove the analogue of the previous problem.

☞ **109** Prove the 'integration by parts' formula

$$\int_M f d\omega = \int_{\partial M} f\omega - \int_M df \wedge \omega,$$

where f is a smooth function and ω a k-form.

110 Compute the integral

$$\int_{S^2} x^2 y dx \wedge dz + x^3 dy \wedge dz + (z - 2x^2) dx \wedge dy,$$

where $S^2 \subset \mathbb{R}^3$ is the standard 2-sphere.

111 Use the standard polar coordinates for the $S^2 \subset \mathbb{R}^3$ with radius r to compute

$$\int_{S^2} \frac{xdy \wedge dz - ydx \wedge dz + zdx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}},$$

and use the answer to compute the volume of the r-ball B_r.

Extra's

112 Use the examples in Section 16 to show that

$$\text{curl grad } f = 0, \qquad \text{and} \qquad \text{div curl } F = 0,$$

where $f : \mathbb{R}^3 \to \mathbb{R}$, and $F : \mathbb{R}^3 \to \mathbb{R}^3$.

☞ **113** Given the mapping $f : \mathbb{R}^2 \to \mathbb{R}^3$, $\omega = dx \wedge dz$, and

$$f(s,t) = \begin{pmatrix} \cos(s)\sin(t) \\ \sqrt{s^2 + t^2} \\ st \end{pmatrix}.$$

Compute $f^*\omega$.

☞ **114** Given the mapping $f : \mathbb{R}^3 \to \mathbb{R}^3$, $\omega = xydx \wedge dy \wedge dz$, and

$$f(s,t,u) = \begin{pmatrix} s\cos(t) \\ s\sin(t) \\ u \end{pmatrix}.$$

Compute $f^*\omega$.

☞ **115** Let $M = \mathbb{R}^3 \backslash \{(0,0,0)\}$, and define the following 2-form on M:

$$\omega = \frac{1}{(x^2 + y^2 + z^2)^{3/2}} \big(xdy \wedge dz + ydz \wedge dx + zdx \wedge dy \big).$$

(***i***) Show that ω is closed ($d\omega = 0$) (Hint: compute $\int_S^2 \omega$).

(***ii***) Prove that ω is not exact on M!

On $N = \mathbb{R}^3 \backslash \{x = y = 0\} \supset M$ consider the 1-form:

$$\eta = \frac{-z}{(x^2 + y^2 + z^2)^{1/2}} \frac{xdy - ydx}{x^2 + y^2}.$$

(***iii***) Show that ω is exact as 2-form on N, and verify that $d\eta = \omega$.

☞ **116** Let $M \subset \mathbb{R}^4$ be given by the parametrization

$$(s,t,u) \mapsto \begin{pmatrix} s \\ t+u \\ t \\ s-u \end{pmatrix}, \qquad (s,t,u) \in [0,1]^3.$$

(***i***) Compute $\int_M dx_1 \wedge dx_2 \wedge dx_4$.

(***ii***) Compute $\int_M x_1 x_3 dx_1 \wedge dx_2 \wedge dx_3 + x_3^2 dx_2 \wedge dx_3 \wedge dx_4$.

☞ **117** Let $C = \partial\Delta$ be the boundary of the triangle OAB in $\mathbb{R}^2$, where $O = (0,0)$, $A = (\pi/2, 0)$, and $B = (\pi/2, 1)$. Orient the traingle by traversing the boundary counter clock wise. Given the integral

$$\int_C (y - \sin(x))dx + \cos(y)dy.$$

(***i***) Compute the integral directly.

(***ii***) Compute the integral via Green's Theorem.

118 Compute the integral $\int_\Sigma F \cdot dS$, where $\Sigma = \partial[0,1]^3$, and

$$F(x,y,z) = \begin{pmatrix} 4xz \\ -y^2 \\ yz \end{pmatrix}$$

(Hint: use the Gauss Divergence Theorem).

De Rham cohomology

Definition of De Rham cohomology

119 Prove Theorem 17.3.

120 Let $M = \bigcup_j M_j$ be a (countable) disjoint union of smooth manifolds. Prove the isomorphism

$$H^k_{\mathrm{dR}}(M) \cong \prod_j H^k_{\mathrm{dR}}(M_j).$$

121 Show that the mapping $\cup : H^k_{\mathrm{dR}}(M) \times H^\ell_{\mathrm{dR}}(M) \to H^{k+\ell}_{\mathrm{dR}}(M)$, called the ***cup-product***, and defined by

$$[\omega] \cup [\eta] := [\omega \wedge \eta],$$

is well-defined.

122 Let $M = S^1$, show that

$$H^0_{\mathrm{dR}}(S^1) \cong \mathbb{R}, \quad H^1_{\mathrm{dR}}(S^1) \cong \mathbb{R}, \quad H^k_{\mathrm{dR}}(S^1) = 0,$$

for $k \geq 2$.

123 Show that

$$H^0_{\mathrm{dR}}(\mathbb{R}^2 \backslash \{(0,0)\}) \cong \mathbb{R}, \quad H^1_{\mathrm{dR}}(\mathbb{R}^2 \backslash \{(0,0)\}) \cong \mathbb{R}, \quad H^k_{\mathrm{dR}}(\mathbb{R}^2 \backslash \{(0,0)\}) = 0,$$

for $k \geq 2$.

124 Find a generator for $H^1_{\mathrm{dR}}(\mathbb{R}^2 \backslash \{(0,0)\})$.

125 Compute the de Rham cohomology of the n-torus $M = \mathbb{T}^n$.

126 Compute the de Rham cohomology of $M = S^2$.

VII. Solutions

1 Note that $(x,y) \in g(\mathbb{R})$ if and only if $x^2+y^2=1$. So $g(\mathbb{R})$ is just the circle S^1 in the plane. The fact that this is a manifold is in the lecture notes.

2 We will show that f has an inverse by showing that the funcrion suggested in the exercise is indeed the inverse of f. It follows that f is a bijection and therefore it is onto. By continuity of f and its inverse, it follows that it is a homeomorphism. We first show that f maps $S^1 \times S^1$ into $\mathbb{T}^2$: We use the parametrisation of $S^1 \times S^1$ given by

$$(s,t) \mapsto (\cos s, \sin s, \cos t, \sin t),$$

where $s,t \in [0,2\pi)$. Suppose that $q = (\cos s, \sin s, \cos t, \sin t) \in S^1 \times S^1$. Then

$$f(q) = (\cos s(2+\cos t), \sin s(2+\cos t), \sin t),$$

so $f(q) \in \mathbb{T}^2$ if and only if **LHS = RHS** where

$$\begin{aligned}
\textbf{(LHS)} &= 16(\cos^2 s(2+\cos t)^2 + \sin^2 s(2+\cos t)^2), \\
\textbf{(RHS)} &= (\cos^2 s(2+\cos t)^2 + \sin^2 s(2+\cos t)^2 + \sin^2 t + 3)^2.
\end{aligned}$$

We have

$$\begin{aligned}
\textbf{LHS} &= 16(2+\cos t)^2 \\
\textbf{RHS} &= ((2+\cos t)^2 + \sin^2 t + 3)^2 \\
&= (4 + 4\cos t + \cos^2 t + \sin^2 t + 3)^2 \\
&= (4(2+\cos t)^2 = 16(2+\cos t).
\end{aligned}$$

This shows that f maps $S^1 \times S^1$ into $\mathbb{T}^2$. Now we let $g : \mathbb{T}^2 \to S^1 \times S^1$ be the suggested inverse; i.e.

$$g(p) = \left(\frac{p_1}{r}, \frac{p_2}{r}, r-2, p_3\right),$$

where $r = \frac{|p|^2+3}{4}$. We will first show that g maps $\mathbb{T}^2$ into $S^1 \times S^1$. So we let $p \in \mathbb{T}^2$ and we will show that $g(p) \in S^1 \times S^1$. But this is the case if and only if $p_1^2 + p_2^2 = r^2$ and $(r-2)^2 + p_3^2 = 1$. The first equality follows from the definition of r and the fact that $p \in \mathbb{T}^2$. For the second equality we note that

$$\begin{aligned}
(r-2)^2 + p_3^2 &= r^2 - 4r + 4 + p_3^2 \\
&= p_1^2 + p_2^2 - 4r + 4 + p_3^2 \\
&= (p_1^2 + p_2^2 + p_3^2 + 3) - (|p|^2 + 3) + 1 \\
&= 1.
\end{aligned}$$

This shows that g maps $\mathbb{T}^2$ into $S^1 \times S^1$. We will now show that g is the inverse of f; let $p \in \mathbb{T}^2$ and $q \in S^1 \times S^1$. Then

$$\begin{aligned}
f \circ g(p) &= f\Big(\frac{p_1}{r}, \frac{p_2}{r}, r-2, p_3\Big) \\
&= \Big(\frac{p_1}{r} \cdot r, \frac{p_2}{r} \cdot r, p_3\Big) = p \\
g \circ f(q) &= g(q_1(2+q_3), q_2(2+q_3), q_4) \\
&= \Big(\frac{q_1(2+q_3)}{r'}, \frac{q_2(2+q_3)}{r'}, r'-2, q_4\Big) \\
\text{where } r' &= \frac{q_1^2(2+q_3)^2 + q_2^2(2+q_3)^2 + q_4^2 + 3}{4} \\
&= {}^1\!/\!_4 \cdot \big\{(2+q_3)^2 + q_4^2 + 3\big\} \\
&= {}^1\!/\!_4 \cdot \big\{4 + 4q_3 + q_3^2 + q_4^2\big\} \\
&= {}^1\!/\!_4 \cdot \big\{4(2+q_3)\big\} = 2 + q_3 \\
\text{and therefore } g \circ f(q) &= (q_1, q_2, q_3, q_4) = q.
\end{aligned}$$

This show that f is a bijection, so in particular it is onto. Since f and g are both continuos, it follows that f is a homeomorphism.

3 (***i***) Given co-ordinate charts (U, φ) and (V, ψ) for S^1, we define a new co-ordinate chart for $\mathbb{T}^2$ as follows; $W = f(U) \times f(V)$, $\xi : W \to \varphi(U) \times \psi(V)$ is defined by $\xi(p) = (\varphi(x), \psi(y))$ where $(x, y) = f^{-1}(p)$ and $p \in W$. This shows how to make an atlas for $\mathbb{T}^2$ using an atlas for S^1.
(***ii***) A parametrisation for $S^1 \times S^1$ was already given in the solution of 2. Using the map f, we obtain the following parametrisation for $\mathbb{T}^2$;

$$(s,t) \mapsto \big(\cos s \cdot (2 + \cos t), \sin s \cdot (2 + \cos t), \sin t\ \big),$$

where $s, t \in [0, 2\pi)$.

5 (***i***) If (V, ψ) is a co-ordinate chart for M, then let $W = U \cap V$. Let $\xi : W \to \psi(W)$ be defined by $\xi(x) = \psi(x)$, i.e. ξ is just the restriction of ψ to W. Then clearly ξ is a homeomorphism from W to an open subset of $\psi(V)$. In particular, if $\psi(V)$ is

m-dimensional, then so is $\xi(W)$. This shows how to make an atlas for U, given an atlas for M.

(***ii***) Let (U,φ) and (V,ψ) be charts for N and M respectively. The map $\xi : U \times V \to \varphi(U)\times\psi(V)$ is defined by $\xi(x,y) = (\varphi(x),\psi(y))$. This is clearly a homeomorphism. This shows how to make an atlas for $N \times M$, given atlases for N and M. Note that if N is n-dimensional and M is m-dimensional, then $N \times M$ is $n+m$-dimensional.

6 (***i***) We always let A be a matrix of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

We let $M = \{A \in M_{2,2}(\mathbb{R}) | \det A = 1\}$ and $U = \{A \in M : a \neq 0\}$. Note that U is an open subset of M. We now define $\varphi : U \to (\mathbb{R}\setminus\{0\})\times\mathbb{R}\times\mathbb{R}$ by

$$\varphi(A) = (a,b,c).$$

The inverse of φ is given by

$$\varphi^{-1}(x,y,z) = \begin{pmatrix} x & y \\ z & \frac{1+yz}{x} \end{pmatrix}.$$

Cleary, φ and its inverse are continuous, and therefore φ is a homeomorphism. So (φ,U) is a co-ordinate chart for M. In a similar way we can define charts on sets in M where $b \neq 0$, $c \neq 0$ and $d \neq 0$. These four sets cover all of M and so we obtain an atlas for M which shows that M is a manifold. Since the range of φ is 3-dimensional, it follows that M has dimension 3.

(***ii***) Note that $\det : M_{n,n}(\mathbb{R}) \to \mathbb{R}$ is a continuous function and that $Gl(n,\mathbb{R}) = \det^{-1}(\mathbb{R}\setminus\{0\})$. It follows that $Gl(n,\mathbb{R})$ is an open subset of the space $M_{n,n}(\mathbb{R})$. But $M_{n,n}(\mathbb{R})$ is homeomorphic to $\mathbb{R}^{n^2}$, which is a manifold, and therefore $Gl(n,\mathbb{R})$ is an open subset of a manifold. It follows from 5 that this space is a manifold. The dimension of $Gl(n,\mathbb{R})$ is n^2.

12 The space M is not second countable; observe that every second countable space is separable. To see that M is not separable, note that it contains an uncountable collection of non-empty pairwise disjoint open subsets.

14 Let M and N be differentiable manifolds with maximal atlases $\mathcal{A} = \{(U_i,\varphi_i)\}_{i\in I}$ and $\mathcal{B} = \{(V_k,\psi_k)\}_{k\in K}$ respectively. A typical co-ordinate chart for $M \times N$ is given by $\varphi_i \times \psi_k : U_i \times V_k \to \mathbb{R}^{m+n}$.

Now, for $i, j \in I$ and $k, l \in K$, consider the transition map $(\varphi_i \times \psi_k) \circ (\varphi_j \times \psi_l)^{-1}$. This os just the product of the maps $\varphi_i \circ \varphi_j^{-1}$ and $\psi_j \circ \psi_k^{-1}$. Since these maps are diffeomorphisms, their product is also a diffeomorphisms.

This shows that the atlas for $M \times N$ given by $\{(U_i \times V_k, \varphi_i \times \psi_k)_{i \in I, k \in K}$ is a C^∞-atlas. Taking a maximal extension of this atlas endows $M \times N$ with a differentiable structure.

15 As usual, $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to P\mathbb{R}^n$ is the quotient mapping, given by $\pi(x) = [x]$. For $i = 1, \ldots, n+1$, we have $V_i = \{x \in \mathbb{R}^{n+1} \setminus \{0\} : x_i \neq 0\}$ and $U_i = \pi(V_i)$. For $[x] \in U_i$ we define

$$\varphi_i([x]) = \left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{n+1}}{x_i}\right),$$

and its inverse is given by

$$\varphi_i^{-1}(z_1, \ldots, z_n) = [(z_1, \ldots, z_{i-1}, 1, z_i, \ldots, z_n)].$$

We will show that for $i < j$, the transition map $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_i)$ is a diffeomorphism. We compute $\varphi_i \circ \varphi_j^{-1}$;

$$\begin{aligned} \varphi_i \circ \varphi_j^{-1}(z_1, \ldots, z_n) &= \varphi_i([z_1, \ldots, z_{j-1}, 1, z_j, \ldots, z_n]) \\ &= \left(\frac{z_1}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_{j-1}}{z_i}, \frac{1}{z_i}, \frac{z_j}{z_i}, \ldots \frac{z_n}{z_i}\right). \end{aligned}$$

This function is clearly a diffeomorphisms. Note that

$$\begin{aligned} \varphi_i(U_i \cap U_j) &= \{z \in \mathbb{R}^n : z_{j-1} \neq 0\}, \\ \varphi_j(U_i \cap U_j) &= \{z \in \mathbb{R}^n : z_i \neq 0\}. \end{aligned}$$

17 All the atlases are compatible. We will compute some of the transition maps. We list three co-ordinate charts, one from each atlas;

(1) $\varphi : U \to U'$, where $U = \{p \in S^1 : p_2 > 0\}$ and $U' = (-1, 1)$.

$$\varphi(p) = p_1 \qquad \text{and} \qquad \varphi^{-1}(x) = (x, \sqrt{1 - x^2}).$$

(2) $\psi : V \to V'$, where $V' = \{p \in S^1 : p \neq (0, 1)\}$ and $V' = \mathbb{R}$.

$$\psi(p) = \frac{2p_1}{1 - p_2} \qquad \text{and} \qquad \psi^{-1}(x) = \left(\frac{4x}{x^2 + 4}, \frac{x^2 - 4}{x^2 + 4}\right).$$

(3) $\xi : W \to W'$, where $W = \{p \in S^1 : p \neq (1, 0)\}$ and $W' = (0, 2\pi)$.

$$\xi(p) = \arccos p_1 \qquad \text{and} \qquad \xi^{-1}(\theta) = (\cos\theta, \sin\theta).$$

Now we compute some of the transition maps;

(1) The map $\psi \circ \varphi^{-1} : (-1,1) \setminus \{0\} \to (-\infty,-2) \cup (2,\infty)$ is given by

$$x \mapsto \frac{2x}{1-\sqrt{1-x^2}}.$$

(2) The map $\varphi \circ \psi^{-1} : (-\infty,-2) \cup (2,\infty) \to (-1,1) \setminus \{0\}$ is given by

$$x \mapsto \frac{4x}{x^2+4}.$$

(3) The map $\varphi \circ \xi^{-1} : (0,\pi) \to (-1,1)$ is given by

$$\theta \mapsto \cos\theta.$$

All the functions mentioned above are C^∞-functions. If we check this for all transition maps, then we have shown that the atlases are compatible.

18 We have proved in 15 that the product of differentiable manifolds is again a differentiable manifolds. The C^∞-atlas constructed there gives a C^∞-atlas for $S^1 \times S^1$. We have seen in 2 that $S^1 \times S^1$ is homeomorphic to the torus $\mathbb{T}^2$ and the map $f : S^1 \times S^1 \to \mathbb{T}^2$ is a homeomorphism. In this exercise we are asked to show that f is even a diffeomorphism. The differentiable structure of $\mathbb{T}^2$ is the structure which it inherits from $\mathbb{R}^3$.

Note that using the map f we can endow $\mathbb{T}^2$ with a differentiable structure inherited from $S^1 \times S^1$. So we have two differentiable structures on $\mathbb{T}^2$; the structure inherited from $\mathbb{R}^3$ and the structure inherited from $S^1 \times S^1$. This exercise amounts to saying that $\mathbb{T}^2$ endowed with the structure inherited from $S^1 \times S^1$ is a smooth embedded submanifold of $\mathbb{R}^3$.

Fixing appropriate charts for $S^1 \times S^1$ and $\mathbb{T}^2$ one can show that f is a diffeomorphism. For example, consider $U \subset S^1 \times S^1$ given by

$$\{q \in S^1 \times S^1 \,|\, q_1 > 0 \,\&\, q_3 > 0\,\}$$

with $\varphi : U \to (-1,1) \times (-1,1)$ given by $\varphi(q) = (q_2,q_4)$. For $\mathbb{T}^2$ we fix a co-ordinate chart (V,ψ) such that $V \subset f(U)$ and ψ is a projection onto the 2^{nd} and 3^{rd} co-ordinate. Computing $\psi \circ f \circ \varphi^{-1}$ gives;

$$\psi \circ f \circ \varphi^{-1}(x,y) = (x(2+\sqrt{1-y^2}),y).$$

This map is a diffeomorphism. For other co-ordinate charts this is similar.

20 Let M, N and O be differentiable manifolds. To show that 'diffeomorphic' is an equivalence relation, we prove ***reflexivity***, ***symmetry*** and ***transitivity***.

For ***reflexivity***: M is diffeomorphic to M since the transition maps are diffeomorphism. So the identity is a diffeomorphism from M onto M.

For ***symmetry***: The inverse of a diffeomorphism is a diffeomorphism.

For ***transitivity***: Suppose $f : M \to N$ and $g : N \to O$ are diffeomorphisms and let $p \in M$. Fix charts (U,φ), (V,ψ), (V',ψ') and (W,ξ) such that $p \in U$, $f(p) \in V \cap V'$ and $g \circ f(p) \in W$. Since f and g are diffeomorphisms, we may asssume that $\psi \circ f \circ \varphi^{-1}$ and $\xi \circ g \circ (\psi')^{-1}$ are diffeomorphims. Also note that since N is a differentiable manifold, the map $\psi' \circ \psi^{-1}$ is a diffeomorphism. Now note that

$$\xi \circ g \circ f \circ \varphi^{-1} = (\xi \circ g \circ (\psi')^{-1}) \circ (\psi' \circ \psi^{-1}) \circ (\psi \circ f \circ \varphi^{-1}).$$

The map on the right hand side is a composition of diffeomorphisms, so it is again a diffeomorphism. It follows that the map on the left hand side is a diffeomorphism. Since $p \in M$ was arbitrary, this show that the composition of diffeomorphisms between manifolds is again a diffeomorphism; i.e. M and O are diffeomorphic. In this proof we have implicitly used part (iii) of Theorem 2.14.

21 See Lee, pg 33&34.

22 Recall that $f : S^1 \times S^1 \to \mathbb{R}^3$. We fix a co-ordinate chart (U,φ) on $S^1 \times S^1$ where

$$U = \{q = (q_1,q_2,q_3,q_4) | q_2 > 0 \ \& \ q_4 > 0\},$$

and $\varphi : U \to V$ is defined by $\varphi(q_1,q_2,q_3,q_4) = (q_1,q_3)$ where $V = (-1,1)\times(-1,1)$. The chart $(\psi,\mathbb{R}^3)$ is just the identity. We express f in local co-ordinates, i.e. $\tilde{f} = \psi \circ f \circ \varphi^{-1}$ where

$$\tilde{f}(x,y) = \big(x(1+y), \sqrt{1-x^2}(2+y), \sqrt{1-y^2}\big).$$

We have that

$$J\tilde{f}_{x=\varphi(p)} = \begin{pmatrix} 1+y & x \\ -x(2+y)(1-x^2)^{1/2} & (1-x^2)^{1/2} \\ 0 & -y(1-y^2)^{1/2} \end{pmatrix}$$

It is not hard to verify that the matrix $J\tilde{f}_{x=\varphi(p)}$ has rank 2 for all $p \in U$. For example, if $p = \varphi^{-1}(0,0)$, then

$$J\tilde{f}_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

This shows that f is of rank 2 at all $p \in U$. Of course, we can also prove this for other similar charts. So it follows that $\text{rk}(f) = 2$, so it is an immersion. Since f is a homeomorphism onto its image, it follows that f is a smooth embedding.

25 We compute g in local co-ordinates. So fix the usual charts (U_i, φ) and (U_j, ψ_j) for $P\mathbb{R}^n$ and $P\mathbb{R}^k$ respectively. For $z \in \mathbb{R}^n$, we let z^i be the point in $\mathbb{R}^{n+1}$ given by $(z_1, \ldots, z_{i-1}, 1, z_i, \ldots z_n)$. If $f_j(z^i) \neq 0$, then

$$\begin{aligned} \tilde{g}(z) = \psi_j \circ g \circ \varphi_i^{-1}(z) &= \psi([f(z^i)]) \\ &= \Big(\frac{f_1(z^i)}{f_j(z^i)}, \ldots, \frac{f_{j-1}(z^i)}{f_j(z^i)}, \frac{f_{j+1}(z^i)}{f_j(z^i)}, \ldots, \frac{f_n(z^i)}{f_j(z^i)}\Big). \end{aligned}$$

By assumption, $f(z^i) \neq 0$, so there is some j such that $f_j(z^i) \neq 0$. The fact that $\tilde{g}$ is smooth follows from the smoothness of all co-ordinate functions f_k of f.

26 The mapping $f : B^n \to S^n$ defined by

$$f(x) = \frac{1}{1 - |x|} \cdot x,$$

is a diffeomorphism. Note that one can chose charts for B^n and $\mathbb{R}^n$ such that the co-ordinate maps are just the identity. So the expression $\tilde{f}$ of f in local co-ordinates is equal to f.

27 We first note that

$$Jf_{(x,y,s)} = \begin{pmatrix} 2x & 1 & 0 \\ 2x & 2y+1 & 2s \end{pmatrix}$$

Now consider a point $p = (x, y, s) \in f^{-1}(q)$. Then $x^2 + y = 0$ and $y^2 + s^2 = 1$. If $s \neq 0$, then the last two columns of Jf_p are independent. If $s = 0$, then $y \in \{-1, 1\}$ and since $x^2 + y = 0$, we have in fact that $y = -1$ and $x^2 = 1$. In this case the first two colums on Jf_p are independent. Sow e have shown that if $p \in f^{-1}(q)$, then $\mathrm{rk}(Jf_p) = 2$ and it follows that q is a regular value.

To show that $f^{-1}(q)$ is diffeomorphic to S^1, consider the set

$$A = \{(x, y, s) \in \mathbb{R}^3 |\ x^4 + s^2 = 1 \ \&\ y = 0\}.$$

Then A is diffeomorphic to S^1, see 23. Now consider the map $g : A \to \mathbb{R}^3$ defined by

$$f(x, y, s) = (x, -x^2, s).$$

Then g is a constant rank map with $\mathrm{rk}(g) = 1$ and g is injective. So g is a smooth embedding of A (and thus of S^1) into $\mathbb{R}^3$. It follows that $g[A] = f^{-1}(q)$ is diffeomorphic to S^1.

32 Let $U_i = \{x \in S^n : x_i > 0\}$ and φ be the projection on all co-ordinates but the i^{th}. We have that

$$\varphi^{-1}(z) = (z_1, \ldots, z_{i-1}, \sqrt{1 - |z|^2}, z_i, \ldots, z_n).$$

where $|z| < 1$. Next we let $V_i = \pi[U_i]$ and

$$\psi_i([x]) = \left(\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_{n+1}}{x_i}\right).$$

We calculate f in local co-ordinates, so $\tilde{f} = \psi \circ f \circ \varphi^{-1}$,

$$\tilde{f}(z) = \frac{1}{\sqrt{1-|z|^2}} \cdot (z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n).$$

We note that

$$\frac{\partial}{\partial z_i} \frac{z_j}{\sqrt{1-|z|^2}} = \begin{cases} \frac{1+2z_i}{(1-|z|^2)^{3/2}} & i = j \\ \\ \frac{2z_i}{(1-|z|^2)^{3/2}} & i \neq j \end{cases}$$

so

$$J\tilde{f}_z = \left(1-|z|^2\right)^{3/2} \begin{pmatrix} 1+2z_1 & 2z_1 & \cdots & 2z_1 \\ 2z_2 & 1+2z_2 & \cdots & 2z_2 \\ \vdots & \vdots & \ddots & \vdots \\ 2z_n & \cdots & 2z_n & 1+2z_n \end{pmatrix}$$

For all z with $|z| < 1$, this matrix has rank n.

33 We let $f_a : \mathbb{R}^2 \to \mathbb{R}$ be given by $f_a(x,y) = y^2 - x(x-1)(x-a)$ and note that:

$$f_a(x,y) = y^2 - x^3 + (a+1)x^2 - ax.$$

Of course, $M_a = f^{-1}(0)$. We also have that:

$$J\tilde{f}_a|_{(x,y)} = (-3x^2 + 2(a+1)x - a, 2y).$$

We compute the rank of this matrix for various values of a and $(x,y) \in M_a$;

$$\begin{array}{lll} \text{If} \quad x \notin \{0,1,a\}: & \text{then } y \neq 0 & \to \text{the rank is 1} \\ & & \\ x = 0: & -3x^2 + 2(a+1)x - a = -a & \to \text{the rank is 1 iff } a \neq 0 \\ x = 1: & -3x^2 + 2(a+1)x - a = a - 1 & \to \text{the rank is 1 iff } a \neq 1 \\ x = a: & -3x^2 + 2(a+1)x - a = a(a-1) & \to \text{the rank is 1 iff } a(a-1) = 0 \end{array}$$

So we conclude that if $a \notin \{0,1\}$, then M_a is an embedded submanifold of $\mathbb{R}^2$.

For (***ii***) : this is possible for all $a \in \mathbb{R}$.

34 So the question really is; which values $q \in \mathbb{R}$ are regular values of f? For this, we compute the Jacobian;

$$J\tilde{f}|_{(x,y)} = (3x^2 + y, 3y^2 + x).$$

Let $q \in \mathbb{R}$ and $(x,y) \in f^{-1}(q)$. The rank of $Jf|_{(x,y)}$ is 0 if and only if:

$$3x^2 + y = 3y^2 + x = 0.$$

In this case we have

$$\begin{aligned} y &= -3x^2 \\ y^2 &= -x/3 \end{aligned}$$

It follows from these equations that $y^3 = x^3$ and therefore $x = y$. Furthermore:

$$\begin{aligned} 0 &= 3x^2 + y = 3(-3y^2)^2 + y = \\ &= 27y^4 + y = y(27y^3 + 1). \end{aligned}$$

We conclude that (x,y) is either $(0,0)$ or $(-1/3,-1/3)$. So $q \in \{0, 1/27\}$.

We conclude that if $q \notin \{0, 1/27\}$, then q is a regular value of f and therefore $f^{-1}(q)$ is a smooth embedded submanifold of $\mathbb{R}^2$. The level sets corresponding to $q = 0$ and $q = 1/27$ are not embedded submanifolds.

36 NO; consider the mapping $f : \mathbb{R}^3 \to \mathbb{R}^2$ given by $f(x,y,z) = (x^2 + y^2, z^2)$. We let $q = (1,0)$ and $S = f^{-1}(q)$. Note that

$$S = \{(x,y,z) \in \mathbb{R}^3 | x^2 + y^2 = 1 \ \& \ z = 0\}.$$

So S is just a circle embedded in $\mathbb{R}^3$. The co-dimension of S is 2. The Jacobian of f in $p = (x,y,z)$ is given by

$$Jf|_p = \begin{pmatrix} 2x & 2y & 0 \\ 0 & 0 & 2z \end{pmatrix}$$

So for example, at the point $p = (1,0,0) \in S$, this matrix is of rank 1 and not of rank 2. So q is not a regular value of f.

37 Let $N = \mathbb{R}^4 \setminus \{p | p_1^2 + p_2^2 = p_3^2 + p_4^2 = 0\}$ and consider the mapping $f : N \to \mathbb{R}^2$ given by $f(p) = (p_1^2 + p_2^2, p_3^2 + p_4^2)$. The Jacobian of f at the point p is given by

$$Jf|_p = \begin{pmatrix} 2p_1 & 2p_2 & 0 & 0 \\ 0 & 0 & 2p_3 & 2p_4 \end{pmatrix}$$

On N, this matrix is always of rank 2, so f is a constant rank mapping with $\mathrm{rk}(f) = 2$. The 2-torus in $\mathbb{R}^4$ is the level set $f^{-1}(1,1)$, so it follows from Theorem 3.27 that this space is a smooth embedded submanifold of N. Since N is an open subset of $\mathbb{R}^4$, it follows that N, and hence the 2-torus, is a smooth embedded submanifold of $\mathbb{R}^4$.

41 Fix $p \in f^{-1}(y)$ and as in the proof of Theorem 3.28, construct a map $g : \mathbb{R}^n \to \mathbb{R}^{n-m} \times \mathbb{R}^m$ where

$$g(\xi) = (L\xi, f(\xi) - y),$$

where $L : \mathbb{R}^n \to \mathbb{R}^{n-m}$ is invertible on $\ker Jf|_p \subset \mathbb{R}^n$. We have that

$$Jg|_p = L \oplus Jf|_p,$$

and this map is onto. As in the proof of Theorem 3.28, g is invertible on a neighbourhood V of $(L(p), 0)$ such that

$$g^{-1} : \left(\mathbb{R}^{n-m} \times \{0\}\right) \cap V \to f^{-1}(y).$$

This gives a local co-ordinate map φ with

$$\varphi^{-1} : \mathbb{R}^{n-m} \cap V|_{n-m} \to f^{-1}(y) \cap U,$$

where $\varphi(\xi) = L\xi$ for all $\xi \in f^{-1}(y) \cap U$. It follows that $\varphi^{-1} = L^{-1}$ and

$$J\varphi^{-1}|_p = JL^{-1}|_p = L^{-1}.$$

So we have that $JL^{-1}|_p(\mathbb{R}^{n-m}) = \ker Jf|_p$ by construction, so it follows that

$$T_pM = \ker Jf|_p,$$

and this is what we wanted to show.

42 Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(p_1, p_2, p_3) = p_1^3 + p_2^3 + p_3^3 - 3p_1p_2p_3$, so that $M = f^{-1}(1)$. The Jacobian of f at the point p is given by

$$Jf|_p = \begin{pmatrix} 3p_1^2 - 3p_2p_3, & 3p_2^2 - 3p_1p_3, & 3p_3^2 - 3p_1p_2 \end{pmatrix}$$

So we have that $\mathrm{rk}(Jf)|_p = 0$ if and only if:

$$p_1^2 = p_2p_3 \;\&\; p_2^2 = p_1p_3 \;\&\; p_3^2 = p_1p_2.$$

So it follows that if $\mathrm{rk}(Jf)|_p = 0$, then $f(p) = 0$. We conclude that $q = 1$ is a regular value of f and therefore M is 2-dimensional smooth embedded submanifold of $\mathbb{R}^3$.

We now compute T_pM at the point $p = (0,0,1)$. We use 41, so $T_pM = \ker Jf|_p$. We have:

$$Jf|_p = \begin{pmatrix} 0 & 0 & 3 \end{pmatrix},$$

so it follows that

$$T_pM = \{(x,y,z) \in \mathbb{R}^3 : z = 0\}.$$

So this is just the (x,y)-plane.

43 Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $f(p_1, p_2, p_3) = (p_1^2 - p_2^2 + p_1p_3 - 2p_2p_3, 2p_1 - p_2 + p_3)$ so that $M = f^{-1}(0,3)$. The Jacobian of f in p is given by:

$$Jf|_p = \begin{pmatrix} 2p_1 + p_3 & -2p_2 - 2p_3 & p_1 - 2p_2 \\ 2 & -1 & 1 \end{pmatrix}$$

Note that if $p \in f^{-1}(0,3)$, then $p_1^2 + p_1 p_3 = p_2^2 + 2p_2 p_3$ and therefore

$$\begin{aligned} p_1(p_1+p_3) &= p_2(p_2+2p_3) \\ 2p_1+p_3 &= p_2+3. \end{aligned}$$

It is left to the reader to verify that in this case, the rank of the Jacobian of f at p is 2.

We now compute T_pM at the point $p = (1,-1,0) \in M$. In this case we have

$$Jf|_p = \begin{pmatrix} 2 & 2 & 3 \\ 2 & -1 & 1 \end{pmatrix}$$

So, using the fact that $T_pM = \ker Jf|_p$, we have

$$T_pM = \{(x,y,z) \in \mathbb{R}^3 : 2x+2y+3z = 0 \ \& \ 2x-y+z = 0\}$$

This is the line spanned by the vector $(5,4,-6)$.

45 We recall that

$$\begin{aligned} \tfrac{\partial}{\partial x} &= (1,0) & \tfrac{\partial}{\partial r} &= (\cos\theta, \sin\theta) \\ \tfrac{\partial}{\partial y} &= (0,1) & \tfrac{\partial}{\partial \theta} &= (-r\sin\theta, r\cos\theta). \end{aligned}$$

So if $r \neq 0$, then

$$\begin{aligned} \tfrac{\partial}{\partial x} &= \cos\theta \cdot \tfrac{\partial}{\partial r} - \tfrac{1}{r}\sin\theta \cdot \tfrac{\partial}{\partial \theta} \\ \tfrac{\partial}{\partial y} &= \sin\theta \cdot \tfrac{\partial}{\partial r} + \tfrac{1}{r}\cos\theta \cdot \tfrac{\partial}{\partial \theta}. \end{aligned}$$

Also recall that $x = r\cos\theta$ and $y = r\sin\theta$. we obtain the following: $(\boldsymbol{i})$

$$\begin{aligned} x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} &= r\cos\theta(\cos\theta \cdot \frac{\partial}{\partial r} - \frac{1}{r}\sin\theta \cdot \frac{\partial}{\partial \theta}) \\ &\quad + r\sin\theta(\sin\theta \cdot \frac{\partial}{\partial \theta} + \frac{1}{r}\cos\theta \cdot \frac{\partial}{\partial \theta}) \\ &= r\cos^2\theta \cdot \frac{\partial}{\partial r} + r\sin^2\theta \cdot \frac{\partial}{\partial r} \\ &= r\frac{\partial}{\partial r} \end{aligned}$$

(***ii***)

$$\begin{aligned}
-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y} &= -r\sin\theta(\cos\theta\cdot\frac{\partial}{\partial r}-\frac{1}{r}\sin\theta\cdot\frac{\partial}{\partial\theta}) \\
&\quad +r\cos\theta(\sin\theta\cdot\frac{\partial}{\partial r}+\frac{1}{r}\cos\theta\cdot\frac{\partial}{\partial\theta}) \\
&= \sin^2\theta\cdot\frac{\partial}{\partial\theta}+\cos^2\theta\frac{\partial}{\partial\theta} \\
&= \frac{\partial}{\partial\theta}
\end{aligned}$$

(***iii***)

$$\begin{aligned}
(x^2+y^2)\frac{\partial}{\partial x} &= (r^2\cos^2\theta+r^2\sin^2\theta)(\cos\theta\cdot\frac{\partial}{\partial r}-\frac{1}{r}\sin\theta\cdot\frac{\partial}{\partial\theta}) \\
&= r^2\cos\theta\cdot\frac{\partial}{\partial r}-r\sin\theta\cdot\frac{\partial}{\partial\theta}
\end{aligned}$$

48 We let $\alpha = u^2+v^2$ and $\beta = (\alpha+1)^2$. We first compute $d(f\circ g)$. Note that $f\circ g : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$\begin{aligned}
f\circ g(u,v) &= \frac{4u^2+4v^2+(u^2+v^2-1)^2}{\beta} \\
&= \frac{4\alpha+\alpha^2-2\alpha+1}{\beta} \\
&= \frac{\alpha^2+2\alpha+1}{\beta} = \frac{(\alpha+1)^2}{\beta} = 1.
\end{aligned}$$

So we have that $d(f\circ g) = 0$. Next we compute g^*df. Note that $df = 2xdx + 2ydy+2zdz$. We have the following;

$$g^*df = 2\frac{2u}{\alpha+1}d\Big(\frac{2u}{\alpha+1}\Big)+2\frac{2v}{\alpha+1}d\Big(\frac{2v}{\alpha+1}\Big)+2\frac{\alpha-1}{\alpha+1}d\Big(\frac{\alpha-1}{\alpha+1}\Big)$$

We first make the following computations:

$$\begin{aligned}
d\Big(\frac{2u}{\alpha+1}\Big) &= \frac{2(\alpha+1)-4u^2}{\beta}du + \frac{-4uv}{\beta}dv \\
&= \frac{2(v^2-u^2+1)}{\beta}du + \frac{-4uv}{\beta}dv \\
d\Big(\frac{2v}{\alpha+1}\Big) &= \frac{-4uv}{\beta}du + \frac{2(\alpha^2+1)-4v}{\beta}dv \\
&= \frac{-4uv}{\beta}du + \frac{2(u^2-v^2+1)}{\beta}dv \\
d\Big(\frac{\alpha^2-1}{\alpha+1}\Big) &= \frac{2u(\alpha^2+1)-2u(\alpha^2-1)}{\beta}du + \frac{2v(\alpha^2+1)-2v(\alpha^2-1)}{\beta}dv \\
&= \frac{4u}{\beta}du + \frac{4v}{\beta}dv
\end{aligned}$$

Combining these results with the above expression for g^*df, one can verify that $g^*df = 0$; i.e. we have that $d(f\circ g) = g^*df$.

49 (***i***)

$$f(p) = \frac{p_1}{p_1^2+p_2^2} \qquad \tilde{f}(x,y) = \frac{x}{x^2+y^2}.$$

$$\begin{aligned}
df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \\
&= \frac{y^2-x^2}{(x^2+y^2)^2}dx + \frac{-2xy}{(x^2+y^2)^2}dy
\end{aligned}$$

(***ii***)

$$f(p) = \frac{p_1}{p_1^2+p_2^2} \qquad \tilde{f}(r,\theta) = \frac{r\cos\theta}{r^2} = \frac{cos\theta}{r}$$

$$\begin{aligned}
df &= \frac{\partial \tilde{f}}{\partial r}dr + \frac{\partial \tilde{f}}{\partial \theta}d\theta \\
&= \frac{-\cos\theta}{r^2}dr + \frac{-\sin\theta}{r}d\theta.
\end{aligned}$$

(***iv***)

$$f(p) = |p|^2 \qquad \tilde{f}(x_1,\ldots,x_n) = x_1^2+\ldots+x_n^2$$

$$df = 2x_i dx_i$$

50 Recall that $x = r\cos\theta$ and $y = r\sin\theta$ and note that

$$\begin{aligned} dx &= \cos\theta dr - r\sin\theta d\theta \\ dy &= \sin\theta dr + r\cos\theta d\theta \end{aligned}$$

(***i***) We obtain:

$$\begin{aligned} \omega &= r^2\cos^2\theta(\cos\theta dr - r\sin\theta d\theta) + (r\cos\theta + r\sin\theta)(\sin\theta dr + r\cos\theta d\theta) \\ &= (r^2\cos^3\theta + r\cos\theta\sin\theta + r\sin^2\theta)dr + (r^2\cos^2\theta + r^2\sin\theta\cos\theta - r^3\cos^2\theta\sin\theta)d\theta \end{aligned}$$

(***ii***) We obtain:

$$\begin{aligned} \omega &= r\cos\theta(\cos\theta dr - r\sin\theta d\theta) + r\sin\theta(\sin\theta dr + r\cos\theta d\theta) \\ &= rdr. \end{aligned}$$

52 We use the fact that $(f\circ\gamma)^* = \gamma^* f^*$, see Lemma 6.3. We have the following sequence of equalities:

$$\int_\gamma f^*\omega = \int_{[a,b]} \gamma^* f^*\omega = \int_{[a,b]} (f\circ\gamma)^*\omega = \int_{f\circ\gamma} \omega.$$

53 (***i***)

$$\begin{aligned} \int_\gamma \alpha = \int_{[0,1]} \gamma^*\alpha &= \int_{[0,1]} \frac{-4t}{(t^2+1)^2}dt + \frac{2t}{t^2+1}dt + \frac{2t}{t^2+1}dt \\ &= \int_{[0,1]} \frac{-4t}{(t^2+1)^2}dt + \int_{[0,1]} \frac{4t}{t^2+1}dt \\ &= \frac{2}{t^2+1}\Big|_0^1 + 2\ln(t^2+1)\Big|_0^1 = 2\ln 2 - 1. \end{aligned}$$

and:

$$\begin{aligned} \int_\gamma \omega = \int_{[0,1]} \gamma^*\omega &= \int_{[0,1]} \frac{-4t^2}{(t^2+1)^2}dt + \frac{2t}{t^2+1}dt + \frac{2}{t^2+1}dt \\ &= \int_{[0,1]} \frac{2(t^2+1) - 4t^2}{(t^2+1)^2}dt + \int_{[0,1]} \frac{2t}{t^2+1}dt \\ &= \frac{2t}{t^2+1}\Big|_0^1 + \ln(t^2+1)\Big|_0^1 = 1 + \ln 2. \end{aligned}$$

(***ii***) We can split γ into three pieces; γ_1, γ_2 and γ_3 all ranging from $[0,1]$ to $\mathbb{R}^3$ where $\gamma_1(t) = (t,0,0)$, $\gamma_2(t) = (1,t,0)$ and $\gamma_3(t) = (1,1,t)$. We now compute the integrals as follows:

$$\begin{aligned}
\int_\gamma \alpha &= \int_{\gamma_1}\alpha+\int_{\gamma_2}\alpha+\int_{\gamma_3}\alpha\\
&= \int_{[0,1]}\frac{4\cdot 0}{(t^2+1)^2}dt+\int_{[0,1]}\frac{2t}{t^2+1}dt+\int_{[0,1]}\frac{2}{1+1}dt\\
&= \int_{[0,1]}\frac{2t}{t^2+1}+\int_{[0,1]}1dt=\ln(t^2+1)\Big|_0^1+t\Big|_0^1=\ln 2+1.
\end{aligned}$$

and:

$$\int_\gamma \omega = \int_{\gamma_1}\omega+\int_{\gamma_2}\omega+\int_{\gamma_3}\omega=\int_{[0,1]}\frac{2t}{t^2+1}dt+\int_{[0,1]}1dt=\ln 2+1.$$

(***iii***) and (***iv***) : Recall that a 1-form is exact if it is of the form dg for some smooth function g. The Fundamental Theorem for Line Integrals states that the integral of an exact 1-form over a path γ only depends on the end-points of γ. So if either α or ω is exact, then the answers in (i) and (ii) do not differ. So we can conclude immediately that α is not exact. So what about ω? Well, we guess that it is exact, but we have to find a function $g:\mathbb{R}^3\to R$ such that $\omega=dg$. This function g is given by

$$g(x,y,z)=\frac{2z}{x^2+1}+\ln(y^2+1).$$

54 Fix $p\in M$ and a homeomorphism $\varphi:U\to V$ where $U,V\subset\mathbb{R}^l$ and

$$\varphi(U\cap M)=V\cap(\mathbb{R}^m\times\{0\}).$$

Suppose $(x_1,\dots,x_l)$ is a co-ordinate representation for φ. Note that $J\varphi^{-1}|_p(\mathbb{R}^l)$ is just $\mathbb{R}^l$. Now define a map $\tilde{\varphi}:U\times\mathbb{R}^l\to V\times\mathbb{R}^l$ as follows;

$$\tilde{\varphi}\Big(q,v^i\frac{\partial}{\partial x^i}\Big|_q\Big)=\big(x_1(q),\dots,x_l(q),v^1,\dots,v^l\big),$$

where $\frac{\partial}{\partial x^i}|_q=J\varphi^{-1}|_p(e_i)$. Now note that

$$\tilde{\varphi}^{-1}(x,v^1,\dots,v^l)=\Big(\varphi^{-1}(x),v^i\frac{\partial}{\partial x^i}\Big|_{\varphi^{-1}(x)}\Big).$$

So $\tilde{\varphi}$ is a homeomorphism from $U\times\mathbb{R}^l$ onto $V\times\mathbb{R}^l$. Since φ is a slice chart for M, we also have that

$$\tilde{\varphi}(\pi^{-1}[U])=\Big[V\cap\Big(\mathbb{R}^m\times\{0\}\Big)\Big]\times\Big[\mathbb{R}^m\times\{0\}\Big]\subset\mathbb{R}^l\times\mathbb{R}^l$$

where $\pi:TM\to M$ is the natural projection map. So $\tilde{\varphi}$ maps a neighbourhood of $(p,v)\in TM$ onto a $2m$-slice in $\mathbb{R}^{2l}$. This shows how to make co-ordinate charts for TM. It remains to verify that if $\tilde{\psi}$ is another such chart, then the transition map $\tilde{\psi}\circ\tilde{\varphi}^{-1}$ is smooth. For this, see Lee pg81.