Shifted symplectic geometry, Calabi–Yau moduli spaces, and generalizations of Donaldson–Thomas theory: our current and future research

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Plan of talk:

1. PTVV's shifted symplectic geometry

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, e.g. $\mathbb{K}=\mathbb{C}$. Work in the context of Toën and Vezzosi's theory of *derived algebraic geometry*. This gives ∞-categories of *derived $\mathbb{K}$-schemes* $\mathbf{dSch}_{\mathbb{K}}$ and *derived stacks* $\mathbf{dSt}_{\mathbb{K}}$, including *derived Artin stacks* $\mathbf{dArt}_{\mathbb{K}}$. Think of a derived $\mathbb{K}$-scheme $\mathbf{X}$ as a geometric space which can be covered by Zariski open sets $\mathbf{Y}\subseteq\mathbf{X}$ with $\mathbf{Y}\simeq\operatorname{Spec}A$ for $A=(A,\mathrm{d})$ a commutative differential graded algebra (cdga) over $\mathbb{K}$.

Cotangent complexes of derived schemes and stacks

Pantev, Toën, Vaquié and Vezzosi (arXiv:1111.3209) defined a notion of *k-shifted symplectic structure* on a derived $\mathbb{K}$-scheme or derived $\mathbb{K}$-stack $\mathbf{X}$, for $k\in\mathbb{Z}$. This is complicated, but here is the basic idea. The *cotangent complex* $\mathbb{L}_{\mathbf{X}}$ of $\mathbf{X}$ is an element of a derived category $L_{\rm qcoh}(\mathbf{X})$ of quasicoherent sheaves on $\mathbf{X}$. It has exterior powers $\Lambda^p\mathbb{L}_{\mathbf{X}}$ for $p=0,1,\ldots.$ The *de Rham differential* $\mathrm{d}_{dR}:\Lambda^p\mathbb{L}_{\mathbf{X}}\rightarrow\Lambda^{p+1}\mathbb{L}_{\mathbf{X}}$ is a morphism of complexes, though not of $\mathcal{O}_{\mathbf{X}}$-modules. Each $\Lambda^p\mathbb{L}_{\mathbf{X}}$ is a complex, so has an internal differential $\mathrm{d}:(\Lambda^p\mathbb{L}_{\mathbf{X}})^k\rightarrow(\Lambda^p\mathbb{L}_{\mathbf{X}})^{k+1}$. We have $\mathrm{d}^2=\mathrm{d}_{dR}^2=\mathrm{d}\circ\mathrm{d}_{dR}+\mathrm{d}_{dR}\circ\mathrm{d}=0$.

p-forms and closed p-forms

A *p-form of degree k* on $\mathbf{X}$ for $k \in \mathbb{Z}$ is an element $[\omega^0]$ of $H^k(\Lambda^p \mathbb{L}_{\mathbf{X}}, \mathrm{d})$. A *closed p-form of degree k* on $\mathbf{X}$ is an element

$$[(\omega^0, \omega^1, \ldots)] \in H^k\bigl(\textstyle\bigoplus_{i=0}^{\infty} \Lambda^{p+i} \mathbb{L}_{\mathbf{X}}[i], \mathrm{d} + \mathrm{d}_{dR}\bigr).$$

There is a projection $\pi : [(\omega^0, \omega^1, \ldots)] \mapsto [\omega^0]$ from closed p-forms $[(\omega^0, \omega^1, \ldots)]$ of degree k to p-forms $[\omega^0]$ of degree k.
Note that a closed p-form *is not a special example of a p-form*, but a p-form with an extra structure. The map π from closed p-forms to p-forms can be neither injective nor surjective.

Nondegenerate 2-forms and symplectic structures

Let $[\omega^0]$ be a 2-form of degree k on $\mathbf{X}$. Then $[\omega^0]$ induces a morphism $\omega^0 : \mathbb{T}_{\mathbf{X}} \to \mathbb{L}_{\mathbf{X}}[k]$, where $\mathbb{T}_{\mathbf{X}} = \mathbb{L}_{\mathbf{X}}^{\vee}$ is the tangent complex of $\mathbf{X}$. We call $[\omega^0]$ *nondegenerate* if $\omega^0 : \mathbb{T}_{\mathbf{X}} \to \mathbb{L}_{\mathbf{X}}[k]$ is a quasi-isomorphism.
If $\mathbf{X}$ is a derived scheme then $\mathbb{L}_{\mathbf{X}}$ lives in degrees $(-\infty, 0]$ and $\mathbb{T}_{\mathbf{X}}$ in degrees $[0, \infty)$. So $\omega^0 : \mathbb{T}_{\mathbf{X}} \to \mathbb{L}_{\mathbf{X}}[k]$ can be a quasi-isomorphism only if $k \leqslant 0$, and then $\mathbb{L}_{\mathbf{X}}$ lives in degrees $[k, 0]$ and $\mathbb{T}_{\mathbf{X}}$ in degrees $[0, -k]$. If $k = 0$ then $\mathbf{X}$ is a smooth classical $\mathbb{K}$-scheme, and if $k = -1$ then $\mathbf{X}$ is quasi-smooth.
A closed 2-form $\omega = [(\omega^0, \omega^1, \ldots)]$ of degree k on $\mathbf{X}$ is called a *k-shifted symplectic structure* if $[\omega^0] = \pi(\omega)$ is nondegenerate.

Calabi–Yau moduli schemes and moduli stacks

Pantev et al. prove that if Y is a Calabi–Yau m-fold over $\mathbb{K}$ and $\boldsymbol{\mathcal{M}}$ is a derived moduli scheme or stack of (complexes of) coherent sheaves on Y, then $\boldsymbol{\mathcal{M}}$ has a natural $(2-m)$-shifted symplectic structure ω. So Calabi–Yau 3-folds give -1-shifted derived schemes or stacks.

We can understand the associated nondegenerate 2-form $[\omega^0]$ in terms of *Serre duality*. At a point $[E]\in\boldsymbol{\mathcal{M}}$, we have $h^i(\mathbb{T}_{\boldsymbol{\mathcal{M}}})|_{[E]}\cong \mathrm{Ext}^{i-1}(E,E)$ and $h^i(\mathbb{L}_{\boldsymbol{\mathcal{M}}})|_{[E]}\cong \mathrm{Ext}^{1-i}(E,E)^*$. The Calabi–Yau condition gives $\mathrm{Ext}^i(E,E)\cong \mathrm{Ext}^{m-i}(E,E)^*$, which corresponds to $h^i(\mathbb{T}_{\boldsymbol{\mathcal{M}}})|_{[E]}\cong h^i(\mathbb{L}_{\boldsymbol{\mathcal{M}}}[2-m])|_{[E]}$. This is the cohomology at $[E]$ of the quasi-isomorphism $\omega^0:\mathbb{T}_{\boldsymbol{\mathcal{M}}}\to\mathbb{L}_{\boldsymbol{\mathcal{M}}}[2-m]$.

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Lagrangians and Lagrangian intersections

Let $(\mathbf{X},\omega)$ be a k-shifted symplectic derived scheme or stack. Then Pantev et al. define a notion of *Lagrangian* $\mathbf{L}$ in $(\mathbf{X},\omega)$, which is a morphism $\mathbf{i}:\mathbf{L}\to\mathbf{X}$ of derived schemes or stacks together with a homotopy $i^*(\omega)\sim 0$ satisfying a nondegeneracy condition, implying that $\mathbb{T}_{\mathbf{L}}\simeq\mathbb{L}_{\mathbf{L}/\mathbf{X}}[k-1]$.

If $\mathbf{L},\mathbf{M}$ are Lagrangians in $(\mathbf{X},\omega)$, then the fibre product $\mathbf{L}\times_{\mathbf{X}}\mathbf{M}$ has a natural $(k-1)$-shifted symplectic structure.

If (S,ω) is a classical smooth symplectic scheme, then it is a 0-shifted symplectic derived scheme in the sense of PTVV, and if $L,M\subset S$ are classical smooth Lagrangian subschemes, then they are Lagrangians in the sense of PTVV. Therefore the (derived) Lagrangian intersection $L\cap M=L\times_S M$ is a -1-shifted symplectic derived scheme.

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2. A Darboux theorem for shifted symplectic schemes

Theorem (Brav, Bussi and Joyce arXiv:1305.6302)

Suppose $(\mathbf{X},\omega)$ *is a* k*-shifted symplectic derived* $\mathbb{K}$*-scheme for* $k<0$. *If* $k\not\equiv 2 \mod 4$, *then each* $x\in\mathbf{X}$ *admits a Zariski open neighbourhood* $\mathbf{Y}\subseteq\mathbf{X}$ *with* $\mathbf{Y}\simeq\operatorname{Spec}A$ *for* (A,d) *an explicit cdga over* $\mathbb{K}$ *generated by graded variables* x_j^{-i},y_j^{k+i} *for* $0\leqslant i\leqslant -k/2$, *and* $\omega|_{\mathbf{Y}}=[(\omega^0,0,0,\ldots)]$ *where* x_j^l,y_j^l *have degree* l, *and*

$$\omega^0=\textstyle\sum_{i=0}^{[-k/2]}\sum_{j=1}^{m_i}\mathrm{d}_{dR}y_j^{k+i}\mathrm{d}_{dR}x_j^{-i}.$$

Also the differential d *in* (A,d) *is given by Poisson bracket with a Hamiltonian* H *in* A *of degree* $k+1$.
If $k\equiv 2 \mod 4$, *we have two statements, one étale local with* ω^0 *standard, and one Zariski local with the components of* ω^0 *in the degree* $k/2$ *variables depending on some invertible functions.*

Sketch of the proof of the theorem

Suppose $(\mathbf{X},\omega)$ is a k-shifted symplectic derived $\mathbb{K}$-scheme for $k<0$, and $x\in\mathbf{X}$. Then $\mathbb{L}_{\mathbf{X}}$ lives in degrees $[k,0]$. We first show that we can build Zariski open $x\in\mathbf{Y}\subseteq\mathbf{X}$ with $\mathbf{Y}\simeq\operatorname{Spec}A$, for $A=\bigoplus_{i\leqslant 0}A^i$ a cdga over $\mathbb{K}$ with A^0 a smooth $\mathbb{K}$-algebra, and such that A is freely generated over A^0 by graded variables x_j^{-i},y_j^{k+i} in degrees $-1,-2,\ldots,k$. We take $\dim A^0$ and the number of x_j^{-i},y_j^{k+i} to be minimal at x.
Using theorems about periodic cyclic cohomology, we show that on $Y\simeq\operatorname{Spec}A$ we can write $\omega|_Y=[(\omega^0,0,0,\ldots)]$, for ω^0 a 2-form of degree k with $\mathrm{d}\omega^0=\mathrm{d}_{dR}\omega^0=0$. Minimality at x implies ω^0 is strictly nondegenerate near x, so we can change variables to write $\omega^0=\sum_{i,j}\mathrm{d}_{dR}y_j^{k+i}\mathrm{d}_{dR}x_j^{-i}$. Finally, we show d in (A,d) is a symplectic vector field, which integrates to a Hamiltonian H.

The case of -1-shifted symplectic derived schemes

When $k=-1$ the Hamiltonian H in the theorem has degree 0. Then the theorem reduces to:

Corollary

Suppose $(\mathbf{X},\omega)$ *is a* -1*-shifted symplectic derived* $\mathbb{K}$*-scheme. Then* $(\mathbf{X},\omega)$ *is Zariski locally equivalent to a derived critical locus* $\mathbf{Crit}(H:U\to\mathbb{A}^1)$*, for* U *a smooth classical* $\mathbb{K}$*-scheme and* $H:U\to\mathbb{A}^1$ *a regular function. Hence, the underlying classical* $\mathbb{K}$*-scheme* $X=t_0(\mathbf{X})$ *is Zariski locally isomorphic to a classical critical locus* $\mathrm{Crit}(H:U\to\mathbb{A}^1)$.

Combining this with results of Pantev et al. from §1 gives interesting consequences in classical algebraic geometry:

Corollary

Let Y *be a Calabi–Yau* 3*-fold over* $\mathbb{K}$ *and* $\mathcal{M}$ *a classical moduli* $\mathbb{K}$*-scheme of coherent sheaves, or complexes of coherent sheaves, on* Y*. Then* $\mathcal{M}$ *is Zariski locally isomorphic to the critical locus* $\mathrm{Crit}(H:U\to\mathbb{A}^1)$ *of a regular function on a smooth* $\mathbb{K}$*-scheme.*

Here we note that $\mathcal{M}=t_0(\boldsymbol{\mathcal{M}})$ for $\boldsymbol{\mathcal{M}}$ the corresponding derived moduli scheme, which is -1-shifted symplectic by PTVV.
A complex analytic analogue of this for moduli of coherent sheaves was proved using gauge theory by Joyce and Song arXiv:0810.5645, and for moduli of complexes was claimed by Behrend and Getzler.
Note that the proof of the corollary is wholly algebro-geometric.

Corollary

Let (S,ω) be a classical smooth symplectic $\mathbb{K}$-scheme, and $L,M\subseteq S$ be smooth algebraic Lagrangians. Then the intersection $L\cap M$, as a $\mathbb{K}$-subscheme of S, is Zariski locally isomorphic to the critical locus $\mathrm{Crit}(H:U\to\mathbb{A}^1)$ of a regular function on a smooth $\mathbb{K}$-scheme.

In real or complex symplectic geometry, where Darboux Theorem holds, the analogue of the corollary is easy to prove, but in classical algebraic symplectic geometry we do not have a Darboux Theorem, so the corollary is not obvious.

3. D-critical loci

Theorem (Joyce arXiv:1304.4508)

Let X be a classical $\mathbb{K}$-scheme. Then there exists a canonical sheaf $\mathcal{S}_X$ of $\mathbb{K}$-vector spaces on X, such that if $R\subseteq X$ is Zariski open and $i:R\hookrightarrow U$ is a closed embedding of R into a smooth $\mathbb{K}$-scheme U, and $I_{R,U}\subseteq\mathcal{O}_U$ is the ideal vanishing on $i(R)$, then

$$\mathcal{S}_X|_R\cong\mathrm{Ker}\left(\frac{\mathcal{O}_U}{I_{R,U}^2}\xrightarrow{\mathrm{d}}\frac{T^*U}{I_{R,U}\cdot T^*U}\right).$$

Also $\mathcal{S}_X$ splits naturally as $\mathcal{S}_X=\mathcal{S}_X^0\oplus\mathbb{K}_X$, where $\mathbb{K}_X$ is the sheaf of locally constant functions $X\to\mathbb{K}$.

The meaning of the sheaves $\mathcal{S}_X, \mathcal{S}_X^0$

If $X = \mathrm{Crit}(f : U \to \mathbb{A}^1)$ then taking $R = X$, $i =$inclusion, we see that $f + I_{X,U}^2$ is a section of $\mathcal{S}_X$. Also $f|_{X^{\mathrm{red}}} : X^{\mathrm{red}} \to \mathbb{K}$ is locally constant, and if $f|_{X^{\mathrm{red}}} = 0$ then $f + I_{X,U}^2$ is a section of $\mathcal{S}_X^0$. Note that $f + I_{X,U} = f|_X$ in $\mathcal{O}_X = \mathcal{O}_U / I_{X,U}$. The theorem means that $f + I_{X,U}^2$ makes sense *intrinsically on* X, without reference to the embedding of X into U.

That is, if $X = \mathrm{Crit}(f : U \to \mathbb{A}^1)$ then we can remember f up to second order in the ideal $I_{X,U}$ as a piece of data on X, not on U.

Suppose $X = \mathrm{Crit}(f : U \to \mathbb{A}^1) = \mathrm{Crit}(g : V \to \mathbb{A}^1)$ is written as a critical locus in two different ways. Then $f + I_{X,U}^2$, $g + I_{X,V}^2$ are sections of $\mathcal{S}_X$, so we can ask whether $f + I_{X,U}^2 = g + I_{X,V}^2$. This gives a way to compare isomorphic critical loci in different smooth classical schemes.

The definition of d-critical loci

Definition (Joyce arXiv:1304.4508)

An (*algebraic*) *d-critical locus* (X, s) is a classical $\mathbb{K}$-scheme X and a global section $s \in H^0(\mathcal{S}_X^0)$ such that X may be covered by Zariski open $R \subseteq X$ with an isomorphism $i : R \to \mathrm{Crit}(f : U \to \mathbb{A}^1)$ identifying $s|_R$ with $f + I_{R,U}^2$, for f a regular function on a smooth $\mathbb{K}$-scheme U.

That is, a d-critical locus (X, s) is a $\mathbb{K}$-scheme X which may Zariski locally be written as a critical locus $\mathrm{Crit}(f : U \to \mathbb{A}^1)$, and the section s remembers f up to second order in the ideal $I_{X,U}$.

We also define *complex analytic d-critical loci*, with X a complex analytic space locally modelled on $\mathrm{Crit}(f : U \to \mathbb{C})$ for U a complex manifold and f holomorphic.

Orientations on d-critical loci

Theorem (Joyce arXiv:1304.4508)

Let (X,s) be an algebraic d-critical locus and $X^{\rm red}$ the reduced $\mathbb{K}$-subscheme of X. Then there is a natural line bundle $K_{X,s}$ on $X^{\rm red}$ called the **canonical bundle**, *such that if (X,s) is locally modelled on $\mathrm{Crit}(f:U\to\mathbb{A}^1)$ then $K_{X,s}$ is locally modelled on $K_U^{\otimes^2}|_{\mathrm{Crit}(f)^{\rm red}}$, for K_U the usual canonical bundle of U.*

Definition

Let (X,s) be a d-critical locus. An *orientation* on (X,s) is a choice of square root line bundle $K_{X,s}^{1/2}$ for $K_{X,s}$ on $X^{\rm red}$.

This is related to *orientation data* in Kontsevich–Soibelman 2008.

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A truncation functor from -1-symplectic derived schemes

Theorem (Brav, Bussi and Joyce arXiv:1305.6302)

Let $(\mathbf{X},\omega)$ be a -1-shifted symplectic derived $\mathbb{K}$-scheme. Then the classical $\mathbb{K}$-scheme $X=t_0(\mathbf{X})$ extends naturally to an algebraic d-critical locus (X,s). The canonical bundle of (X,s) satisfies $K_{X,s}\cong\det\mathbb{L}_{\mathbf{X}}|_{X^{\rm red}}$.

That is, we define a *truncation functor* from -1-shifted symplectic derived $\mathbb{K}$-schemes to algebraic d-critical loci. Examples show this functor is not full. Think of d-critical loci as *classical truncations* of -1-shifted symplectic derived $\mathbb{K}$-schemes.
An alternative semi-classical truncation, used in D–T theory, is *schemes with symmetric obstruction theory*. D-critical loci appear to be better, for both categorified and motivic D–T theory.

The corollaries in §2 imply:

Corollary

Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a classical moduli $\mathbb{K}$-scheme of coherent sheaves, or complexes of coherent sheaves, on Y. Then $\mathcal{M}$ extends naturally to a d-critical locus $(\mathcal{M}, s)$. The canonical bundle satisfies $K_{\mathcal{M},s} \cong \det(\mathcal{E}^\bullet)|_{\mathcal{M}^{\mathrm{red}}}$, where $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$ is the (symmetric) obstruction theory on $\mathcal{M}$ defined by Thomas or Huybrechts and Thomas.

Corollary

Let (S, ω) be a classical smooth symplectic $\mathbb{K}$-scheme, and $L, M \subseteq S$ be smooth algebraic Lagrangians. Then $X = L \cap M$ extends naturally to a d-critical locus (X, s). The canonical bundle satisfies $K_{X,s} \cong K_L|_{X^{\mathrm{red}}} \otimes K_M|_{X^{\mathrm{red}}}$. Hence, choices of square roots $K_L^{1/2}, K_M^{1/2}$ give an orientation for (X, s).

4. Categorification using perverse sheaves

Theorem (Brav, Bussi, Dupont, Joyce, Szendrői arXiv:1211.3259)

Let (X, s) be an algebraic d-critical locus over $\mathbb{K}$, with an orientation $K_{X,s}^{1/2}$. Then we can construct a canonical perverse sheaf $P_{X,s}^\bullet$ on X, such that if (X, s) is locally modelled on $\mathrm{Crit}(f : U \to \mathbb{A}^1)$, then $P_{X,s}^\bullet$ is locally modelled on the perverse sheaf of vanishing cycles $\mathcal{PV}_{U,f}^\bullet$ of (U, f).
Similarly, we can construct a natural $\mathscr{D}$-module $D_{X,s}^\bullet$ on X, and when $\mathbb{K} = \mathbb{C}$ a natural mixed Hodge module $M_{X,s}^\bullet$ on X.

Sketch of the proof of the theorem

Roughly, we prove the theorem by taking a Zariski open cover $\{R_i : i \in I\}$ of X with $R_i \cong \mathrm{Crit}(f_i : U_i \to \mathbb{A}^1)$, and showing that $\mathcal{PV}^\bullet_{U_i,f_i}$ and $\mathcal{PV}^\bullet_{U_j,f_j}$ are canonically isomorphic on $R_i \cap R_j$, so we can glue the $\mathcal{PV}^\bullet_{U_i,f_i}$ to get a global perverse sheaf $P^\bullet_{X,s}$ on X.
In fact things are more complicated: the (local) isomorphisms $\mathcal{PV}^\bullet_{U_i,f_i} \cong \mathcal{PV}^\bullet_{U_j,f_j}$ are only canonical *up to sign*. To make them canonical, we use the orientation $K^{1/2}_{X,s}$ to define natural principal $\mathbb{Z}_2$-bundles Q_i on R_i, such that $\mathcal{PV}^\bullet_{U_i,f_i} \otimes_{\mathbb{Z}_2} Q_i \cong \mathcal{PV}^\bullet_{U_j,f_j} \otimes_{\mathbb{Z}_2} Q_j$ is canonical, and then we glue the $\mathcal{PV}^\bullet_{U_i,f_i} \otimes_{\mathbb{Z}_2} Q_i$ to get $P^\bullet_{X,s}$.

The first corollary in §2 implies:

Corollary

Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a classical moduli $\mathbb{K}$-scheme of coherent sheaves, or complexes of coherent sheaves, on Y, with (symmetric) obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$. Suppose we are given a square root $\det(\mathcal{E}^\bullet)^{1/2}$ *for* $\det(\mathcal{E}^\bullet)$ *(i.e.* **orientation data**, *K–S). Then we have a natural perverse sheaf $P^\bullet_{\mathcal{M},s}$ on $\mathcal{M}$.*

The *hypercohomology* $\mathbb{H}^*(P^\bullet_{\mathcal{M},s})$ is a finite-dimensional graded vector space. The pointwise Euler characteristic $\chi(P^\bullet_{\mathcal{M},s})$ is the *Behrend function* $\nu_{\mathcal{M}}$ of $\mathcal{M}$. Thus

$$\textstyle\sum_{i\in\mathbb{Z}}(-1)^i \dim \mathbb{H}^i(P^\bullet_{\mathcal{M},s}) = \chi(\mathcal{M},\nu_{\mathcal{M}}).$$

Now by Behrend 2005, the Donaldson–Thomas invariant of $\mathcal{M}$ is $DT(\mathcal{M}) = \chi(\mathcal{M},\nu_{\mathcal{M}})$. So, $\mathbb{H}^*(P^\bullet_{\mathcal{M},s})$ is a graded vector space with dimension $DT(\mathcal{M})$, that is, a *categorification* of $DT(\mathcal{M})$.

Categorifying Lagrangian intersections

The second corollary in §2 implies:

Corollary

Let (S,ω) be a classical smooth symplectic $\mathbb{K}$-scheme of dimension $2n$, and $L, M \subseteq S$ be smooth algebraic Lagrangians, with square roots $K_L^{1/2}, K_M^{1/2}$ of their canonical bundles. Then we have a natural perverse sheaf $P^\bullet_{L,M}$ on $X = L \cap M$.

This is related to Behrend and Fantechi 2009. We think of the hypercohomology $\mathbb{H}^*(P^\bullet_{L,M})$ as being morally related to the *Lagrangian Floer cohomology* $HF^*(L,M)$ by

$$\mathbb{H}^i(P^\bullet_{L,M}) \approx HF^{i+n}(L,M).$$

We are working on defining 'Fukaya categories' for algebraic/complex symplectic manifolds using these ideas.

5. Motivic Milnor fibres

By similar arguments to those used to construct the perverse sheaves $P^\bullet_{X,s}$ in §4, we prove:

Theorem (Bussi, Joyce and Meinhardt arXiv:1305.6428)

Let (X,s) be an algebraic d-critical locus over $\mathbb{K}$, with an orientation $K_{X,s}^{1/2}$. Then we can construct a natural motive $MF_{X,s}$ in a certain ring of $\hat\mu$-equivariant motives $\bar{\mathcal{M}}^{\hat\mu}_X$ on X, such that if (X,s) is locally modelled on $\mathrm{Crit}(f : U \to \mathbb{A}^1)$, then $MF_{X,s}$ is locally modelled on $\mathbb{L}^{-\dim U/2}\bigl([X] - MF^{\mathrm{mot}}_{U,f}\bigr)$, where $MF^{\mathrm{mot}}_{U,f}$ is the **motivic Milnor fibre** *of f.*

Vittoria Bussi's talk will give more details.

Relation to motivic D–T invariants

The first corollary in §2 implies:

Corollary

Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a classical moduli $\mathbb{K}$-scheme of coherent sheaves, or complexes of coherent sheaves, on Y, with (symmetric) obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$. Suppose we are given a square root $\det(\mathcal{E}^\bullet)^{1/2}$ *for* $\det(\mathcal{E}^\bullet)$ *(i.e.* **orientation data**, *K–S). Then we have a natural motive $MF^\bullet_{\mathcal{M},s}$ on $\mathcal{M}$.*

This motive $MF^\bullet_{\mathcal{M},s}$ is essentially the motivic Donaldson–Thomas invariant of $\mathcal{M}$ defined (partially conjecturally) by Kontsevich and Soibelman 2008. K–S work with motivic Milnor fibres of formal power series at each point of $\mathcal{M}$. Our results show the formal power series can be taken to be a regular function, and clarify how the motivic Milnor fibres vary in families over $\mathcal{M}$.

6. Extension to Artin stacks

In Ben-Bassat, Bussi, Brav and Joyce (in progress) we extend the material of §2–§6 from (derived) schemes to (derived) Artin stacks. We define a *derived Artin stack* $\mathbf{X}$ to be 'strongly 1-geometric' in the sense of Toën and Vezzosi. Then the cotangent complex $\mathbb{L}_{\mathbf{X}}$ lives in degrees $(-\infty, 1]$, and $X = t_0(\mathbf{X})$ is a classical Artin stack (in particular, not a higher stack). A derived Artin stack $\mathbf{X}$ admits a smooth atlas $\varphi : \mathbf{U} \to \mathbf{X}$ with $\mathbf{U}$ a derived scheme. If Y is a smooth projective scheme and $\boldsymbol{\mathcal{M}}$ is a derived moduli stack of coherent sheaves F on Y, or of complexes $F^\bullet$ in $D^b \operatorname{coh}(Y)$ with $\operatorname{Ext}^{<0}(F^\bullet, F^\bullet) = 0$, then $\boldsymbol{\mathcal{M}}$ is a derived Artin stack.

A 'Darboux Theorem' for shifted symplectic derived stacks

Theorem (Ben-Bassat, Bussi, Brav, Joyce)

Let $(\mathbf{X},\omega_{\mathbf{X}})$ be a k-shifted symplectic derived Artin stack for $k<0$, and $p\in\mathbf{X}$. Then there exist 'standard form' affine derived schemes $\mathbf{U}=\operatorname{Spec}A$, $\mathbf{V}=\operatorname{Spec}B$, points $u\in\mathbf{U}$, $v\in\mathbf{V}$ with A,B minimal at u,v, morphisms $\varphi:\mathbf{U}\to\mathbf{X}$ and $\mathbf{i}:\mathbf{U}\to\mathbf{V}$ with $\varphi(u)=p$, $\mathbf{i}(u)=v$, such that φ is smooth of relative dimension $\dim H^1(\mathbb{L}_{\mathbf{X}}|_p)$, and $t_0(\mathbf{i}):t_0(\mathbf{U})\to t_0(\mathbf{V})$ is an isomorphism on classical schemes, and $\mathbb{L}_{\mathbf{U}/\mathbf{V}}\simeq\mathbb{T}_{\mathbf{U}/\mathbf{X}}[1-k]$, and a 'Darboux form' k-shifted symplectic form ω_B on $\mathbf{V}=\operatorname{Spec}B$ such that $\mathbf{i}^*(\omega_B)\sim\varphi^*(\omega_{\mathbf{X}})$ in k-shifted closed 2-forms on $\mathbf{U}$.

Discussion of the 'Darboux Theorem' for stacks

Let $(\mathbf{X},\omega_{\mathbf{X}})$ be a k-shifted symplectic derived Artin stack for $k<0$, and $p\in\mathbf{X}$. Although we do not know how to give a complete, explicit 'standard model' for $(\mathbf{X},\omega_{\mathbf{X}})$ near p, we can give standard models for a smooth atlas $\varphi:\mathbf{U}\to\mathbf{X}$ for $\mathbf{X}$ near p with $\mathbf{U}=\operatorname{Spec}A$ a derived scheme, and for the pullback 2-form $\varphi^*(\omega_{\mathbf{X}})$. We may think of $\varphi:\mathbf{U}\to\mathbf{X}$ as an open neighbourhood of p in the smooth topology, rather than the Zariski topology. Now $(\mathbf{U},\varphi^*(\omega_{\mathbf{X}}))$ is not k-shifted symplectic, as $\varphi^*(\omega_{\mathbf{X}})$ is closed, but not nondegenerate. However, there is a way to modify $\mathbf{U},A$ to get another derived scheme $\mathbf{V}=\operatorname{Spec}B$, where A has generators in degrees $0,-1,\ldots,-k-1$, and $B\subseteq A$ is the dg-subalgebra generated by the generators in degrees $0,-1,\ldots,-k$ only.

Then $\mathbf{V}$ has a natural k-shifted symplectic form ω_B, which we may take to be in 'Darboux form' as in §2, with $\mathbf{i}^*(\omega_B) \sim \varphi^*(\omega_{\mathbf{X}})$. In terms of cotangent complexes, $\mathbb{L}_{\mathbf{U}}$ is obtained from $\varphi^*(\mathbb{L}_{\mathbf{X}})$ by deleting a vector bundle $\mathbb{L}_{\mathbf{U}/\mathbf{X}}$ in degree 1. Also $\mathbb{L}_{\mathbf{V}}$ is obtained from $\mathbb{L}_{\mathbf{U}}$ by deleting the dual vector bundle $\mathbb{T}_{\mathbf{U}/\mathbf{X}}$ in degree $k-1$. As these two deletions are dual under $\varphi^*(\omega_{\mathbf{X}})$, the symplectic form descends to $\mathbf{V}$.

An example in which we have this picture $(\mathbf{V}, \omega_B) \xleftarrow{\mathbf{i}} \mathbf{U} \xrightarrow{\varphi} (\mathbf{X}, \omega_{\mathbf{X}})$ is a 'k-shifted symplectic quotient', when an algebraic group G acts on a k-shifted symplectic derived scheme $(\mathbf{V}, \omega_B)$ with 'moment map' $\mu \in H^k(\mathbf{V}, \mathfrak{g}^* \otimes \mathcal{O}_{\mathbf{V}})$, and $\mathbf{U} = \mu^{-1}(0)$, and $X = [\mathbf{U}/G]$.

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-1-shifted symplectic derived stacks

When $k = -1$, $(\mathbf{V}, \omega_B)$ is a derived critical locus $\mathbf{Crit}(f : S \to \mathbb{A}^1)$ for S a smooth scheme. Then $t_0(\mathbf{V}) \cong t_0(\mathbf{U})$ is the classical critical locus $\mathrm{Crit}(f : S \to \mathbb{A}^1)$, and $U = t_0(\mathbf{U})$ is a smooth atlas for the classical Artin stack $X = t_0(\mathbf{X})$. Thus we deduce:

Corollary

Let $(\mathbf{X}, \omega_{\mathbf{X}})$ be a -1-shifted symplectic derived stack. Then the classical Artin stack $X = t_0(\mathbf{X})$ locally admits smooth atlases $\varphi : U \to X$ with $U = \mathrm{Crit}(f : S \to \mathbb{A}^1)$, for S a smooth scheme and f a regular function.

Dominic Joyce, Oxford University Shifted symplectic geometry

Calabi–Yau 3-fold moduli stacks

If Y is a Calabi–Yau 3-fold and $\mathcal{M}$ a moduli stack of of coherent sheaves F on Y, or complexes $F^\bullet$ in $D^b \operatorname{coh}(Y)$ with $\operatorname{Ext}^{<0}(F^\bullet, F^\bullet) = 0$, then by PTVV the corresponding derived moduli stack $\boldsymbol{\mathcal{M}}$ with $t_0(\boldsymbol{\mathcal{M}}) = \mathcal{M}$ has a -1-shifted symplectic structure $\omega_{\boldsymbol{\mathcal{M}}}$. So the previous corollary gives:

Corollary

Suppose Y is a Calabi–Yau 3-fold and $\mathcal{M}$ a classical moduli stack of coherent sheaves F on Y, or of complexes $F^\bullet$ in $D^b \operatorname{coh}(Y)$ with $\operatorname{Ext}^{<0}(F^\bullet, F^\bullet) = 0$. Then $\mathcal{M}$ locally admits smooth atlases $\varphi : U \to X$ with $U = \operatorname{Crit}(f : S \to \mathbb{A}^1)$, for S a smooth scheme.

A holomorphic version of this was proved by Joyce and Song using gauge theory, and is important in D-T theory.

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Sheaves on Artin stacks

To generalize the d-critical loci in §3, and the perverse sheaves in §4, to Artin stacks, we need a good notion of sheaves on Artin stacks. This is already well understood. Roughly, a sheaf $\mathcal{S}$ on an Artin stack X assigns a sheaf $\mathcal{S}(U, \varphi)$ on U (in the usual sense for schemes) for each smooth morphism $\varphi : U \to X$ with U a scheme, and a morphism $\mathcal{S}(\alpha, \eta) : \alpha^*(\mathcal{S}(V, \psi)) \to \mathcal{S}(U, \varphi)$ (often an isomorphism) for each 2-commutative diagram

$$\begin{tikzcd} & V \arrow[dr, "\psi"] & \\ U \arrow[ur, "\alpha"] \arrow[rr, "\varphi"'] \arrow[rr, phantom, shift left=3, "\eta\Uparrow"] & & X \end{tikzcd} \qquad (1)$$

with U, V schemes and φ, ψ smooth, such that $\mathcal{S}(\alpha, \eta)$ have the obvious associativity properties. So, we pass from stacks X to schemes U by working with smooth atlases $\varphi : U \to X$.

D-critical stacks

Generalizing d-critical loci to stacks is now straightforward. As in §3, on each scheme U we have a canonical sheaf $\mathcal{S}_U^0$. If $\alpha : U \to V$ is a morphism of schemes we have pullback morphisms $\alpha^\star : \alpha^{-1}(\mathcal{S}_V^0) \to \mathcal{S}_U^0$ with associativity properties.

So, for any classical Artin stack X, we define a sheaf $\mathcal{S}_X^0$ on X by $\mathcal{S}_X(U,\varphi) = \mathcal{S}_U^0$ for all smooth $\varphi : U \to X$ with U a scheme, and $\mathcal{S}(\alpha,\eta) = \alpha^\star$ for all diagrams (1).

A global section $s \in H^0(\mathcal{S}_X^0)$ assigns $s(U,\varphi) \in H^0(\mathcal{S}_U^0)$ for all smooth $\varphi : U \to X$ with $\alpha^\star[\alpha^{-1}(s(V,\psi))] = s(U,\varphi)$ for all diagrams (1). We call (X,s) a *d-critical stack* if $\big(U, s(U,\varphi)\big)$ is a d-critical locus for all smooth $\varphi : U \to X$.

That is, if X is a d-critical stack then any smooth atlas $\varphi : U \to X$ for X is a d-critical locus.

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A truncation functor from -1-symplectic derived stacks

As for the scheme case in BBJ, we prove:

Theorem (Ben-Bassat, Brav, Bussi, Joyce)

Let $(\mathbf{X},\omega)$ be a -1-shifted symplectic derived Artin stack. Then the classical Artin stack $X = t_0(\mathbf{X})$ extends naturally to a d-critical stack (X,s), with canonical bundle $K_{X,s} \cong \det \mathbb{L}_{\mathbf{X}}|_{X^{\rm red}}$.

Corollary

Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a classical moduli stack of coherent sheaves F on Y, or complexes $F^\bullet$ in $D^b\mathrm{coh}(Y)$ with $\mathrm{Ext}^{<0}(F^\bullet,F^\bullet)=0$. Then $\mathcal{M}$ extends naturally to a d-critical locus $(\mathcal{M},s)$ with canonical bundle $K_{\mathcal{M},s} \cong \det(\mathcal{E}^\bullet)|_{\mathcal{M}^{\rm red}}$, where $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$ is the natural obstruction theory on $\mathcal{M}$.

Canonical bundles and orientations

For schemes, a d-critical locus (U, s) has a canonical bundle $K_{U,s} \to U^{\mathrm{red}}$, and an orientation on (U, s) is a square root $K_{U,s}^{1/2}$. Similarly, a d-critical stack (X, s) has a *canonical bundle* $K_{X,s} \to X^{\mathrm{red}}$. For any smooth $\varphi : U \to X$ with U a scheme we have $K_{X,s}(U^{\mathrm{red}}, \varphi^{\mathrm{red}}) = K_{U,s(U,\varphi)} \otimes (\det \mathbb{L}_{U/X})^{\otimes^{-2}}$. An *orientation* on (X, s) is a choice of square root $K_{X,s}^{1/2}$ for $K_{X,s}$. Note that as $(\det \mathbb{L}_{U/X})^{\otimes^{-2}}$ has a natural square root, an orientation for (X, s) gives an orientation for $\bigl(U, s(U, \varphi)\bigr)$ for any smooth atlas $\varphi : U \to X$.

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Categorification using perverse sheaves

Let (X, s) be a d-critical stack, with an orientation $K_{X,s}^{1/2}$. Then for any smooth $\varphi : U \to X$ with U a scheme, $\bigl(U, s(U, \varphi)\bigr)$ is an oriented d-critical locus, so as in §4, BBDJS constructs a perverse sheaf $P^\bullet_{U,\varphi}$ on U. Given a diagram

$$\begin{tikzcd} & V \arrow[dr, "\psi"] & \\ U \arrow[ur, "\alpha"] \arrow[rr, "\varphi"'] & \arrow[u, phantom, "\eta\Uparrow"] & X \end{tikzcd}$$

with U, V schemes and φ, ψ smooth, we can construct a natural isomorphism $P^\bullet_{\alpha,\eta} : \alpha^*(P^\bullet_{V,\psi})[\dim \varphi - \dim \psi] \to P^\bullet_{U,\varphi}$.
All this data $P^\bullet_{U,\varphi}$, $P^\bullet_{\alpha,\eta}$ is equivalent to a perverse sheaf on X.

Thus we prove:

Theorem (Ben-Bassat, Brav, Bussi, Joyce)

Let (X,s) be a d-critical stack, with an orientation $K_{X,s}^{1/2}$. Then we can construct a canonical perverse sheaf $P_{X,s}^\bullet$ on X.

Corollary

Suppose Y is a Calabi–Yau 3-fold and $\mathcal{M}$ a classical moduli stack of coherent sheaves F on Y, or of complexes $F^\bullet$ in $D^b\operatorname{coh}(Y)$ with $\operatorname{Ext}^{<0}(F^\bullet,F^\bullet)=0$, with (symmetric) obstruction theory $\phi:\mathcal{E}^\bullet\to\mathbb{L}_{\mathcal{M}}$. Suppose we are given a square root $\det(\mathcal{E}^\bullet)^{1/2}$ for $\det(\mathcal{E}^\bullet)$. Then we construct a natural perverse sheaf $P_{\mathcal{M},s}^\bullet$ on $\mathcal{M}$.

The *hypercohomology* $\mathbb{H}^*(P_{\mathcal{M},s}^\bullet)$ is a categorification of the Donaldson–Thomas theory of Y.

Motivic Milnor fibres for d-critical stacks

We can also generalize BJM to d-critical stacks:

Theorem (Ben-Bassat, Brav, Bussi, Joyce)

Let (X,s) be a d-critical stack, with an orientation $K_{X,s}^{1/2}$. Then we can construct a natural motive $MF_{X,s}$ in a certain ring of $\hat\mu$-equivariant motives $\overline{\mathcal{M}}_X^{\hat\mu}$ on X, such that if $\varphi:U\to X$ is smooth and U is a scheme then

$$\varphi_*(MF_{U,s(U,\varphi)})=MF_{X,s}\cdot[\varphi:U\to X]\cdot\mathbb{L}^{-\dim\varphi/2},$$

where $MF_{U,s(U,\varphi)}$ for the scheme case is as in BJM, §5.

For CY3 moduli stacks, these $MF_{X,s}$ are basically Kontsevich–Soibelman motivic Donaldson–Thomas invariants.

7. Current and future projects

(A) Extension of Donaldson–Thomas theory to the derived category (Vittoria Bussi, work in progress)

The material on writing atlases of Calabi-Yau 3-fold moduli stacks as critical loci, ought to enable us to generalize the Joyce–Song identities for Behrend functions of such stacks from the coherent sheaves to complexes in $D^b \operatorname{coh}(Y)$. We could then deduce wall-crossing formulae for D–T invariants under change of stability conditions in the derived category, with interesting consequences.

(B) Computation of examples

We now know that Calabi–Yau 3-fold moduli schemes and stacks carry d-critical structures, and (given orientation data) natural perverse sheaves/MHMs and motives. But we have not really computed any examples yet.

Compute these in interesting examples (e.g. Hilbert schemes of points on a Calabi–Yau 3-fold, Balázs Szendrői work in progress), find the hypercohomology of the perverse sheaf/MHM, and so on.

(C) Torus localization

Let (X, s) be a d-critical locus, and $\rho : \mathbb{G}_m \times X \to X$ a $\mathbb{G}_m$-action on X which scales the d-critical structure by $\rho(\lambda)^\star(s) = \lambda^d s$ for some $d \in \mathbb{Z}$. (The case $d = 0$ is special.)

Question: if X is proper, can you give an expression for the motive $MF_{X,s}$ (or possibly some reduction of it) in terms of data solely on the fixed locus $X^{\rho(\mathbb{G}_m)}$ of ρ in X?

I have a likely-looking conjecture for $d = 0$. Case $d > 0$ looks not so nice, but may simplify when $d = 1$.

Is something similar possible for the perverse sheaf $P^\bullet_{X,s}$, and its hypercohomology? (Balázs Szendrői, work in progress.)

Extend to d-critical stacks?

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(D) Shifted Lagrangian Neighbourhood Theorem

Let $(\mathbf{X}, \omega_{\mathbf{X}})$ be a k-shifted symplectic derived scheme (or stack) for $k < 0$. Then PTVV define *Lagrangians* $\mathbf{i} : \mathbf{L} \to \mathbf{X}$.

Problem: We already have a 'Darboux Theorem' putting $(\mathbf{X}, \omega_{\mathbf{X}})$ locally into a standard form (BBJ). Prove a 'Lagrangian Neighbourhood Theorem' putting $\mathbf{L}$ locally into a standard form, relative to the BBJ standard form for $(\mathbf{X}, \omega_{\mathbf{X}})$.

I already know the answer, I think. I just need everyone else to go away for a month while I write it down, or a helpful postdoc.

Dominic Joyce, Oxford University Shifted symplectic geometry

(E) Extensions of the d-critical story

D-critical loci are 'classical truncations' of -1-shifted symplectic derived schemes. I believe there are very similar 'classical truncations' of PTVV-style Lagrangians in 0-shifted symplectic schemes – these would be derived Lagrangians in classical complex/algebraic symplectic manifolds – and of PTVV-style Lagrangians in -1-shifted symplectic schemes, which would give a notion of Lagrangian in a d-critical locus.

Again, I already know the answer, I think.

One can then ask which parts of the theory factor via the classical truncations (e.g. perverse sheaves do, matrix factorizations don't).

(F) A conjecture on morphisms of perverse sheaves

Let $(\mathbf{X}, \omega_{\mathbf{X}})$ be a -1-shifted symplectic derived scheme, and $\mathbf{i} : \mathbf{L} \to \mathbf{X}$ a Lagrangian. Or else work with the classical truncations (a d-critical locus (X, s) for $(\mathbf{X}, \omega_{\mathbf{X}})$).
Choose an orientation $K_{X,s}^{1/2}$ for $(\mathbf{X}, \omega_{\mathbf{X}})$. There is then a notion of relative orientation for $\mathbf{i} : \mathbf{L} \to \mathbf{X}$, choose one of these.
We get a perverse sheaf $P^\bullet_{\mathbf{X},\omega_{\mathbf{X}}}$ on $\mathbf{X}$ (BBDJS).
Conjecture: there is a natural morphism in $D^b_c(\mathbb{L})$

$$\mu_{\mathbf{L}} : \mathbb{Q}_{\mathbf{L}}[\mathrm{vdim}\,\mathbf{L}] \longrightarrow \mathbf{i}^!(P^\bullet_{\mathbf{X},\omega_{\mathbf{X}}}), \tag{2}$$

with given local models in 'Darboux form' presentations for $\mathbf{X}, \mathbf{L}$.
This Conjecture has important consequences.

I already know local models for $\mu_{\mathbf{L}}$ in (2). What makes this difficult is that local models are not enough: $\mu_{\mathbf{L}}$ is a morphism of complexes, not of (perverse) sheaves, and such morphisms do not glue like sheaves. For instance, one could imagine $\mu_{\mathbf{L}}$ to be globally nonzero, but zero on the sets of an open cover of $\mathbf{L}$.
So to construct $\mu_{\mathbf{L}}$, we have to do a gluing problem in an ∞-category, probably using hypercovers. I have a sketch of one way to do this (over $\mathbb{C}$). It is not easy.
Maybe gluing local models naïvely is not the best approach for this problem, need some more advanced Lurie-esque technology?

(G) 'Fukaya categories' of complex symplectic manifolds

Let (S,ω) be a complex symplectic manifold, with $\dim_{\mathbb{C}} S = 2n$, and $L, M \subset X$ be complex Lagrangians (not supposed compact or closed). The intersection $L \cap M$, as a complex analytic space, has a d-critical structure s (Vittoria). Given square roots of canonical bundles $K_L^{1/2}, K_M^{1/2}$, we get an orientation on $(L \cap M, s)$, and so a perverse sheaf $P^\bullet_{L,M}$ on $L \cap M$.
I claim that we should think of the shifted hypercohomology $\mathbb{H}^{*-n}(P^\bullet_{L,M})$ as a substitute for the Lagrangian Floer cohomology $HF^*(L,M)$ in symplectic geometry. But $HF^*(L,M)$ is the morphisms in the derived Fukaya category $D^b\mathrm{Fuk}(S,\omega)$.

Problem: given a complex symplectic manifold (S,ω), build a 'Fukaya category' with objects $(L, K_L^{1/2})$ for L a complex Lagrangian, and graded morphisms $\mathbb{H}^{*-n}(P^\bullet_{L,M})$.
Extend to *derived* Lagrangians L in (S,ω).
Work out the 'right' way to form a 'derived Fukaya category' for (S,ω) out of this, as a (Calabi–Yau?) triangulated category.
Show that (derived) Lagrangian correspondences induce functors between these derived Fukaya categories.
Question: can we include complex coisotropic submanifolds as objects? Maybe using $\mathscr{D}$-modules?

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The Conjecture in (F) is what we need to define composition of morphisms in this 'Fukaya category', as follows. If L, M, N are Lagrangians in (S,ω), then $M\cap L, N\cap M, L\cap N$ are -1-shifted symplectic / d-critical loci, and $L\cap M\cap N$ is Lagrangian in the product $(M\cap L)\times(N\cap M)\times(L\cap N)$ (Oren). Applying the Conjecture to $L\cap M\cap N$ and rearranging gives a morphism of constructible complexes

$$\mu_{L,M,N} : P^\bullet_{L,M} \overset{L}{\otimes} P^\bullet_{M,N}[n] \longrightarrow P^\bullet_{L,N}.$$

Taking hypercohomology gives the multiplication
$\mathrm{Hom}^*(L,M)\times\mathrm{Hom}^*(M,N)\to\mathrm{Hom}^*(L,N)$.

Dominic Joyce, Oxford University Shifted symplectic geometry

(H) Cohomological Hall Algebras

Let Y be a Calabi–Yau 3-fold, and $\mathcal{M}$ the moduli stack of coherent sheaves (or suitable complexes) on Y. Then BBBJ makes $\mathcal{M}$ into a d-critical stack $(\mathcal{M}, s)$. Suppose we have 'orientation data' for Y, i.e. an orientation $K^{1/2}_{\mathcal{M},s}$, with compatibility condition on exact sequences.
Then we have a perverse sheaf $P^\bullet_{\mathcal{M},s}$, with hypercohomology $\mathbb{H}^*(P^\bullet_{\mathcal{M},s})$.
We would like to define an associative multiplication on $\mathbb{H}^*(P^\bullet_{\mathcal{M},s})$, making it into a *Cohomological Hall Algebra*, Kontsevich–Soibelman style (arXiv:1006.2706).

Let $\mathcal{E}xact$ be the stack of short exact sequences $0 \to F_1 \to F_2 \to F_3 \to 0$ in $\mathrm{coh}(Y)$ (or distinguished triangles in $D^b \mathrm{coh}(Y)$), with projections $\pi_1, \pi_2, \pi_3 : \mathcal{E}xact \to \mathcal{M}$.
Claim: $\pi_1 \times \pi_2 \times \pi_3 : \mathcal{E}xact \to \mathcal{M} \times \bar{\mathcal{M}} \times \mathcal{M}$ is Lagrangian, where the central term $\bar{\mathcal{M}}$ has the opposite sign -1-shifted symplectic structure / d-critical structure. (Oren Ben-Bassat, work in progress.)
Then apply stack version of Conjecture in (F) to get COHA multiplication, in a similar way to the Fukaya category case.

(I) Talk to Representation Theorists

If we can complete (H) we will have a large source of algebras coming from Calabi–Yau 3-folds, plus ways of getting representations of these (e.g. by looking at moduli schemes of pairs, such as Hilbert schemes).

Do these have an interesting representation theory? What can we say about it?

(J) Gluing matrix factorization categories

Suppose $f : U \to \mathbb{A}^1$ is a regular function on a smooth scheme U. The *matrix factorization category* $\mathrm{MF}(U, f)$ is a $\mathbb{Z}_2$-periodic triangulated category. It depends only on U, f in a neighbourhood of $\mathrm{Crit}(f)$, and we can think of it as a sheaf of triangulated categories on $\mathrm{Crit}(f)$.
By BBJ, -1-shifted symplectic derived schemes $(X, \omega_{\mathbf{X}})$ are locally modelled on $\mathrm{Crit}(f : U \to \mathbb{A}^1)$.
Problem: Given a -1-shifted symplectic derived scheme $(X, \omega_{\mathbf{X}})$ with extra data (orientation and spin structure), construct a sheaf of $\mathbb{Z}_2$-periodic triangulated categories $\mathrm{MF}_{\mathbf{X},\omega_{\mathbf{X}}}$ on $\mathbf{X}$, such that if $(\mathbf{X}, \omega_{\mathbf{X}})$ is locally modelled on $\mathrm{Crit}(f : U \to \mathbb{A}^1)$, then $\mathrm{MF}_{\mathbf{X},\omega_{\mathbf{X}}}$ is locally modelled on $\mathrm{MF}(U, f)$.

Although d-critical loci (X, s) are also locally modelled on $\mathrm{Crit}(f : U \to \mathbb{A}^1)$, I do not expect the analogue for d-critical loci to work; $\mathrm{MF}_{\mathbf{X},\omega_{\mathbf{X}}}$ will encode derived data in $(\mathbf{X}, \omega_{\mathbf{X}})$ which is forgotten by the d-critical locus (X, s).

I expect that a Lagrangian $\mathbf{i} : \mathbf{L} \to \mathbf{X}$ (plus extra data) should define an object (global section of sheaf of objects) in $\mathrm{MF}_{\mathbf{X},\omega_{\mathbf{X}}}$, with nice properties.

It is conceivable that one could actually *define* $\mathrm{MF}_{\mathbf{X},\omega_{\mathbf{X}}}$ as a derived category of Lagrangians $\mathbf{i} : \mathbf{L} \to \mathbf{X}$ in $(\mathbf{X}, \omega_{\mathbf{X}})$

(K) Kapustin–Rozansky 2-category for complex symplectic manifolds

Given a complex symplectic manifold (S, ω), Kapustin and Rozansky conjecture the existence of an interesting 2-category, with objects complex Lagrangians L with $K_L^{1/2}$, such that $\mathrm{Hom}(L, M)$ is a $\mathbb{Z}_2$-periodic triangulated category (or sheaf of such on $L \cap M$), and if $L \cap M$ is locally modelled on $\mathrm{Crit}(f : U \to \mathbb{A}^1)$ then $\mathrm{Hom}(L, M)$ is locally modelled on $\mathrm{MF}(U, f)$.

A lot of this K–R Conjecture would follow by combining (G) Fukaya categories and (J) Gluing matrix factorization categories above.

Seeing what the rest of the K–R Conjecture requires should tell us some interesting properties to expect of $\mathrm{MF}_{\mathbf{X},\omega_{\mathbf{X}}}$.

(L) Categorifying Cohomological Hall Algebras?

Question: do COHAs of Calabi–Yau 3-folds admit a categorification using matrix factorization categories, in a similar way to the K–R conjectured categorification of the 'Fukaya categories' of complex symplectic manifolds?
Compare Kontsevich and Soibelman arXiv:1006.2706, §8.1, 'Categorification of critical COHA'.