Riemannian holonomy groups and calibrated geometry

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Lecture 12. The holonomy groups G_2 and Spin(7)

These slides available at

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Geometry of G_2

The action of G_2 on $\mathbb{R}^7$ preserves the metric g_0 and a 3-form φ_0 on $\mathbb{R}^7$. Let g be a metric and φ a 3-form on M^7. We call (φ, g) a G_2-*structure* if $(\varphi, g) \cong (\varphi_0, g_0)$ at each $x \in M$. We call $\nabla\varphi$ the *torsion* of (φ, g).

If $\nabla\varphi = 0$ then (φ, g) is *torsion-free*.

We have $\nabla\varphi = 0$ iff $d\varphi = d^*\varphi = 0$. If (φ, g) is torsion-free then $\mathrm{Hol}(g) \subseteq G_2$, and g is Ricci-flat. Conversely, if g is a metric on M^7 then $\mathrm{Hol}(g) \subseteq G_2$ iff there is a G_2-structure (φ, g) with $\nabla\varphi = 0$. If M is compact and $\mathrm{Hol}(g) \subseteq G_2$ then $\mathrm{Hol}(g) = G_2$ iff $\pi_1(M)$ is finite.

Let M be compact and oriented, and (φ, g) a torsion-free G_2-structure on M. Then $\mathrm{d}\varphi = \mathrm{d} * \varphi = 0$, so $[\varphi] \in H^3(M, \mathbb{R})$ and $[*\varphi] \in H^4(M, \mathbb{R})$. Let $\mathcal{M}$ be the *moduli space* of oriented torsion-free G_2-structures on M up to diffeomorphisms isotopic to the identity.

Theorem. $\mathcal{M}$ is smooth of dimension $b^3(M)$. The maps $\mathcal{M} \to H^3(M,\mathbb{R})$ and $\mathcal{M} \to H^4(M,\mathbb{R})$ taking $[(\varphi,g)] \mapsto [\varphi]$, $[(\varphi,g)] \mapsto [*\varphi]$ are local diffeomorphisms. The map $\mathcal{M} \to H^3(M,\mathbb{R})\times H^4(M,\mathbb{R})$ taking $[(\varphi,g)] \mapsto ([\varphi],[*\varphi])$ has image a Lagrangian submanifold in $H^3(M,\mathbb{R}) \times H^4(M,\mathbb{R})$.

Constructing compact G_2-manifolds

I constructed the first compact 7-manifolds with holonomy G_2 in 1993-4. This is difficult as such manifolds have only finite symmetry groups, and are not algebraic.

It is interesting as such manifolds are Ricci-flat, they are important in String Theory, and they have beautiful geometrical properties.

The construction, 1

First we choose a compact 7-manifold M. We write down an explicit G_2-structure (φ, g) on M with *small torsion*. Then we use analysis to deform to a nearby G_2-structure $(\tilde{\varphi}, \tilde{g})$ with *zero torsion.* If $\pi_1(M)$ is finite then $\mathrm{Hol}(\tilde{g}) = G_2$ as we want.

The construction, 2

It is not easy to find G_2-structures with small torsion! Here is one way to do it, in 4 steps.

Step 1. Choose a finite group Γ of isometries of the 7-torus T^7, and a flat, Γ-invariant G_2-structure (φ_0, g_0) on T^7. Then T^7/Γ is compact, with a torsion-free G_2-structure (φ_0, g_0).

Step 2. However, T^7/Γ is an *orbifold*. We repair its singularities to get a compact 7-manifold M. We can resolve *complex* orbifolds using algebraic geometry.

If the singularities of T^7/Γ locally resemble $\mathcal{S}^1 \times \mathbb{C}^3/G$ for $G \subset SU(3)$, then we model M on a *crepant resolution* X of $\mathbb{C}^3/G$.

Step 3. M is made by gluing patches $\mathcal{S}^1 \times X$ into T^7/Γ. Now X carries ALE metrics of holonomy $SU(3)$. As $SU(3) \subset G_2$, these give torsion-free G_2-structures on $\mathcal{S}^1 \times X$.

We join them to (φ_0, g_0) on T^7/Γ to get a family $\{(\varphi_t, g_t) : t \in (0, \epsilon)\}$ of G_2-structures on M.

Step 4. This (φ_t, g_t) has $\nabla\varphi_t = O(t^4)$. So $\nabla\varphi_t$ is small for small t. But $R(g_t) = O(t^{-2})$ and the injectivity radius $\delta(g_t) = O(t)$, since g_t becomes singular as $t \to 0$.

For small t we deform (φ_t, g_t) to $(\tilde{\varphi}_t, \tilde{g}_t)$ with $\nabla\tilde{\varphi}_t = 0$, using analysis. Then $\mathrm{Hol}(\tilde{g}_t) = G_2$ if $\pi_1(M)$ is finite.

Steps in the analysis proof:

- Arrange that $d\varphi_t = 0$ and $d^*\varphi_t = d^*\psi_t$, where $\psi_t = O(t^4)$.
- Set $\tilde{\varphi}_t = \varphi_t + d\eta_t$, where $d^*\eta_t = 0$.
- Then $(\tilde{\varphi}_t, \tilde{g}_t)$ is torsion-free iff

$$(d^*d + dd^*)(\eta_t) = d^*\psi_t + dF(d\eta_t)$$

where F is nonlinear with $F(\chi) = O\bigl(|\chi|^2\bigr)$.

- Integrating by parts gives $\|\mathrm{d}\eta_t\|_{L^2} \leqslant 2\|\psi_t\|_{L^2}$ when $\|\mathrm{d}\eta_t\|_{C^0}$ is small.
- Solve by contraction method in $L^{14}_2(\Lambda^2 T^*M)$, using elliptic regularity of $\mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*$, balls of radius t and Sobolev embedding.

The construction, 3

Using different groups Γ acting on T^7, and resolving T^7/Γ in more than one way, we get many compact manifolds with holonomy G_2.

Geometry of Spin(7)

The action of Spin(7) on $\mathbb{R}^8$ preserves the metric g_0 and a 4-form Ω_0 on $\mathbb{R}^8$. Let g be a metric and Ω a 4-form on M^8. We call (Ω, g) a Spin(7)-*structure* if $(\Omega, g) \cong (\Omega_0, g_0)$ at each $x \in M$. We call $\nabla\Omega$ the *torsion* of (Ω, g).

If $\nabla\Omega = 0$ then (Ω, g) is *torsion-free*. Also $\nabla\Omega = 0$ iff $\mathrm{d}\Omega = 0$. If $\nabla\Omega = 0$ then $\mathrm{Hol}(g) \subseteq \mathrm{Spin}(7)$. If g is a metric on M^8 then $\mathrm{Hol}(g) \subseteq \mathrm{Spin}(7)$ iff there is a $\mathrm{Spin}(7)$-structure (Ω, g) with $\nabla\Omega = 0$. If M is compact and $\mathrm{Hol}(g) \subseteq \mathrm{Spin}(7)$ then g has holonomy $\mathrm{Spin}(7)$ iff $\pi_1(M) = \{1\}$, $\hat{A}(M) = 1$.

Compact examples

The first examples of compact 8-manifolds with holonomy Spin(7) were constructed by me in 1995. Here is how. Let T^8 be a torus with flat Spin(7)-structure (Ω_0, g_0), and let Γ be a finite group acting on T^8 preserving (Ω_0, g_0). Then T^8/Γ is an *orbifold*.

We choose Γ so that the singularities of T^8/Γ are locally modelled on $\mathbb{C}^4/G$, for $G \subset SU(4)$.

Then we use complex algebraic geometry to resolve T^8/Γ, giving a compact 8-manifold M.

Finally we use analysis to construct metrics on M with holonomy Spin(7).

A second construction

Another way to make compact 8-manifolds with holonomy Spin(7) is to start not with T^8 but with a *Calabi-Yau 4-orbifold* Y with isolated singular points $p_1, \ldots, p_k$. Find the complex orbifold Y using algebraic geometry, and the metric by the orbifold Calabi Conjecture.

Instead of a group Γ we use an antiholomorphic, isometric involution σ on Y fixing only the p_j. Then $Z = Y/\langle\sigma\rangle$ is a real 8-orbifold with singular points $p_1, \ldots, p_k$. We resolve the p_j to get a compact 8-manifold M, and construct holonomy Spin(7) metrics on M.