Riemannian holonomy groups and calibrated geometry

Dominic Joyce, Oxford

Lecture 14. Calibrated m-folds in $\mathbb{R}^n$

These slides available at

www.maths.ox.ac.uk/∼joyce/talks.html

7. Calibrated m-folds in $\mathbb{R}^n$

7.1 Special Lagrangian submanifolds

Let $\mathbb{C}^m$ have complex coordinates $(z_1, \ldots, z_m)$, metric $g = \sum_{j=1}^m |\mathrm{d}z_j|^2$, Kähler form $\omega = \frac{i}{2}\sum_{j=1}^m \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j$, and complex volume form $\Omega = \bigwedge_{j=1}^m \mathrm{d}z_j$. An oriented real m-submanifold L in $\mathbb{C}^m$ is called *special Lagrangian* if it is calibrated w.r.t. $\mathrm{Re}\,\Omega$.

More generally, L is special Lagrangian with *phase* $e^{i\theta}$ if it is calibrated with respect to $\cos\theta\,\mathrm{Re}\,\Omega + \sin\theta\,\mathrm{Im}\,\Omega$. The subgroup of $\mathrm{GL}(2m,\mathbb{R})$ preserving g,ω and Ω is $\mathrm{SU}(m)$. Define $U = \mathbb{R}^m$ in $\mathbb{C}^m$. Then U is calibrated w.r.t. $\mathrm{Re}\,\Omega$. Any real vector subspace V in $\mathbb{C}^m$ calibrated w.r.t. $\mathrm{Re}\,\Omega$ is of the form $V = \gamma \cdot U$ for some $\gamma \in \mathrm{SU}(m)$. The stabilizer of U in $\mathrm{SU}(m)$ is $\mathrm{SO}(m)$.

This proves:

Proposition. *The family $\mathcal{F}$ of oriented real m-dimensional vector subspaces V in $\mathbb{C}^m$ with* $\operatorname{Re}\Omega|_V = \operatorname{vol}_V$ *is isomorphic to* $\mathrm{SU}(m)/\mathrm{SO}(m)$, *and has dimension* $\frac{1}{2}(m^2+m-2)$.

An m-submanifold L in $\mathbb{C}^m$ is special Lagrangian iff $T_xL \in \mathcal{F}$ for all $x \in L$.

Now $\omega|_U = \operatorname{Im}\Omega|_U = 0$. As $\mathrm{SU}(m)$ preserves ω and $\operatorname{Im}\Omega$ and acts transitively on $\mathcal{F}$, we have $\omega|_V = \operatorname{Im}\Omega|_V = 0$ for any $V \in \mathcal{F}$. Conversely, if $V \cong \mathbb{R}^m$ and $\omega|_V = \operatorname{Im}\Omega|_V = 0$, then $V \in \mathcal{F}$. This proves:

Proposition. *Let L be a real m-submanifold of $\mathbb{C}^m$. Then L is special Lagrangian, with some orientation, iff $\omega|_L \equiv 0$ and $\operatorname{Im}\Omega|_L \equiv 0$.*

7.2 SL 2-folds and the quaternions

Let $\mathbb{C}^2$ have its standard complex structure I. A 2-fold L in $\mathbb{C}^2$ is special Lagrangian iff it is *holomorphic* with respect to a second complex structure J on $\mathbb{C}^2$. Here I, J and $K = IJ$ are the complex structures on the *quaternions* $\mathbb{H}$. So SL 2-folds are well understood.

7.3 SL m-folds as graphs

Let $f : \mathbb{R}^m \to \mathbb{R}$ be smooth, and define

$$\Gamma_f = \Big\{ (x_1 + i\frac{\partial f}{\partial x_1}, \ldots, x_m + i\frac{\partial f}{\partial x_m}) : x_1, \ldots, x_m \in \mathbb{R} \Big\}.$$

Then Γ_f is a Lagrangian m-fold in $\mathbb{C}^m$, the graph of $\mathrm{d}f$.

It is *special* Lagrangian iff $\mathrm{Im}\,\Omega|_{\Gamma_f} \equiv 0$, which holds iff

$$\mathrm{Im}\,\mathrm{det}_{\mathbb{C}}\big(I + i\,\mathrm{Hess}\,f\big) \equiv 0$$

on $\mathbb{C}^m$. This is a *second-order nonlinear elliptic p.d.e.* on f.

7.4 Local deformations of SL m-folds

What do special Lagrangian graphs Γ_f in $\mathbb{C}^m$ look like when $f \approx 0$? For small f,

$$\operatorname{Im}\det_{\mathbb{C}}\bigl(I + i\,\mathrm{Hess}\, f\bigr) \approx \mathrm{Tr}\,\mathrm{Hess}\, f = \Delta f,$$

where Δ is the *Laplacian*. Thus, SL m-folds near $\Gamma_0 = \mathbb{R}^m$ in $\mathbb{C}^m$ are roughly parametrized by small *harmonic functions* on $\mathbb{R}^m$.

But Γ_f is the graph of $\mathrm{d}f$, and if f is harmonic then $\mathrm{d}f$ is a closed, coclosed 1-form on $\mathbb{R}^m$. This gives:

Principle. *Small special Lagrangian deformations of a special Lagrangian m-fold L are approximately parametrized by closed and coclosed 1-forms α on L.*

This is the idea behind McLean's Theorem (next lecture).

Written using graphs, deforming SL m-folds gives $\Delta f=0$, one equation on one function. But written using submanifolds, it is $\frac{1}{2}(m-1)(m+2)$ equations on m functions, and looks *overdetermined*. As $\mathrm{d}\omega = 0$, these $\frac{1}{2}(m-1)(m+2)$ equations are dependent, and the problem is not overdetermined. So $\mathrm{d}\omega = 0$ is an *integrability condition* for the existence of many SL m-folds.

7.5 Associative 3-folds and coassociative 4-folds

Define a 3-form φ on $\mathbb{R}^7$ by

$$\varphi = \mathrm{d}\mathbf{x}_{123} + \mathrm{d}\mathbf{x}_{145} + \mathrm{d}\mathbf{x}_{167} + \mathrm{d}\mathbf{x}_{246} - \mathrm{d}\mathbf{x}_{257} - \mathrm{d}\mathbf{x}_{347} - \mathrm{d}\mathbf{x}_{356}.$$

The subgroup of $\mathrm{GL}(7,\mathbb{R})$ preserving φ is the *holonomy group* G_2. It also fixes the 4-form $*\varphi$, the Euclidean metric $g = \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_7^2$, and the orientation on $\mathbb{R}^7$. Both φ and $*\varphi$ are calibrations on $\mathbb{R}^7$.

Define an *associative 3-fold* to be a 3-fold in $\mathbb{R}^7$ calibrated w.r.t. φ, and a *coassociative 4-fold* to be a 4-fold in $\mathbb{R}^7$ calibrated w.r.t. $*\varphi$.

Define an *associative 3-plane* to be an oriented subspace $V \cong \mathbb{R}^3$ in $\mathbb{R}^7$ with $\varphi|_V = \mathrm{vol}_V$, and a *coassociative 4-plane* to be an oriented subspace $V \cong \mathbb{R}^4$ in $\mathbb{R}^7$ with $*\varphi|_V = \mathrm{vol}_V$.

Then we have:

Proposition. *The families* $\mathcal{F}^3$ *of associative 3-planes in* $\mathbb{R}^7$ *and* $\mathcal{F}^4$ *of coassociative 4-planes in* $\mathbb{R}^7$ *are both isomorphic to* $G_2/SO(4)$, *with dimension 8.*

Also, we can prove:

Proposition. *Let* L *be a real 4-submanifold in* $\mathbb{R}^7$. *Then* L *is coassociative, with some orientation, iff* $\varphi|_L \equiv 0$.

As $\mathcal{F}^3$ is codimension 4 in the set of all 3-planes in $\mathbb{R}^7$, for a 3-fold L to be associative is 4 equations. But the freedom to vary L is 4 functions. So, deforming associative 3-folds involves 4 equations on 4 functions, and is *determined*. The equation is *elliptic*, a Dirac operator on L. So the deformation theory of associative 3-folds is quite well-behaved.

7.6 Cayley 4-folds in $\mathbb{R}^8$

The *holonomy group* Spin(7) is the stabilizer of a 4-form Ω on $\mathbb{R}^8$. It also preserves the orientation and the Euclidean metric $g = \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_8^2$ on $\mathbb{R}^8$. The 4-form Ω is a calibration, and 4-folds in $\mathbb{R}^8$ calibrated w.r.t. Ω are called *Cayley 4-folds*.

Oriented subspaces $V \cong \mathbb{R}^4$ in $\mathbb{R}^8$ with $\Omega|_V = \mathrm{vol}_V$ are called *Cayley 4-planes*.

The family of Cayley 4-planes has codimension 4 in the set of all 4-planes in $\mathbb{R}^8$. Thus, the deformation problem for a Cayley 4-fold L may be written as 4 real equations on 4 real functions, a determined problem. In fact this is an elliptic equation, essentially the positive Dirac equation upon L. So the deformation theory of Cayley 4-folds is quite well-behaved.