# Riemannian holonomy groups and calibrated geometry

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# Lectures 15 and 16. Compact calibrated $k$-folds and special Lagrangian $m$-folds

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## 8. Compact calibrated $k$-folds in special holonomy $m$-folds

Let $(M, J, g)$ be a Calabi–Yau $m$-fold with complex volume form $\Omega$. Then $\mathrm{Re}\,\Omega$ is a *calibration* on $M$. Its calibrated submanifolds are called *special Lagrangian $m$-folds*, or *SL $m$-folds* for short.

What can we say about *compact* SL $m$-folds in $M$?

Let $(M, J, g, \Omega)$ be a Calabi–Yau $m$-fold and $N$ a compact SL $m$-fold in $M$. Let $\mathcal{M}_N$ be the moduli space of *SL deformations* of $N$. We ask:

**1.** Is $\mathcal{M}_N$ a manifold, and of what dimension?

**2.** Does $N$ persist under deformations of $(J, g, \Omega)$?

**3.** Can we compactify $\mathcal{M}_N$ by adding a 'boundary' of *singular* SL $m$-folds? If so, what are the singularities like?

These questions concern the *deformations* of SL $m$-folds, *obstructions* to their existence, and their *singularities*.

Questions 1 and 2 are fairly well understood, and we shall discuss them in this lecture.

Question 3 is an active area of research, and will be discussed next lecture.

## 8.1 Deformations of compact SL $m$-folds

Robert McLean proved the following result.

**Theorem.** *Let* $(M, J, g, \Omega)$ *be a Calabi–Yau* $m$*-fold, and* $N$ *a compact SL* $m$*-fold in* $M$*. Then the moduli space* $\mathcal{M}_N$ *of SL deformations of* $N$ *is a smooth manifold of dimension* $b^1(N)$*, the first Betti number of* $N$*.*

Here is a sketch of the proof. Let $\nu \longrightarrow N$ be the *normal bundle* of $N$ in $M$. Then $J$ identifies $\nu \cong TN$ and $g$ identifies $TN \cong T^*N$. So $\nu \cong T^*N$. We can identify a small *tubular neighbourhood* $T$ of $N$ in $M$ with a neighbourhood of the zero section in $\nu$, identifying $\omega$ on $M$ with the symplectic structure on $T^*N$.

Let $\pi : T \longrightarrow N$ be the obvious projection.

Then graphs of small 1-forms $\alpha$ on $N$ are identified with sub-manifolds $N'$ in $T \subset M$ close to $N$. Which $\alpha$ correspond to *SL $m$-folds* $N'$?

Well, $N'$ is special Lagrangian iff $\omega|_{N'} \equiv \operatorname{Im}\Omega|_{N'} \equiv 0$.

Now $\pi|_{N'} : N' \to N$ is a diffeo-morphism, so this holds iff $\pi_*(\omega|_{N'}) = \pi_*(\operatorname{Im}\Omega|_{N'}) = 0$.

We regard $\pi_*(\omega|_{N'})$ and $\pi_*(\operatorname{Im}\Omega|_{N'})$ as functions of $\alpha$.

Calculation shows that $\pi_*(\omega|_{N'}) = \mathrm{d}\alpha$ and $\pi_*(\mathrm{Im}\,\Omega|_{N'}) = F(\alpha, \nabla\alpha)$, where $F$ is nonlinear. Thus, $\mathcal{M}_N$ is locally the set of small 1-forms $\alpha$ on $N$ with $\mathrm{d}\alpha \equiv 0$ and $F(\alpha, \nabla\alpha) \equiv 0$. Now $F(\alpha, \nabla\alpha) \approx \mathrm{d}(*\alpha)$ for small $\alpha$. So $\mathcal{M}_N$ is locally approximately the set of 1-forms $\alpha$ with $\mathrm{d}\alpha = \mathrm{d}(*\alpha) = 0$. But by Hodge theory this is the de Rham group $H^1(N, \mathbb{R})$, of dimension $b^1(N)$.

## 8.2 Natural coordinates on $\mathcal{M}_N$

Let $N$ be a compact SL $m$-fold in a Calabi–Yau $m$-fold $(M, J, g, \Omega)$. Let $\mathcal{M}_N$ be the moduli space of SL deformations of $N$. Then $\dim \mathcal{M}_N = b^1(N) = b^{m-1}(N)$. There are natural local identifications $\Phi, \Psi$ between $\mathcal{M}_N$ and $H^1(N, \mathbb{R}), H^{m-1}(N, \mathbb{R})$. Effectively these are *natural coordinate systems* on $\mathcal{M}_N$.

Let $U \subset \mathcal{M}_N$ be connected and simply-connected with $N \in U$. For each $N' \in U$, choose $\gamma : [0,1] \to U$ with $\gamma(0) = N$ and $\gamma(1) = N'$. Lift to $\Gamma : N \times [0,1] \to M$ with $\Gamma(N \times \{t\}) = \gamma(t)$. As $\omega|_{\gamma(t)} \equiv 0$ for all $t \in [0,1]$ we have $\Gamma^*(\omega) = \alpha_t \wedge \mathrm{d}t$, for $\alpha_t$ a closed 1-form on $N$. Define $\Phi(N') = \left[\int_0^1 \alpha_t \, \mathrm{d}t\right]$ in $H^1(N,\mathbb{R})$. It is independent of choices. We define $\Psi$ in the same way.

## 8.3 Obstructions to existence of SL $m$-folds

Let $M$ be a C-Y $m$-fold. Then an $m$-fold $N$ in $M$ is SL iff $\omega|_N \equiv \operatorname{Im}\Omega|_N = 0$. This holds only if $[\omega|_N] = [\operatorname{Im}\Omega|_N] = 0$ in $H^*(N,\mathbb{R})$. So we have:

**Lemma.** *Let $M$ be a Calabi–Yau $m$-fold, and $N$ a compact $m$-fold in $M$. Then $N$ is isotopic to an SL $m$-fold $N'$ in $M$ only if $[\omega|_N] = 0$ and $[\operatorname{Im}\Omega|_N] = 0$ in $H^*(N,\mathbb{R})$.*

The Lemma is a *necessary* condition for a C-Y $m$-fold to have an SL $m$-fold in a given deformation class. Locally, it is also *sufficient*.

**Theorem.** *Let* $M_t : t \in (-\epsilon, \epsilon)$ *be a family of Calabi–Yau* $m$-*folds, and* $N_0$ *a compact SL* $m$-*fold of* $M_0$. *If* $[\omega_t|_{N_0}] = [\mathrm{Im}\,\Omega_t|_{N_0}] = 0$ *in* $H^*(N_0, \mathbb{R})$ *for all* $t$, *then* $N_0$ *extends to a family* $N_t : t \in (-\delta, \delta)$ *of SL* $m$-*folds in* $M_t$, *for* $0 < \delta \leqslant \epsilon$.

## 8.4 Coassociative 4-folds

Let $(M, g)$ have holonomy $G_2$. Then $M$ has a constant 3-form $\varphi$ and 4-form $*\varphi$. They are calibrations, whose calibrated submanifolds are called *associative 3-folds* and *coassociative 4-folds*. A 4-fold $N$ in $M$ is coassociative iff $\varphi|_N \equiv 0$. Also, if $N$ is coassociative then the normal bundle $\nu$ is isomorphic to $\Lambda^2_+ T^*N$, the self-dual 2-forms.

Using this, McLean proved:

**Theorem.** *Let* $(M, g)$ *be a 7-manifold with holonomy* $G_2$*, and* $N$ *a compact coassociative 4-fold in* $M$*. Then the moduli space* $\mathcal{M}_N$ *of coassociative deformations of* $N$ *is a smooth manifold of dimension* $b^2_+(N)$*.*

Roughly, nearby coassociative 4-folds correspond to small closed forms in $\Lambda^2_+ T^*N$, which are $H^2_+(N, \mathbb{R})$ by Hodge theory.

## 8.5 Associative 3-folds and Cayley 4-folds

*Associative 3-folds* in 7-manifolds with holonomy $G_2$, and *Cayley 4-folds* in 8-manifolds with holonomy Spin(7), cannot be defined by the vanishing of closed forms. This gives their deformation theory a different character. Here is how the theories work.

Let $N$ be a compact associative 3-fold or Cayley 4-fold in $M$. Then there are vector bundles $E, F \to N$ and a first order elliptic operator
$D_N : C^\infty(E) \to C^\infty(F)$.
The *kernel* $\operatorname{Ker} D_N$ is the set of *infinitesimal deformations* of $N$. The *cokernel* $\operatorname{Coker} D_N$ is the *obstruction space*. The *index* of $D_N$ is $\operatorname{ind}(D_N) = \dim \operatorname{Ker} D_N - \dim \operatorname{Coker} D_N$.

In the associative case $\text{ind}(D_N) = 0$, and in the Cayley case $\text{ind}(D_N) = \tau(N) - \frac{1}{2}\chi(N) - \frac{1}{2}[N] \cdot [N]$, where $\tau$ is the signature and $\chi$ the Euler characteristic. Generically $\text{Coker}\, D_N = 0$, and then $\mathcal{M}_N$ is locally a manifold with dimension $\text{ind}(D_N)$. If $\text{Coker}\, D_N \neq 0$, then $\mathcal{M}_N$ may be singular, or have a different dimension.

Note that the special Lagrangian and coassociative cases are unusual: there are *no* obstructions, and the moduli space is *always* a manifold of given dimension, without genericity assumptions.
This is a minor mathematical miracle.

## 9. Almost Calabi-Yau $m$-folds

An *almost Calabi-Yau $m$-fold* $(M, J, g, \Omega)$ is a compact complex $m$-fold $(M, J)$ with a Kähler metric $g$ with Kähler form $\omega$, and a nonvanishing holomorphic $(m, 0)$-form $\Omega$, the *holomorphic volume form*.

It is a *Calabi-Yau $m$-fold* if $|\Omega|^2 \equiv 2^m$. Then $\nabla\Omega = 0$ and $g$ is Ricci-flat.

## 9.1 Special Lagrangian $m$-folds

Let $(M, J, g, \Omega)$ be an almost Calabi-Yau $m$-fold. Let $N$ be a real $m$-submanifold of $M$. We call $N$ *special Lagrangian* (*SL*) if $\omega|_N \equiv \operatorname{Im}\Omega|_N \equiv 0$.

If $(M, J, g, \Omega)$ is a Calabi-Yau $m$-fold then $\operatorname{Re}\Omega$ is a *calibration* on $(M, g)$, and $N$ is an SL $m$-fold iff it is calibrated with respect to $\operatorname{Re}\Omega$.

## 9.2 Singular SL $m$-folds

General singularities of SL $m$-folds may be very bad, and difficult to study. Would like a class of singular SL $m$-folds with nice, well-behaved singularities to study in depth. Would like these to occur often in real life, i.e. of finite codimension in the space of all SL $m$-folds. SL $m$-folds with *isolated conical singularities* (*ICS*) are such a class.

Let $N$ be an SL $m$-fold in $M$ whose only singular points are $x_1, \ldots, x_n$. Near $x_i$ we can identify $M$ with $\mathbb{C}^m \cong T_{x_i}M$, and $N$ near $x_i$ approximates an SL $m$-fold in $\mathbb{C}^m$ with singularity at 0. We say $N$ has *isolated conical singularities* if near $x_i$ it converges with order $O(r^{\mu_i})$ for $\mu_i > 1$ to an SL cone $C_i$ in $\mathbb{C}^m$ nonsingular except at 0.

SL $m$-folds with ICS have a rich theory.

• **Examples.** Many examples of SL cones $C_i$ in $\mathbb{C}^m$ have been constructed. Rudiments of classification for $m = 3$.

• **Regularity near $x_1, \ldots, x_n$.** Let $\iota : N \to M$ be the inclusion. If $\nabla^k \iota$ converges to $C_i$ near $x_i$ with order $O(r^{\mu_i - k})$ for $k = 0, 1$ then it does so for all $k \geqslant 0$.

• **Deformation theory.** The moduli space $\mathcal{M}_N$ of deformations of $N$ is locally homeomorphic to $\Phi^{-1}(0)$, for smooth $\Phi : \mathcal{I} \to \mathcal{O}$ and fin. dim. vector spaces $\mathcal{I}, \mathcal{O}$ with $\mathcal{I}$ the image of $H^1_{\mathrm{cs}}(N', \mathbb{R})$ in $H^1(N', \mathbb{R})$, $N' = N \setminus \{x_1, \ldots, x_n\}$, and $\dim \mathcal{O} = \sum_{i=1}^{n}$s-ind$(C_i)$. Here s-ind$(C_i) \in \mathbb{N}$ is the *stability index*, the obstructions from $C_i$. If s-ind$(C_i) = 0$ for all $i$ then $\mathcal{M}_N$ is smooth.

• **Desingularization.** Let $C$ be an SL cone in $\mathbb{C}^m$, nonsingular except at 0. A nonsingular SL $m$-fold $L$ in $\mathbb{C}^m$ is *Asymptotically Conical* (*AC*) $C$ if $L$ converges to $C$ at infinity with order $O(r^\lambda)$ for $\lambda < 1$. Then $tL$ converges to $C$ as $t \rightarrow 0_+$. Thus, AC SL $m$-folds model how families of nonsingular SL $m$-folds develop singularities modelled on $C$.

If $N$ is an SL $m$-fold with ICS at $x_1, \ldots, x_n$ and cones $C_i$, and $L_1, \ldots, L_n$ are AC SL $m$-folds in $\mathbb{C}^m$ with cones $C_i$, then under cohomological conditions we can construct a family of compact nonsingular SL $m$-folds $\tilde{N}_t$ for small $t > 0$ converging to $N$ as $t \to 0$, by gluing $tL_i$ into $N$ at $x_i$, all $i$.

• **Generic codimension of singularities.** Given an SL $m$-fold $N$ with ICS in $M$, we have moduli spaces $\mathcal{M}_N$ of deformations of $N$, and $\mathcal{M}_{\tilde{N}}$ of desingularizations $\tilde{N}$ of $N$ made by gluing in $L_1, \ldots, L_n$. Here $\mathcal{M}_N$ is part of the *boundary* of $\mathcal{M}_{\tilde{N}}$. If $M$ is a *generic* almost C-Y $m$-fold then $\mathcal{M}_N$, $\mathcal{M}_{\tilde{N}}$ are smooth with known dimension.

Call dim $\mathcal{M}_{\tilde{N}}$−dim $\mathcal{M}_N$ the *index* of the singularities of $N$. It is the sum over $i$ of s-ind$(C_i)$ and topological terms from $L_i$. In a dim $k$ family $\mathcal{B}$ of SL $m$-folds in a generic almost C-Y $m$-fold $M$, only singularities with index $\leqslant k$ occur. For SYZ in generic $M$ we need to know about singularities with index 1,2,3 (and 4).

**Problem:** classify singularities with small index.

# 10. The SYZ Conjecture and SL singularities

## 10.1 String Theory and Mirror Symmetry

String Theory is a branch of physics which models particles as 1-dimensional objects – 'strings' – propagating in a space-time $M$. String theorists aim to *quantize* the string's motion.

This string quantum theory is very complicated, and poorly understood. For it to work, the universe must (supposedly) be 10-dimensional.
String Theorists say that our universe looks locally like $M = \mathbb{R}^4 \times X$, where $\mathbb{R}^4$ is Minkowski space, and $X$ is a compact Riemannian 6-manifold with radius of order $10^{-33}$cm, the Planck length.

By supersymmetry, $X$ has to be a *Calabi–Yau* 3-*fold*. String Theorists believe that each Calabi–Yau 3-fold $X$ has a quantization, a *Super Conformal Field Theory* (SCFT). Invariants of $X$ such as the Dolbeault groups $H^{p,q}(X)$ and the number of holomorphic curves in $X$ translate to properties of the SCFT.

Two different Calabi–Yau 3-folds $X, \hat{X}$ may have the *same* SCFT. Then the invariants of $X$ and $\hat{X}$ are related via properties of the SCFT. There is an automorphism of SCFT's which does *not* correspond to a classical automorphism of Calabi–Yau 3-folds. We say that $X, \hat{X}$ are *mirror* Calabi–Yau 3-folds if their SCFT's are related by this automorphism.

One can argue using String Theory that $H^{1,1}(X) \cong H^{2,1}(\hat{X})$ and $H^{2,1}(X) \cong H^{1,1}(\hat{X})$. The mirror transform exchanges even- and odd-dimensional cohomology. This is surprising! The Mirror Transform exchanges things to do with the complex structure of $X$, such as numbers of holomorphic '$\mathbb{CP}^1$'s in $X$, with things to do with the symplectic structure of $\hat{X}$, and vice versa.

Because the quantization process is poorly understood and not at all rigorous — it involves non-convergent path-integrals over horrible infinite-dimensional spaces — String Theory generates only conjectures about Mirror Symmetry, not proofs. However, many of these conjectures have been verified in particular cases.

## 10.2 Interpretations: Kontsevich and SYZ

There are two conjectural theories which explain Mirror Symmetry fairly mathematically. The first was due to Kontsevich in 1994. It says that for mirror Calabi–Yau 3-folds $X$ and $\hat{X}$, the derived category of coherent sheaves on $X$ is equivalent to the derived category of the Fukaya category of $\hat{X}$, and vice versa.

The second was due to Strominger, Yau and Zaslow in 1996.

**The SYZ Conjecture.** *Let $X, \hat{X}$ be mirror Calabi–Yau 3-folds. There is a compact 3-manifold $B$ and continuous, surjective $f : X \to B$ and $\hat{f} : \hat{X} \to B$, such that*

(i) *For $b$ in a dense $B_0 \subset B$, the fibres $f^{-1}(b), \hat{f}^{-1}(b)$ are dual SL 3-tori $T^3$ in $X, \hat{X}$.*

(ii) *For $b \notin B_0$, $f^{-1}(b)$ and $\hat{f}^{-1}(b)$ are singular SL 3-folds in $X, \hat{X}$.*

We call $f, \hat{f}$ *special Lagrangian fibrations*, and $\Delta = B \backslash B_0$ the *discriminant*.

In (i), the nonsingular fibres $T, \hat{T}$ of $f, \hat{f}$ are supposed to be *dual tori*. Topologically, this means an isomorphism $H^1(T, \mathbb{Z}) \cong H_1(\hat{T}, \mathbb{Z})$. But the metrics on $T, \hat{T}$ should really be dual as well. This only makes sense in the 'large complex structure limit', when the fibres are small and nearly flat.

## 10.3 U(1)-invariant SL 3-folds

Let U(1) act on $\mathbb{C}^3$ by $(z_1, z_2, z_3) \mapsto (e^{i\theta} z_1, e^{-i\theta} z_2, z_3)$. Let $N$ be a U(1)-invariant SL 3-fold. Then locally we can write $N$ in the form

$$\Big\{(z_1, z_2, z_3) : |z_1|^2 - |z_2|^2 = 2a, \quad z_1 z_2 = v(x,y) + iy, \quad z_3 = x + iu(x,y), \; x, y \in \mathbb{R}\Big\},$$

where $u, v : \mathbb{R}^2 \to \mathbb{R}$ satisfy

$$u_x = v_y \quad \text{and} \quad v_x = -2(v^2 + y^2 + a^2)^{1/2} u_y. \qquad (*)$$

Since $u_x = v_y$, there exists a potential function $f$ with $u = f_y$ and $v = f_x$. The 2nd equation of $(*)$ becomes

$$f_{xx}+2(f_x^2+y^2+a^2)^{1/2}f_{yy} = 0. \qquad (+)$$

This is a second-order quasi-linear equation. When $a \neq 0$ it is locally uniformly elliptic. When $a=0$ it is non-uniformly elliptic, except at *singular points* $f_x = y = 0$.

**Theorem A.** Let $S$ be a compact domain in $\mathbb{R}^2$ satisfying some convexity conditions.
Let $\phi \in C^{3,\alpha}(\partial S)$.
If $a \neq 0$ there exists a unique $f \in C^{3,\alpha}(S)$ satisfying $(+)$ with $f|_{\partial S} = \phi$. If $a = 0$ there exists a unique $f \in C^1(S)$ satisfying $(+)$ with weak second derivatives, with $f|_{\partial S} = \phi$.
Also $f$ depends continuously in $C^1(S)$ on $a, \phi$.

Theorem A shows that the Dirichlet problem for (+) is uniquely solvable in certain convex domains. The induced solutions $u, v \in C^0(S)$ of $(*)$ yield U(1)-invariant SL 3-folds in $\mathbb{C}^3$ satisfying certain boundary conditions over $\partial S$. When $a \neq 0$ these SL 3-folds are nonsingular, when $a = 0$ they are singular when $v = y = 0$.

**Theorem B.**
Let $\phi, \phi' \in C^{3,\alpha}(\partial S)$, let $a \in \mathbb{R}$ and let $f, f' \in C^{3,\alpha}(S)$ or $C^1(S)$ be the solutions of (+) from Theorem A with $f|_{\partial S} = \phi$, $f'|_{\partial S} = \phi'$. Let $u = f_y$, $v = f_x$, $u' = f'_y$, $v' = f'_x$. Suppose $\phi - \phi'$ has $k+1$ local maxima and $k+1$ local minima on $\partial S$. Then $(u, v) - (u', v')$ has no more than $k$ zeroes in $S^\circ$, counted with multiplicity.

**Theorem C.**
Let $u, v \in C^0(S)$ be a singular solution of $(*)$ with $a = 0$, e.g. from Theorem A. Then **either** $u(x, y) \equiv u(x, -y)$ and $v(x, y) \equiv -v(x, -y)$, so that $u, v$ is singular on the $x$-axis, **or** the singularities $(x, 0)$ of $u, v$ in $S^\circ$ are *isolated*, with a *multiplicity* $n > 0$. Multiplicity $n$ singularities occur in codimension $n$ of boundary data. All multiplicities occur.

**Theorem D.**
Let $U \subset \mathbb{R}^3$ be open, $S$ as above, and $\Phi : U \to C^{3,\alpha}(\partial S)$ continuous such that if $(a,b,c) \neq (a,b',c') \in U$ then $\Phi(a,b,c) - \Phi(a,b',c')$ has 1 local maximum and 1 local minimum.
For $\alpha = (a,b,c) \in U$, let $f_\alpha \in C^1(S)$ be the solution of $(+)$ from Theorem A with $f_\alpha|_{\partial S} = \Phi(\alpha)$.

Set $u_\alpha = (f_\alpha)_y$ and $v_\alpha = (f_\alpha)_x$. Let $N_\alpha$ be the SL 3-fold

$$\bigl\{(z_1, z_2, z_3) : |z_1|^2 - |z_2|^2 = 2a,\ z_1 z_2 = v_\alpha(x, y) + iy,\ z_3 = x + iu_\alpha(x, y),\ (x, y) \in S^\circ\bigr\}.$$

Then there exists an open $V \subset \mathbb{C}^3$ and a continuous map $F : V \to U$ with $F^{-1}(\alpha) = N_\alpha$. This is a U(1)-*invariant special Lagrangian fibration*. It can include *singular fibres*, of every multiplicity $n > 0$.

**Example.** Define $f : \mathbb{C}^3 \to \mathbb{R} \times \mathbb{C}$ by $f(z_1, z_2, z_3) = (a, b)$, where $2a = |z_1|^2 - |z_2|^2$ and

$$b = \begin{cases} z_3, & z_1 = z_2 = 0, \\ z_3 + \bar{z}_1 \bar{z}_2 / |z_1|, & a \geqslant 0,\ z_1 \neq 0, \\ z_3 + \bar{z}_1 \bar{z}_2 / |z_2|, & a < 0. \end{cases}$$

Then $f$ is a piecewise-smooth SL fibration of $\mathbb{C}^3$. It is not smooth on $|z_1| = |z_2|$.

The fibres $f^{-1}(a, b)$ are $T^2$-cones when $a = 0$, and non-singular $\mathcal{S}^1 \times \mathbb{R}^2$ when $a \neq 0$.

## 10.4 Conclusions

Using these SL fibrations as local models, if $X$ is a *generic* ACY 3-fold and $f : X \to B$ an SL fibration, I predict:

- $f$ is only piecewise smooth.
- All fibres have finitely many singular points.
- $\Delta$ is codim 1 in $B$. Generic singularities are modelled on the example above.
- Some codim 2 singularities are also locally U(1)-invariant.

• Codim 3 singularities are not locally U(1)-invariant.

• If $f : X \to B$, $\hat{f} : \hat{X} \to B$ are dual SL fibrations of mirror C-Y 3-folds, the discriminants $\Delta, \hat{\Delta}$ have different topology near codim 3 singular fibres, so $\Delta \neq \hat{\Delta}$.

This contradicts some statements of the SYZ Conjecture.

I regard SYZ as primarily a limiting statement about the 'large complex structure limit'.