Riemannian holonomy groups and calibrated geometry

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Lectures 3 and 4. Introduction to holonomy groups

These slides available at

`www.maths.ox.ac.uk/~joyce/talks.html`

2. Holonomy groups

2.1 Parallel transport

Let ∇^E be a connection on a vector bundle $E \rightarrow M$. Let $\gamma : [0,1] \rightarrow M$ be a smooth curve with $\gamma(0) = x$ and $\gamma(1) = y$. Then $\gamma^*(\nabla^E)$ is a connection on $\gamma^*(E) \rightarrow [0,1]$.

For each $e \in E_x$ there is a unique section s of $\gamma^*(E)$ with $s(0) = e$ and $\gamma^*(\nabla^E)s \equiv 0$. Define $P_\gamma(e) = s(1)$. Then $P_\gamma : E_x \rightarrow E_y$ is the *parallel transport map*.

Think of a connection ∇^E on $E \to M$ as identifying nearby fibres $E_x, E_{x'}$ for x, x' close together in M.

Parallel transport identifies the fibres of E all along a curve γ, so we can drag vectors along γ.

2.2 Holonomy groups

Let ∇^E be a connection on a vector bundle $E \to M$. Fix $x \in M$. Let $\gamma : [0,1] \to M$ be a piecewise-smooth loop based at x, so that $\gamma(0) = \gamma(1) = x$. Then P_γ is an invertible linear map $E_x \to E_x$. The *holonomy group* $\mathrm{Hol}_x(\nabla^E)$ of ∇^E is the set of parallel transports P_γ for all piecewise-smooth loops γ based at x.

Some properties of $\mathrm{Hol}_x(\nabla^E)$:

- It's a *Lie subgroup* of $\mathrm{GL}(E_x)$.
- Identify $E_x \cong \mathbb{R}^n$, so that $\mathrm{Hol}_x(\nabla^E) \subseteq \mathrm{GL}(n,\mathbb{R})$. Then $\mathrm{Hol}_x(\nabla^E)$ is *independent of basepoint* $x \in M$, *up to conjugation in* $\mathrm{GL}(n,\mathbb{R})$.
- If M is simply-connected, then $\mathrm{Hol}_x(\nabla^E)$ is connected.
- Let $\mathfrak{hol}_x(\nabla^E)$ be the *Lie algebra* of $\mathrm{Hol}_x(\nabla^E)$. Then $R(\nabla^E)_x \in \mathfrak{hol}_x(\nabla^E) \otimes \Lambda^2 T^*M$ in $\mathrm{End}(E_x) \otimes \Lambda^2 T^*M$.

Now let ∇ be a connection on TM. Then ∇ also acts on $\bigotimes^k TM \otimes \bigotimes^l T^*M$. A *constant tensor* S satisfies $\nabla S = 0$.

If S is constant then S_x is invariant under the action of $\mathrm{Hol}_x(\nabla)$ on $\bigotimes^k T_xM \otimes \bigotimes^l T_x^*M$. Any S_x in $\bigotimes^k T_xM \otimes \bigotimes^l T_x^*M$ invariant under $\mathrm{Hol}_x(\nabla)$ extends to a unique constant tensor S on M by parallel transport.

So *the constant tensors on* M *are determined by* $\mathrm{Hol}_x(\nabla)$.

2.3 Riemannian geometry

Let g be a *Riemannian metric* on M. The *Levi-Civita connection* is the unique, torsion-free connection ∇ on TM with $\nabla g = 0$. The *Riemann curvature* $R(\nabla)$ is a tensor $R^a{}_{bcd}$. The *Ricci curvature* of g is $R_{ab} = R^c{}_{acb}$. It satisfies $R_{ab} = R_{ba}$. We call g *Einstein* if $R_{ab} = \lambda g_{ab}$ for some $\lambda \in \mathbb{R}$, and *Ricci-flat* if $R_{ab} = 0$.

Define $R_{abcd} = g_{ae} R^e{}_{bcd}$. Then R_{abcd} and $\nabla_e R_{abcd}$ satisfy the equations

$$R_{abcd} + R_{adbc} + R_{acdb} = 0, \qquad (1)$$

$$\nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec} = 0, \qquad (2)$$

$$R_{abcd} = -R_{abdc} = -R_{bacd} = R_{cdab}. \qquad (3)$$

Eqns (1) and (2) are the *first* and *second Bianchi identities*, and hold as ∇ is torsion-free. In (3), $R_{abcd} = -R_{abdc}$ holds as curvature is a 2-form, and $R_{abcd} = -R_{bacd}$ as $\nabla g = 0$. The last part follows from (1).

2.4 Riemannian holonomy

Let g be a Riemannian metric on M, and $x \in M$. The *holonomy group* $\mathrm{Hol}_x(g)$ of g is the holonomy group $\mathrm{Hol}_x(\nabla)$ of its Levi-Civita connection. It is a closed Lie subgroup of $\mathrm{O}(n)$, which up to conjugation in $\mathrm{O}(n)$ is independent of basepoint x.

Riemannian holonomy groups have stronger properties than the general case.

Regard the Lie algebra $\mathfrak{hol}_x(g)$ as a vector subspace of $\Lambda^2 T_x^* M$. Using symmetries of R_{abcd}, eqn (3) of §2.3, we find that R_{abcd} lies in the vector subspace $S^2\,\mathfrak{hol}_x(g)$ in $\Lambda^2 T_x^* M \otimes \Lambda^2 T_x^* M$ at each $x \in M$.

Thus, the holonomy group imposes strong restrictions on the curvature tensor R_{abcd} of g. These are the basis of the classification of Riemannian holonomy groups.

2.5 Reducible metrics

Let (P, g) and (Q, h) be Riemannian manifolds with $\dim P, \dim Q > 0$. The *product metric* $g \times h$ on $P \times Q$ is given by $g \times h|_{(p,q)} = g|_p + h|_q$ for $p \in P$ and $q \in Q$.

Proposition 2.1 The holonomy groups satisfy $\mathrm{Hol}(g \times h) = \mathrm{Hol}(g) \times \mathrm{Hol}(h)$.

We call (M, g) *irreducible* if it is not locally isometric to a Riemannian product.

Theorem 2.2 *Let* (M, g) *be an irreducible Riemannian* n*-manifold. Then the representation of* Hol(g) *on* $\mathbb{R}^n$ *is irreducible.*

Proof. If $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^l$ for $\mathbb{R}^k, \mathbb{R}^l$ subrepresentations of Hol(g), can define a local isometry $M \cong P \times Q$ with $\dim P = k$, $\dim Q = l$, so M is reducible.

2.6 Symmetric spaces

A Riemannian manifold (M,g) is a *symmetric space* if for each $p \in M$ there is an isometry $s_p : M \to M$ with $s_p^2 = 1$ such that p is an isolated fixed point of s_p.

Let G be the group of isometries of (M,g) generated by $s_q \circ s_r$ for all $q,r \in M$. Then G is a connected Lie group and $M = G/H$ for some closed Lie subgroup H of G.

Symmetric spaces can be classified completely using Lie groups.

We call (M, g) *locally symmetric* if it is locally isometric to a symmetric space, and *non-symmetric* otherwise.

Theorem 2.5 *Let* (M, g) *have Levi-Civita connection* ∇ *and Riemann curvature* R. *Then* (M, g) *is locally symmetric if and only if* $\nabla R = 0$.

2.7 Berger's classification

Let M be a simply-connected n-manifold and g an irreducible, nonsymmetric Riemannian metric on M. Then either

(i) $\mathrm{Hol}(g)=\mathrm{SO}(n)$,

(ii) $n=2m$ and $\mathrm{Hol}(g)=\mathrm{U}(m)$,

(iii) $n=2m$ and $\mathrm{Hol}(g)=\mathrm{SU}(m)$,

(iv) $n=4m$ and $\mathrm{Hol}(g)=\mathrm{Sp}(m)$,

(v) $n=4m$ and

$$\mathrm{Hol}(g)=\mathrm{Sp}(m)\,\mathrm{Sp}(1),$$

(vi) $n=7$ and $\mathrm{Hol}(g)=G_2$, or

(vii) $n=8$ and $\mathrm{Hol}(g)=\mathrm{Spin}(7)$.

There are three assumptions in Berger's Theorem.

- As M is simply-connected, Hol(g) is connected.
- As g is irreducible, Hol(g) acts irreducibly on $\mathbb{R}^n$.
- As g is nonsymmetric, $\nabla R \not\equiv 0$.

Each excludes some possible holonomy groups. Without them, the list of holonomy groups would be much longer.

2.8 A sketch proof

Let M be simply-connected and g irreducible and nonsymmetric, and let $H = \text{Hol}(g)$. Then H is a closed, connected Lie subgroup of $\text{SO}(n)$ acting irreducibly on $\mathbb{R}^n$.

Berger made a list of all such subgroups up to conjugation, and applied two tests to see if each could be a holonomy group. Berger's list are the groups passing both tests.

Berger's first test

Let R_{abcd} be the Riemann curvature of g, and $\mathfrak{h}$ the Lie algebra of H. Then $R_{abcd} \in S^2\mathfrak{h}$. Also we have

$R_{abcd}+R_{adbc}+R_{acdb}=0$, (1)

the first Bianchi Identity. Let $\mathfrak{R}^H$ be the subset of $S^2\mathfrak{h}$ satisfying (1). Now $\mathfrak{R}^H$ must be big enough to generate $\mathfrak{h}$. That is, a generic element of $\mathfrak{R}^H$ cannot lie in $S^2\mathfrak{g}$ for $\mathfrak{g} \subset \mathfrak{h}$ a proper Lie subalgebra.

If $\mathfrak{R}^H$ is too small, H fails the first test.

Berger's second test

Now $\nabla_e R_{abcd}$ lies in $(\mathbb{R}^n)^* \otimes \mathfrak{R}^H$, and also satisfies

$$\nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec} = 0, \quad (2)$$

the second Bianchi identity. If these two requirements force $\nabla R = 0$, then g is locally symmetric. This excludes such H, the second test.

2.9 Simons' proof

The list of closed, connected Lie subgroups of SO(n) acting transitively on $\mathcal{S}^{n-1}$ is known; it consists of Berger's list together with Sp(m)U(1) in SO($4m$) and Spin(9) in SO(16). Simons gave a general proof that for M simply-connected and g irreducible and nonsymmetric, $H = \mathrm{Hol}(g)$ acts transitively on $\mathcal{S}^{n-1}$. It starts like this: if H is not transitive we can choose orthonormal $x, z \in \mathbb{R}^n$ with z orthogonal to $T_x(H \cdot x)$. Then $R \cdot (x \wedge z) = 0$ in $\mathrm{End}(\mathbb{R}^n)$ for all $R \in \mathfrak{R}^H$. The proof is very algebraic.

We then exclude the cases Sp(m) U (1) and Spin(9) to get an alternative proof of Berger's theorem.

2.10 Understanding Berger's list

The four *inner product algebras* are

$\mathbb{R}$ — *real numbers*.

$\mathbb{C}$ — *complex numbers*.

$\mathbb{H}$ — *quaternions*.

$\mathbb{O}$ — *octonions*,

or *Cayley numbers*.

Here $\mathbb{C}$ is not ordered,
$\mathbb{H}$ is not commutative,
and $\mathbb{O}$ is not associative.
Also we have $\mathbb{C} \cong \mathbb{R}^2$, $\mathbb{H} \cong \mathbb{R}^4$
and $\mathbb{O} \cong \mathbb{R}^8$, with $\operatorname{Im}\mathbb{O} \cong \mathbb{R}^7$.

Group	**Acts on**
$SO(m)$	$\mathbb{R}^m$
$O(m)$	$\mathbb{R}^m$
$SU(m)$	$\mathbb{C}^m$
$U(m)$	$\mathbb{C}^m$
$Sp(m)$	$\mathbb{H}^m$
$Sp(m)Sp(1)$	$\mathbb{H}^m$
G_2	$\mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$
$Spin(7)$	$\mathbb{O} \cong \mathbb{R}^8$

Thus there are two holonomy groups for each of $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.

Remarks on Berger's list

(i) SO(n) is the holonomy group of generic metrics.

(ii) Metrics g with Hol(g) $\subseteq$ U(m) are called *Kähler metrics*, a natural class of metrics on *complex manifolds*.

(iii) Metrics g with Hol(g) $\subseteq$ SU(m) are called *Calabi–Yau metrics*. They are Ricci-flat and Kähler.

(iv) Metrics g with $\mathrm{Hol}(g) \subseteq \mathrm{Sp}(m)$ are called *hyperkähler metrics*. They are also Ricci-flat and Kähler.

(v) Metrics g with holonomy group $\mathrm{Sp}(m)\,\mathrm{Sp}(1)$ for $m \geqslant 2$ are called *quaternionic Kähler metrics*. They are Einstein, but not Kähler.

(vi) and (vii) G_2 and Spin(7) are the *exceptional holonomy groups*.

Common features

• The *Kähler holonomy groups* are U(m), SU(m) and Sp(m). Any Riemannian manifold with one of these holonomy groups is Kähler, and thus a complex manifold.

• The *Ricci-flat holonomy groups* are SU(m), Sp(m), G_2 and Spin(7). Metrics with these holonomy groups are Ricci-flat, as $\mathfrak{R}^H$ has zero Ricci component.

2.11 Principal bundles and G-structures

Let M be a manifold and G a Lie group. A *principal bundle* over M with fibre G is a manifold P with a free (left) G-action and a smooth, surjective map $\pi : P \to M$ whose fibres are G-orbits, such that each $x \in M$ has an open neighbourhood $U \subseteq M$ with a diffeomorphism $\pi^{-1}(U) \cong U \times G$ identifying π and the G-action with the obvious projection $U \times G \to G$ and G and G-action on $U \times G$.

Let M be a smooth manifold of dimension n. The *frame bundle* F of M is a principal bundle over M with fibre $\mathrm{GL}(n,\mathbb{R})$. The points of F are $(n+1)$-tuples $(x, e_1, \ldots, e_n)$, where $x \in M$ and $e_1, \ldots, e_n$ is a basis for T_xM. We have $\pi : (x, e_1, \ldots, e_n) \mapsto x$, and $\mathrm{GL}(n,\mathbb{R})$ fixes x and acts on $e_1, \ldots, e_n$ by change of basis,

$A : (x, e_1, \ldots, e_n) \mapsto (x, \tilde{e}_1, \ldots, \tilde{e}_n)$,

where $\tilde{e}_i = \sum_{j=1}^{n} A_{ij} e_j$.

Let M be a manifold, P a principal bundle over M with fibre G and projection $\pi : P \longrightarrow M$, and H a Lie subgroup of G. A *principal subbundle* Q of P with fibre H is a submanifold Q of P closed under the action of H on P, such that the H-action on Q and the restriction $\pi|_Q : Q \longrightarrow M$ make Q into a principal bundle over M with fibre H.

Let M be a manifold of dimension n, and G be a Lie subgroup of $\mathrm{GL}(n,\mathbb{R})$. A G-*structure* on M is a principal subbundle P of the frame bundle F of M with fibre G.

For example, if (M,g) is a Riemannian manifold, let P be the subset of $(x,e_1,\ldots,e_n)$ in F with $e_1,\ldots,e_n$ an *orthonormal* basis for T_xM w.r.t. $g|_x$. All such bases are related by matrices in $\mathrm{O}(n)$, so P is an $\mathrm{O}(n)$-*structure*.

2.12 G-structures and holonomy groups

Let M be an n-manifold and ∇ a connection on TM. Fix $x \in M$ and a basis $(e_1, \ldots, e_n)$ for T_xM. This identifies $T_xM \cong \mathbb{R}^n$, so the holonomy group $\mathrm{Hol}_x(\nabla)$ lies in $\mathrm{GL}(T_xM) \cong \mathrm{GL}(n, \mathbb{R})$. Let G be a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$ containing $\mathrm{Hol}_x(\nabla)$. Define Q to be the set of $(y, f_1, \ldots, f_n)$ in the frame bundle F of M, such that if $\gamma : [0, 1] \to M$ is a smooth path with $\gamma(0) = x$, $\gamma(1) = y$, then there exists $g \in G$ with $(P_\gamma \circ g)e_i = f_i$ for $i = 1, \ldots, n$.

As $\mathrm{Hol}_x(\nabla) \subseteq G$ this is independent of choice of γ, and P is a G-*structure* on M.

Thus, a connection ∇ on TM with holonomy in G induces a G-structure on M. Can take $G = \mathrm{Hol}_x(\nabla)$.

Let (M, g) be a Riemannian manifold with $\mathrm{Hol}(g) = H \subseteq \mathrm{O}(n) \subset \mathrm{GL}(n, \mathbb{R})$. Then M has a natural H-structure Q, which is a principal subbundle of the $\mathrm{O}(n)$-structure P constructed before.

There is a notion of *connection* on principal bundles. A (vector bundle) connection on TM is equivalent to a (principal bundle) connection on the frame bundle F.
A connection ∇ on TM or F has holonomy contained in G iff there exists a G-structure on M preserved by (closed under) ∇.

A G-structure Q is called *torsion-free* if there exists a torsion-free connection ∇ on TM preserving Q. If $G \subseteq \mathrm{O}(n)$ this ∇ is unique, and is the Levi-Civita connection of the Riemannian metric associated to Q. Studying torsion-free G-structures for $G \subseteq \mathrm{O}(n)$ is equivalent to studying metrics g with $\mathrm{Hol}(g) \subseteq G$.