Shifted Symplectic Derived Algebraic Geometry and generalizations of Donaldson–Thomas Theory

Lecture 1 of 3: Classical Donaldson–Thomas Theory, Derived Algebraic Geometry

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These slides, and references, etc., available at
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Plan of talk:

References for §1

R.P. Thomas, *'A holomorphic Casson invariant for Calabi– Yau 3-folds, and bundles on K3 fibrations'*, J. Diff. Geom. 54 (2000), 367–438. math.AG/9806111.
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D. Joyce and Y. Song, *'A theory of generalized Donaldson-Thomas invariants'*, Memoirs of the AMS 217 (2012), arXiv:0810.5645.
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1.1. Introduction
Calabi–Yau manifolds

A *Calabi–Yau m-fold* is a compact $2m$-dimensional manifold X equipped with four geometric structures:

- a Riemannian metric g;
- a complex structure J;
- a symplectic form (Kähler form) ω; and
- a complex volume form Ω.

These satisfy pointwise compatibility conditions: $\omega(u,v)=g(Ju,v)$, $|\Omega|_g\equiv 2^{m/2}$, Ω is of type $(m,0)$ w.r.t. J, and p.d.e.s: J is integrable, and $\mathrm{d}\omega\equiv\mathrm{d}\Omega\equiv 0$. Usually we also require $H^1(X;\mathbb{R})=0$. This is a rich geometric structure, and very interesting from several points of view.

Complex algebraic geometry: (X, J) is a projective complex manifold. That is, we can embed X as a complex submanifold of $\mathbb{CP}^N$ for some $N \gg 0$, and then X is the zero set of finitely many homogeneous polynomials on $\mathbb{C}^{N+1}$. Also Ω is a holomorphic section of the canonical bundle K_X, so K_X is trivial, and $c_1(X) = 0$.
Analysis: For fixed (X, J), Yau's solution of the Calabi Conjecture by solving a nonlinear elliptic p.d.e. shows that there exists a family of Kähler metrics g on X making X Calabi–Yau.
Combining complex algebraic geometry and analysis proves the existence of huge numbers of examples of Calabi–Yau m-folds.

Riemannian geometry: (X, g) is a Ricci-flat Riemannian manifold with holonomy group $\mathrm{Hol}(g) \subseteq \mathrm{SU}(m)$.
Symplectic geometry: (X, ω) is a symplectic manifold with $c_1(X) = 0$.
Calibrated geometry: there is a distinguished class of minimal submanifolds in (X, g) called *special Lagrangian m-folds*.
String Theory: a branch of theoretical physics aiming to combine Quantum Theory and General Relativity. String Theorists believe that space-time is not 4 dimensional, but 10-dimensional, and is locally modelled on $\mathbb{R}^{3,1} \times X$, where $\mathbb{R}^{3,1}$ is Minkowski space, our observed universe, and X is a Calabi–Yau 3-fold with radius of order 10^{-33}cm, the Planck length.

String Theory and Mirror Symmetry

String Theorists believe that each Calabi–Yau 3-fold X has a quantization, a *Super Conformal Field Theory* (SCFT), not yet rigorously defined. Invariants of X such as the Dolbeault groups $H^{p,q}(X)$ and the Gromov–Witten invariants of X translate to properties of the SCFT. Using physical reasoning they made amazing predictions about Calabi–Yau 3-folds, an area known as *Mirror Symmetry*, conjectures which are slowly turning into theorems.

Part of the picture is that Calabi–Yau 3-folds should occur in pairs $X, \hat{X}$, such that $H^{p,q}(X) \cong H^{3-p,q}(\hat{X})$, and the complex geometry of X is somehow equivalent to the symplectic geometry of $\hat{X}$, and vice versa. This is very strange. It is an exciting area in which to work.

Invariants in Geometry

When geometers talk about *invariants*, they tend to have a particular, quite complex set-up in mind:

- Let X be a manifold (usually compact).
- Let $\mathcal{G}$ be a geometric structure on X that we are interested in.
- Let $\mathcal{A}$ be some auxiliary geometric structure on X.
- Let α be some topological invariant, e.g. a homology class on X.

We define a *moduli space* $\mathfrak{M}(\mathcal{G}, \mathcal{A}, \alpha)$ which parametrizes isomorphism classes of some kind of geometric object on X (e.g. submanifolds, or bundles with connection) which satisfy a p.d.e. depending on $\mathcal{G}$ and $\mathcal{A}$, and have topological invariant α.

Then we define $I(\mathcal{G}, \alpha)$ in $\mathbb{Z}$ or $\mathbb{Q}$ or $H_*(X; \mathbb{Q})$ which 'counts' the number of points in $\mathfrak{M}(\mathcal{G}, \mathcal{A}, \alpha)$. The 'counting' often has to be done in a complicated way. Usually we need $\mathfrak{M}(\mathcal{G}, \mathcal{A}, \alpha)$ compact.

Sometimes one can prove $I(\mathcal{G},\alpha)$ is *independent of the choice of auxiliary geometric structure* $\mathcal{A}$, even though $\mathfrak{M}(\mathcal{G},\mathcal{A},\alpha)$ depends very strongly on $\mathcal{A}$, and even though we usually have no way to define $I(\mathcal{G},\alpha)$ without choosing $\mathcal{A}$. Then we call $I(\mathcal{G},\alpha)$ an *invariant*. Invariants are interesting as they may be part of some deep underlying structure, perhaps some kind of Quantum Geometry coming from String Theory. Some examples:

- *Donaldson invariants* and *Seiberg–Witten invariants* of 4-manifolds 'count' self-dual connections. They are independent of the Riemannian metric used to define them. They can distinguish homeomorphic, non-diffeomorphic 4-manifolds.
- *Gromov–Witten invariants* of a compact symplectic manifold (X,ω) 'count' J-holomorphic curves in X for an almost complex structure J compatible with ω, but are independent of J.
- *Donaldson–Thomas invariants* of a Calabi–Yau 3-fold (X,J,g,Ω) 'count' coherent sheaves on X, and are independent of the complex structure J up to deformation.

1.2. Donaldson–Thomas invariants

Let X be a Calabi–Yau 3-fold. A *holomorphic vector bundle* $\pi : E \to X$ of *rank* r is a complex manifold E with a holomorphic map $\pi : E \to X$ whose fibres are complex vector spaces $\mathbb{C}^r$. A *morphism* $\phi : E \to F$ of holomorphic vector bundles $\pi : E \to X$, $\pi' : F \to X$ is a holomorphic map $\phi : E \to F$ with $\pi' \circ \phi \equiv \pi$, that is linear on the vector space fibres. Then $\mathrm{Hom}(E,F)$ is a finite-dimensional vector space. Holomorphic vector bundles form an exact category $\mathrm{Vect}(X)$.

A holomorphic vector bundle E has topological invariants, the *Chern character* $\mathrm{ch}_*(E)$ in $H^{\mathrm{even}}(X,\mathbb{Q})$, with $\mathrm{ch}_0(E) = r$, the rank of E. Holomorphic vector bundles are very natural objects to study. Roughly speaking, D–T invariants are integers which 'count' (semi)stable holomorphic vector bundles. But we actually consider a larger category, the *coherent sheaves* $\mathrm{coh}(X)$ on X.

A coherent sheaf is a (possibly singular) vector bundle $E \to Y$ on a complex submanifold (subscheme) Y in X. We need coherent sheaves for two reasons:
Firstly, moduli spaces of semistable holomorphic vector bundles are generally noncompact; to get compact moduli spaces, we have to allow singular vector bundles, that is, coherent sheaves.
Secondly, if $\phi : E \to F$ is a morphism of vector bundles then $\mathrm{Ker}\,\phi$ and Cokerϕ are generally coherent sheaves, not vector bundles.
The category $\mathrm{coh}(X)$ is better behaved than Vect(X) (it is an *abelian category*, has kernels and cokernels).

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One cannot define invariants 'counting' all coherent sheaves with a fixed Chern character α, as the number would be infinite (the moduli spaces are not of finite type). Instead, one restricts to *(semi)stable* coherent sheaves. A coherent sheaf E is Gieseker (semi)stable if all subsheaves $F \subset E$ satisfy some numerical conditions. These conditions depend on an ample line bundle on X; essentially, on the cohomology class $[\omega] \in H^2(X;\mathbb{R})$ of the Kähler form ω of X. We will write τ for Gieseker stability.
Every coherent sheaf can be decomposed into τ-semistable sheaves in a unique way, the *Harder–Narasimhan filtration*. So counting τ-semistable sheaves is related to counting all sheaves.

Thomas' definition of Donaldson–Thomas invariants

Let X be a Calabi–Yau 3-fold. The *Donaldson–Thomas invariants* $DT^\alpha(\tau)$ of X were defined by Richard Thomas in 1998. Fix a Chern character α in $H^{\text{even}}(X;\mathbb{Q})$. Then one can define *coarse moduli schemes* $\mathfrak{M}_{\text{st}}^\alpha(\tau)$, $\mathfrak{M}_{\text{ss}}^\alpha(\tau)$ parametrizing equivalence classes of τ-(semi)stable sheaves with Chern character α. They are not manifolds, but schemes which may have bad singularities. Two good properties:

- $\mathfrak{M}_{\text{ss}}^\alpha(\tau)$ is a projective $\mathbb{C}$-scheme, so in particular it is compact and Hausdorff.
- $\mathfrak{M}_{\text{st}}^\alpha(\tau)$ is an open subset in $\mathfrak{M}_{\text{ss}}^\alpha(\tau)$, and has an extra structure, a *symmetric obstruction theory*, which does not extend to $\mathfrak{M}_{\text{ss}}^\alpha(\tau)$ in general.

If $\mathfrak{M}_{\text{ss}}^\alpha(\tau)=\mathfrak{M}_{\text{st}}^\alpha(\tau)$, that is, there are no strictly τ-semistable sheaves in class α, then $\mathfrak{M}_{\text{st}}^\alpha(\tau)$ is compact with a symmetric obstruction theory. Thomas used the *virtual class* of Behrend and Fantechi to define the 'number' $DT^\alpha(\tau)\in\mathbb{Z}$ of points in $\mathfrak{M}_{\text{st}}^\alpha(\tau)$, and showed $DT^\alpha(\tau)$ is unchanged under deformations of the complex structure of X.

Virtual classes are *non-local*. But Behrend (2005) showed that $DT^\alpha(\tau)$ can be written as a *weighted Euler characteristic*

$$DT^\alpha(\tau)=\int_{\mathfrak{M}_{\text{st}}^\alpha(\tau)}\nu\,\mathrm{d}\chi, \tag{1.1}$$

where ν is the 'Behrend function', a $\mathbb{Z}$-valued constructible function on $\mathfrak{M}_{\text{st}}^\alpha(\tau)$ depending only on $\mathfrak{M}_{\text{st}}^\alpha(\tau)$ as a $\mathbb{C}$-scheme. We think of ν as a *multiplicity function*, so (1.1) counts points with multiplicity.

Donaldson–Thomas invariants are of interest in String Theory. The *MNOP Conjecture*, an important problem, relates the rank 1 Donaldson–Thomas invariants to the Gromov–Witten invariants counting holomorphic curves in X.

Thomas' definition of $DT^\alpha(\tau)$ has two disadvantages:

- $DT^\alpha(\tau)$ is undefined if $\mathfrak{M}_{\rm ss}^\alpha(\tau)\neq\mathfrak{M}_{\rm st}^\alpha(\tau)$.
- It was not understood how $DT^\alpha(\tau)$ depends on the choice of stability condition τ (effectively, on the Kähler class $[\omega]$ of X).

I will explain a theory which solves these two problems (joint work with Yinan Song).

1.3. Joyce–Song's generalized D–T invariants

We will define *generalized Donaldson–Thomas invariants* $\bar{DT}{}^\alpha(\tau)\in\mathbb{Q}$ for all Chern characters α, such that:

- $\bar{DT}{}^\alpha(\tau)$ is unchanged by deformations of the underlying CY3.
- If $\mathfrak{M}_{\rm st}^\alpha(\tau)=\mathfrak{M}_{\rm ss}^\alpha(\tau)$ then $\bar{DT}{}^\alpha(\tau)=DT^\alpha(\tau)$ in $\mathbb{Z}\subset\mathbb{Q}$.
- The $\bar{DT}{}^\alpha(\tau)$ transform according to a known transformation law under change of stability condition.
- For 'generic' τ, we have a conjecture rewriting the $\bar{DT}{}^\alpha(\tau)$ in terms of $\mathbb{Z}$-valued 'BPS invariants' $\hat{DT}{}^\alpha(\tau)$. (Cf. Gromov– Witten and Gopakumar–Vafa invariants). Now proved Davison–Meinhardt.
- The theory generalizes to invariants counting representations of a quiver with relations coming from a superpotential. (Cf. 'noncommutative D–T invariants').

On the face of it, the problem is just to decide how to 'count' strictly τ-semistable sheaves with the correct multiplicity, which sounds simple. But the solution turns out to be very long and complex. I will just explain a few of the key ideas involved.

Key idea 1: work with Artin stacks

Kinds of space used in complex algebraic geometry, in decreasing order of 'niceness':

- complex manifolds (very nice)
- varieties (nice)
- schemes (not bad): Thomas' $DT^\alpha(\tau)$.
- algebraic spaces (getting worse)
- Deligne–Mumford stacks (not nice)
- Artin stacks (horrible): our $\bar{DT}^\alpha(\tau)$.
- derived stacks (deeply horrible).

For classical D–T theory we work with moduli spaces which are *Artin stacks*, rather than coarse moduli schemes as Thomas does. One reason is that strictly τ-semistable sheaves can have nontrivial automorphism groups, and Artin stacks keep track of automorphism groups, but schemes do not.
For generalizations of D–T theory, we will need to work with derived stacks, and the theory of Pantev–Toën–Vaquié–Vezzosi.

Key idea 2: Ringel–Hall algebras

Write $\mathfrak{M}$ for the moduli stack of coherent sheaves on X. The 'stack functions' $\mathrm{SF}(\mathfrak{M})$ is the $\mathbb{Q}$-vector space generated by isomorphism classes $[(\mathfrak{R},\rho)]$ of morphisms $\rho:\mathfrak{R}\to\mathfrak{M}$ for $\mathfrak{R}$ a finite type Artin $\mathbb{C}$-stack, with the relation

$$[(\mathfrak{R},\rho)]=[(\mathfrak{S},\rho)]+[(\mathfrak{R}\setminus\mathfrak{S},\rho)]$$

for $\mathfrak{S}$ a closed substack of $\mathfrak{R}$.
There is an interesting associative, noncommutative product $*$ on $\mathrm{SF}(\mathfrak{M})$ defined using short exact sequences in $\mathrm{coh}(X)$; for $f,g\in\mathrm{SF}(\mathfrak{M})$, think of $(f*g)(F)$ as the 'integral' of $f(E)g(G)$ over all exact sequences $0\to E\to F\to G\to 0$ in $\mathrm{coh}(X)$.

The substack $\mathfrak{M}_{\rm ss}^\alpha(\tau)$ of $\mathfrak{M}$ of τ-semistable sheaves with Chern character α has finite type, so $\bar\delta_{\rm ss}^\alpha(\tau)=[(\mathfrak{M}_{\rm ss}^\alpha(\tau),{\rm inc})]\in{\rm SF}(\mathfrak{M})$. There is a Lie subalgebra ${\rm SF}^{\rm ind}(\mathfrak{M})$ of ${\rm SF}(\mathfrak{M})$ of stack functions 'supported on virtual indecomposables'. Define elements

$$\bar\epsilon^\alpha(\tau)=\sum_{n\geqslant 1,\;\alpha_1+\cdots+\alpha_n=\alpha,\;\tau(\alpha_i)=\tau(\alpha),\text{ all }i}(-1)^{n-1}/n\cdot\bar\delta_{\rm ss}^{\alpha_1}(\tau)*\bar\delta_{\rm ss}^{\alpha_2}(\tau)*\cdots*\bar\delta_{\rm ss}^{\alpha_n}(\tau).$$

Then $\bar\epsilon^\alpha(\tau)\in{\rm SF}^{\rm ind}(\mathfrak{M})$.
There are many important *universal identities* in the Ringel–Hall algebra ${\rm SF}(\mathfrak{M})$. For instance, if $\tau,\tilde\tau$ are different stability conditions, we have

$$\bar\delta_{\rm ss}^\alpha(\tilde\tau)=\sum_{n\geqslant 1,\;\alpha_1+\cdots+\alpha_n=\alpha}S(\alpha_1,\ldots,\alpha_n;\tau,\tilde\tau)\cdot\bar\delta_{\rm ss}^{\alpha_1}(\tau)*\cdots*\bar\delta_{\rm ss}^{\alpha_n}(\tau),\qquad(1.2)$$

$$\bar\epsilon^\alpha(\tilde\tau)=\sum_{n\geqslant 1,\;\alpha_1+\cdots+\alpha_n=\alpha}U(\alpha_1,\ldots,\alpha_n;\tau,\tilde\tau)\cdot\bar\epsilon^{\alpha_1}(\tau)*\cdots*\bar\epsilon^{\alpha_n}(\tau),\qquad(1.3)$$

for combinatorial coefficients $S,U(\cdots;\tau,\tilde\tau)$.

Key idea 3: local structure of the moduli stack

We prove that the moduli stack of coherent sheaves $\mathfrak{M}$ can be written locally in the complex analytic topology as $[{\rm Crit}(f)/G]$, where G is a complex Lie group, U a complex manifold acted on by G, and $f:U\to\mathbb{C}$ a G-invariant holomorphic function.
This is a complex analytic analogue for $\mathfrak{M}$ of the fact that $\mathfrak{M}_{\rm st}^\alpha(\tau)$ has a symmetric obstruction theory.
It requires X to be a Calabi–Yau 3-fold. The proof in Joyce–Song from 2008 is non-algebraic, using gauge theory on complex vector bundles over X, and works only over the field $\mathbb{C}$. However, as in §3, more recently Ben-Bassat–Bussi–Brav–Joyce used PTVV's shifted symplectic derived algebraic geometry to give an algebraic proof, in the Zariski/smooth topologies, which works over fields $\mathbb{K}$ of characteristic zero.

Key idea 4: Behrend function identities

For each Artin $\mathbb{C}$-stack $\mathfrak{M}$ we can define a *Behrend function* $\nu_{\mathfrak{M}}$, a $\mathbb{Z}$-valued constructible function we interpret as a *multiplicity function*. If we can write $\mathfrak{M}$ locally as $[\mathrm{Crit}(f)/G]$ for $f : U \to \mathbb{C}$ holomorphic and U a complex manifold then $\nu_{\mathfrak{M}}(uG) = (-1)^{\dim U - \dim G}(1 - \chi(MF_f(u)))$ for $u \in \mathrm{Crit}(f)$, where $MF_f(u)$ is the *Milnor fibre* of f at u.

Using Key idea 3 we prove two identities on the Behrend function of the moduli stack $\mathfrak{M}$:

$$\nu_{\mathfrak{M}}(E_1 \oplus E_2) = (-1)^{\bar{\chi}([E_1],[E_2])}\nu_{\mathfrak{M}}(E_1)\nu_{\mathfrak{M}}(E_2), \tag{1.4}$$

$$\begin{aligned}\int_{\substack{[\lambda]\in\mathbb{P}(\mathrm{Ext}^1(E_2,E_1)):\\ \lambda \Leftrightarrow 0\to E_1\to F\to E_2\to 0}} \nu_{\mathfrak{M}}(F)\mathrm{d}\chi - \int_{\substack{[\lambda']\in\mathbb{P}(\mathrm{Ext}^1(E_1,E_2)):\\ \lambda' \Leftrightarrow 0\to E_2\to F'\to E_1\to 0}} \nu_{\mathfrak{M}}(F')\mathrm{d}\chi \\ = \bigl(\dim \mathrm{Ext}^1(E_2,E_1) - \dim \mathrm{Ext}^1(E_1,E_2)\bigr)\nu_{\mathfrak{M}}(E_1 \oplus E_2).\end{aligned} \tag{1.5}$$

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Key idea 5: A morphism from a Ringel–Hall Lie algebra

Let $K(X) \subset H^{\mathrm{even}}(X;\mathbb{Q})$ be the lattice of Chern characters of coherent sheaves. Then $K(X) \cong \mathbb{Z}^l$, and there is an antisymmetric *Euler form* $\bar{\chi} : K(X) \times K(X) \to \mathbb{Z}$.

Define a Lie algebra $L(X)$ to have basis, as a $\mathbb{Q}$-vector space, symbols λ^α for $\alpha \in K(X)$, and Lie bracket

$$[\lambda^\alpha, \lambda^\beta] = (-1)^{\bar{\chi}(\alpha,\beta)}\bar{\chi}(\alpha,\beta)\,\lambda^{\alpha+\beta}.$$

We define a *Lie algebra morphism* $\Psi : \mathrm{SF}^{\mathrm{ind}}(\mathfrak{M}) \to L(X)$. Roughly speaking this is given by

$$\Psi\bigl([\mathfrak{R},\rho]\bigr) = \sum_{\alpha\in K(X)} \chi^{\mathrm{stk}}\bigl(\mathfrak{R}\times_{\mathfrak{M}}\mathfrak{M}^\alpha, \rho^*(\nu_{\mathfrak{M}})\bigr)\,\lambda^\alpha,$$

where χ^{stk} is a kind of stack-theoretic weighted Euler characteristic.

However, Euler characteristics of stacks are not well-defined: we want $\chi([X/G])=\chi(X)/\chi(G)$ for X a scheme and G a Lie group, but $\chi(G)=0$ whenever $\mathrm{rank}\, G>0$.
The point of using $\mathrm{SF}^{\mathrm{ind}}(\mathfrak{M})$ is that it is generated by elements $[(U\times[\mathrm{Spec}\,\mathbb{C}/\mathbb{C}^*],\rho)]$ for U a $\mathbb{C}$-variety, and we set

$$\begin{aligned}\Psi\bigl([(U\times[\mathrm{Spec}\,\mathbb{C}/\mathbb{C}^*],\rho)]\bigr)\\ =\textstyle\sum_{\alpha\in K(X)}\chi(U\times_{\mathfrak{M}}\mathfrak{M}^\alpha,\rho^*(\nu_{\mathfrak{M}}))\,\lambda^\alpha,\end{aligned}$$

which is well-defined as $U\times_{\mathfrak{M}}\mathfrak{M}^\alpha$ is a variety. We do not yet know how to extend Ψ from $\mathrm{SF}^{\mathrm{ind}}(\mathfrak{M})$ to $\mathrm{SF}(\mathfrak{M})$. To prove Ψ is a Lie algebra morphism we use the Behrend function identities (1.4)–(1.5).

We can now define *generalized Donaldson–Thomas invariants* $\bar{DT}^\alpha(\tau)\in\mathbb{Q}$: we set $\Psi(\bar\epsilon^\alpha(\tau))=\bar{DT}^\alpha(\tau)\lambda^\alpha$ for all $\alpha\in K(\mathcal{A})$. The transformation law (1.3) for the $\bar\epsilon^\alpha(\tau)$ under change of stability condition can be written as a Lie algebra identity in $\mathrm{SF}^{\mathrm{ind}}(\mathfrak{M})$. So applying the Lie algebra morphism Ψ yields a transformation law for the $\bar{DT}^\alpha(\tau)$:

$$\bar{DT}^\alpha(\tilde\tau)=\sum_{\substack{\text{iso. classes}\\ \text{of }\Gamma,I,\kappa}}\pm\, U(\Gamma,I,\kappa;\tau,\tilde\tau)\cdot \prod_{i\in I}\bar{DT}^{\kappa(i)}(\tau)\cdot\prod_{\substack{\text{edges}\\ i-j\text{ in }\Gamma}}\bar\chi(\kappa(i),\kappa(j)). \quad (1.6)$$

Here Γ is a connected, simply-connected undirected graph with vertices I, $\kappa:I\to K(\mathcal{A})$ has $\sum_{i\in I}\kappa(i)=\alpha$, and $U(\Gamma,I,\kappa;\tau,\tilde\tau)$ in $\mathbb{Q}$ are explicit combinatorial coefficients.

Key idea 6: pair invariants $PI^{\alpha,N}(\tau')$

We define an auxiliary invariant $PI^{\alpha,N}(\tau') \in \mathbb{Z}$ counting 'stable pairs' (E,s) with E a semistable sheaf in class α and $s \in H^0(E(N))$, for $N \gg 0$. The moduli space of stable pairs is a projective $\mathbb{C}$-scheme with a symmetric obstruction theory, so $PI^{\alpha,N}(\tau')$ is unchanged by deformations of X.
By a similar proof to (1.6) we show that $PI^{\alpha,N}(\tau')$ can be written in terms of the $\bar{DT}^{\beta}(\tau)$ by

$$PI^{\alpha,N}(\tau') = \sum_{\substack{\alpha_1,\ldots,\alpha_n \in K(\mathcal{A}):\\ \alpha_1+\cdots+\alpha_n=\alpha,\\ \tau(\alpha_i)=\tau(\alpha)\ \forall i}} \frac{(-1)^n}{n!} \cdot \prod_{i=1}^{n} (-1)^{\bar\chi([\mathcal{O}_X(-N)]-\alpha_1-\cdots-\alpha_{i-1},\alpha_i)}\cdot \bar\chi([\mathcal{O}_X(-N)]-\alpha_1-\cdots-\alpha_{i-1},\alpha_i)\bar{DT}^{\alpha_i}(\tau). \tag{1.7}$$

Since the $PI^{\alpha,N}(\tau')$ are deformation- invariant, we use (1.7) and induction on $\mathrm{rank}\,\alpha$ to prove that $\bar{DT}^{\alpha}(\tau)$ is *unchanged under deformations of* X for all $\alpha \in K(X)$.
The $PI^{\alpha,N}(\tau')$ are similar to Pandharipande–Thomas invariants.
Note that $\bar{DT}^{\alpha}(\tau)$ counts strictly semistables E in a complicated way: there are $\mathbb{Q}$-valued contributions from every filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$ with E_i τ-semistable and $\tau(E_i) = \tau(E)$, weighted by $\nu_M(E)$. One can show by example that more obvious, simpler definitions of $\bar{DT}^{\alpha}(\tau)$ do not give deformation-invariant answers.

Integrality properties of the invariants

Suppose E is stable and rigid in class α. Then $kE = E \oplus \cdots \oplus E$ is strictly semistable in class $k\alpha$, for $k \geqslant 2$. Calculations show that E contributes 1 to $\bar{DT}^{\alpha}(\tau)$, and kE contributes $1/k^2$ to $\bar{DT}^{k\alpha}(\tau)$. So we do not expect the $\bar{DT}^{\alpha}(\tau)$ to be integers, in general.

Define new invariants $\hat{DT}^{\alpha}(\tau) \in \mathbb{Q}$ by

$$\bar{DT}^{\alpha}(\tau) = \sum_{k \geqslant 1 : k \text{ divides } \alpha} \frac{1}{k^2} \hat{DT}^{\alpha/k}(\tau).$$

Then the kE for $k \geqslant 1$ above contribute 1 to $\hat{DT}^{\alpha}(\tau)$ and 0 to $\hat{DT}^{k\alpha}(\tau)$ for $k > 1$.

Conjecture (Joyce–Song, now proved Davison–Meinhardt)

Suppose τ is generic, in the sense that $\tau(\alpha) = \tau(\beta)$ implies $\bar{\chi}(\alpha, \beta) = 0$. Then $\hat{DT}^{\alpha}(\tau) \in \mathbb{Z}$ for all $\alpha \in K(X)$.

These $\hat{DT}^{\alpha}(\tau)$ should coincide with invariants conjectured by Kontsevich–Soibelman, and in String Theory should perhaps be interpreted as 'numbers of BPS states'.

2. Derived Algebraic Geometry
References for §2

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2.1. Derived Algebraic Geometry for dummies

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, e.g. $\mathbb{K} = \mathbb{C}$. Work in the context of Toën and Vezzosi's theory of *Derived Algebraic Geometry* (*DAG*). This gives ∞-categories of *derived* $\mathbb{K}$*-schemes* $\mathbf{dSch}_{\mathbb{K}}$ and *derived stacks* $\mathbf{dSt}_{\mathbb{K}}$. In this talk, for simplicity, we will mostly discuss derived schemes, though the results also extend to derived stacks.
This is a very technical subject. It is not easy to motivate DAG, or even to say properly what a derived scheme is, in an elementary talk. So I will lie a little bit.

What is a derived scheme?

$\mathbb{K}$-schemes in classical algebraic geometry are geometric spaces X which can be covered by Zariski open sets $Y \subseteq X$ with $Y \cong \operatorname{Spec} A$ for A a commutative $\mathbb{K}$-algebra. General $\mathbb{K}$-schemes are very singular, but *smooth* $\mathbb{K}$*-schemes* X are very like smooth manifolds over $\mathbb{K}$, many differential geometric ideas like cotangent bundles TX, T^*X work nicely for them.
Think of a derived $\mathbb{K}$-scheme $\boldsymbol{X}$ as a geometric space which can be covered by Zariski open sets $\boldsymbol{Y} \subseteq \boldsymbol{X}$ with $\boldsymbol{Y} \simeq \operatorname{Spec} A^\bullet$ for $A^\bullet = (A, \mathrm{d})$ a commutative differential graded algebra (cdga) over $\mathbb{K}$, in degrees $\leqslant 0$.
We require $\boldsymbol{X}$ to be *locally finitely presented*, that is, we can take the $A^\bullet$ to be finitely presented, a strong condition.

Why derived algebraic geometry?

One reason derived algebraic geometry can be a powerful tool, is the combination of two facts:

(A) Many algebro-geometric spaces one wants to study, such as moduli spaces of coherent sheaves, or complexes, or representations, etc., which in classical algebraic geometry may be very singular, also have an incarnation as (locally finitely presented) derived schemes (or derived stacks).

(B) Within the framework of DAG, one can treat (locally finitely presented) derived schemes or stacks very much like smooth, nonsingular objects (Kontsevich's 'hidden smoothness philosophy'). Some nice things work in the derived world, which do not work in the classical world.

2.2. Tangent and cotangent complexes

In going from classical to derived geometry, we always replace vector bundles, sheaves, representations, ..., by *complexes* of vector bundles, ... A classical smooth $\mathbb{K}$-scheme X has a tangent bundle TX and dual cotangent bundle T^*X, which are vector bundles on X, of rank the dimension $\dim X \in \mathbb{N}$.

Similarly, a derived $\mathbb{K}$-scheme $\boldsymbol{X}$ has a *tangent complex* $\mathbb{T}_{\boldsymbol{X}}$ and a dual *cotangent complex* $\mathbb{L}_{\boldsymbol{X}}$, which are perfect complexes of coherent sheaves on $\boldsymbol{X}$, of rank the virtual dimension $\mathrm{vdim}\,\boldsymbol{X} \in \mathbb{Z}$. A complex $\mathcal{E}^\bullet$ on $\boldsymbol{X}$ is called *perfect in the interval* $[a, b]$ if locally on $\boldsymbol{X}$ it is quasi-isomorphic to a complex $\cdots 0 \to E_a \to E_{a+1} \to \cdots \to E_b \to 0 \to \cdots$, with E_i a vector bundle in position i. For $\boldsymbol{X}$ a derived scheme, $\mathbb{T}_{\boldsymbol{X}}$ is perfect in $[0, \infty)$ and $\mathbb{L}_{\boldsymbol{X}}$ perfect in $(-\infty, 0]$; for $\boldsymbol{X}$ a derived Artin stack, $\mathbb{T}_{\boldsymbol{X}}$ is perfect in $[-1, \infty)$ and $\mathbb{L}_{\boldsymbol{X}}$ perfect in $(-\infty, 1]$.

Tangent complexes of moduli stacks

Suppose X is a smooth projective scheme, and $\boldsymbol{\mathcal{M}}$ is a derived moduli stack of coherent sheaves E on X. Then for each point $[E]$ in $\boldsymbol{\mathcal{M}}$ and each $i \in \mathbb{Z}$ we have natural isomorphisms

$$H^i(\mathbb{T}_{\boldsymbol{\mathcal{M}}}|_{[E]}) \cong \mathrm{Ext}^{i-1}(E,E). \tag{2.1}$$

In effect, the derived stack $\boldsymbol{\mathcal{M}}$ remembers the entire deformation theory of sheaves on X. In contrast, if $\mathcal{M} = t_0(\boldsymbol{\mathcal{M}})$ is the corresponding classical moduli scheme, (2.1) holds when $i \leqslant 1$ only. This shows that the derived structure on a moduli scheme/stack can remember useful information forgotten by the classical moduli scheme/stack, e.g. the Ext groups $\mathrm{Ext}^i(E,E)$ for $i \geqslant 2$.

If X has dimension n then (2.1) implies that $H^i(\mathbb{T}_{\boldsymbol{\mathcal{M}}}|_{[E]}) = 0$ for $i \geqslant n$, so $\mathbb{T}_{\boldsymbol{\mathcal{M}}}$ is perfect in $[-1, n-1]$.

Quasi-smooth derived schemes and virtual cycles

A derived scheme $\boldsymbol{X}$ is called *quasi-smooth* if $\mathbb{T}_{\boldsymbol{X}}$ is perfect in $[0,1]$, or equivalently $\mathbb{L}_{\boldsymbol{X}}$ is perfect in $[-1,0]$.

A proper quasi-smooth derived scheme $\boldsymbol{X}$ has a *virtual cycle* $[\boldsymbol{X}]_{\mathrm{virt}}$ in the Chow homology $A_*(X)$, where $X = t_0(\boldsymbol{X})$ is the classical truncation. This is because the natural morphism $\mathbb{L}_{\boldsymbol{X}}|_X \to \mathbb{L}_X$ induced by the inclusion $X \hookrightarrow \boldsymbol{X}$ is a 'perfect obstruction theory' in the sense of Behrend and Fantechi.

Most theories of invariants in algebraic geometry – e.g. Gromov–Witten invariants, Mochizuki invariants counting sheaves on surfaces, Donaldson–Thomas invariants – can be traced back to the existence of quasi-smooth derived moduli schemes.

For an (ordinary) derived moduli scheme $\boldsymbol{\mathcal{M}}$ of coherent sheaves E on X to be quasi-smooth, we need $\mathrm{Ext}^i(E,E) = 0$ for $i \geqslant 3$. This is automatic if $\dim X \leqslant 2$. For Calabi–Yau 3-folds X, you would expect a problem with $\mathrm{Ext}^3(E,E) \neq 0$, but stable sheaves E with fixed determinant have trace-free Ext groups $\mathrm{Ext}^3(E,E)_0 = 0$.

An example of nice behaviour in the derived world

Here is an example of the 'hidden smoothness philosophy'. Suppose we have a Cartesian square of smooth $\mathbb{K}$-schemes (or indeed, smooth manifolds)

$$\begin{array}{ccc} W & \xrightarrow{\quad f \quad} & Y \\ \downarrow e & & h \downarrow \\ X & \xrightarrow{\quad g \quad} & Z, \end{array}$$

with g, h transverse. Then we have an exact sequence of vector bundles on W, which we can use to compute TW:

$$0 \rightarrow TW \xrightarrow{Te \oplus Tf} e^*(TX) \oplus f^*(TY) \xrightarrow{e^*(Tg) \oplus -f^*(Th)} (g \circ e)^*(TZ) \rightarrow 0.$$

Similarly, if we have a homotopy Cartesian square of derived $\mathbb{K}$-schemes

$$\begin{array}{ccc} \boldsymbol{W} & \xrightarrow{\quad \boldsymbol{f} \quad} & \boldsymbol{Y} \\ \downarrow \boldsymbol{e} & & \boldsymbol{h} \downarrow \\ \boldsymbol{X} & \xrightarrow{\quad \boldsymbol{g} \quad} & \boldsymbol{Z}, \end{array}$$

with no transversality, we have a distinguished triangle on $\boldsymbol{W}$

$$\mathbb{T}_{\boldsymbol{W}} \xrightarrow{\mathbb{T}_{\boldsymbol{e}} \oplus \mathbb{T}_{\boldsymbol{f}}} \boldsymbol{e}^*(\mathbb{T}_{\boldsymbol{X}}) \oplus \boldsymbol{f}^*(\mathbb{T}_{\boldsymbol{Y}}) \xrightarrow{\boldsymbol{e}^*(\mathbb{T}_{\boldsymbol{g}}) \oplus -\boldsymbol{f}^*(\mathbb{T}_{\boldsymbol{h}})} (\boldsymbol{g} \circ \boldsymbol{e})^*(\mathbb{T}_{\boldsymbol{Z}}) \rightarrow \mathbb{T}_{\boldsymbol{W}}[+1],$$

which we can use to compute $\mathbb{T}_{\boldsymbol{W}}$. This is false for classical schemes. So, derived schemes with arbitrary morphisms, have good behaviour analogous to smooth classical schemes with transverse morphisms, and are better behaved than classical schemes.