Shifted Symplectic Derived Algebraic Geometry and generalizations of Donaldson–Thomas Theory

Lecture 2 of 3: PTVV's shifted symplectic geometry. D-critical loci and perverse sheaves

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These slides, and references, etc., available at
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Plan of talk:

3. Shifted symplectic geometry
References for §3

T. Pantev, B. Toën, M. Vaquié and G. Vezzosi, *Shifted symplectic structures*, Publ. Math. I.H.E.S. 117 (2013), 271–328. arXiv:1111.3209.

C. Brav, V. Bussi and D. Joyce, *A Darboux theorem for derived schemes with shifted symplectic structure*, arXiv:1305.6302, 2013.

O. Ben-Bassat, C. Brav, V. Bussi, and D. Joyce, *A 'Darboux Theorem' for shifted symplectic structures on derived Artin stacks, with applications*, Geometry and Topology 19 (2015), 1287–1359. arXiv:1312.0090.

D. Joyce and P. Safronov, *A Lagrangian neighbourhood theorem for shifted symplectic derived schemes*, to appear in Annales de la Faculté des Sciences de Toulouse, 2018. arXiv:1506.04024.

E. Bouaziz and I. Grojnowski, *A d-shifted Darboux theorem*, arXiv:1309.2197, 2013.

Classical symplectic geometry

Let M be a smooth manifold. Then M has a tangent bundle and cotangent bundle T^*M. We have k-forms $\omega \in C^\infty(\Lambda^k T^*M)$, and the de Rham differential $\mathrm{d}_{dR} : C^\infty(\Lambda^k T^*M) \to C^\infty(\Lambda^{k+1} T^*M)$. A k-form ω is *closed* if $\mathrm{d}_{dR}\omega = 0$.

A 2-form ω on M is *nondegenerate* if $\omega \cdot : TM \to T^*M$ is an isomorphism. This is possible only if $\dim M = 2n$ for $n \geqslant 0$. A *symplectic structure* is a closed, nondegenerate 2-form ω on M.

Symplectic geometry is the study of symplectic manifolds (M, ω).

A *Lagrangian* in (M, ω) is a submanifold $i : L \to M$ such that $\dim L = n$ and $i^*(\omega) = 0$.

3.1. PTVV's shifted symplectic geometry

Pantev, Toën, Vaquié and Vezzosi (arXiv:1111.3209) defined a version of symplectic geometry in the derived world.
Let $\boldsymbol{X}$ be a derived $\mathbb{K}$-scheme. The cotangent complex $\mathbb{L}_{\boldsymbol{X}}$ has exterior powers $\Lambda^p\mathbb{L}_{\boldsymbol{X}}$. The *de Rham differential* $\mathrm{d}_{dR}:\Lambda^p\mathbb{L}_{\boldsymbol{X}}\rightarrow\Lambda^{p+1}\mathbb{L}_{\boldsymbol{X}}$ is a morphism of complexes. Each $\Lambda^p\mathbb{L}_{\boldsymbol{X}}$ is a complex, so has an internal differential $\mathrm{d}:(\Lambda^p\mathbb{L}_{\boldsymbol{X}})^k\rightarrow(\Lambda^p\mathbb{L}_{\boldsymbol{X}})^{k+1}$. We have $\mathrm{d}^2=\mathrm{d}_{dR}^2=\mathrm{d}\circ\mathrm{d}_{dR}+\mathrm{d}_{dR}\circ\mathrm{d}=0$.
A *p-form of degree k* on $\boldsymbol{X}$ for $k\in\mathbb{Z}$ is an element $[\omega^0]$ of $H^k\bigl(\Lambda^p\mathbb{L}_{\boldsymbol{X}},\mathrm{d}\bigr)$. A *closed p-form of degree k* on $\boldsymbol{X}$ is an element

$$[(\omega^0,\omega^1,\ldots)]\in H^k\bigl(\textstyle\bigoplus_{i=0}^{\infty}\Lambda^{p+i}\mathbb{L}_{\boldsymbol{X}}[i],\mathrm{d}+\mathrm{d}_{dR}\bigr).$$

There is a projection $\pi:[(\omega^0,\omega^1,\ldots)]\mapsto[\omega^0]$ from closed p-forms $[(\omega^0,\omega^1,\ldots)]$ of degree k to p-forms $[\omega^0]$ of degree k.

Nondegenerate 2-forms and symplectic structures

Let $[\omega^0]$ be a 2-form of degree k on $\boldsymbol{X}$. Then $[\omega^0]$ induces a morphism $\omega^0:\mathbb{T}_{\boldsymbol{X}}\rightarrow\mathbb{L}_{\boldsymbol{X}}[k]$, where $\mathbb{T}_{\boldsymbol{X}}=\mathbb{L}_{\boldsymbol{X}}^\vee$ is the tangent complex of $\boldsymbol{X}$. We call $[\omega^0]$ *nondegenerate* if $\omega^0:\mathbb{T}_{\boldsymbol{X}}\rightarrow\mathbb{L}_{\boldsymbol{X}}[k]$ is a quasi-isomorphism.
If $\boldsymbol{X}$ is a derived scheme then the complex $\mathbb{L}_{\boldsymbol{X}}$ lives in degrees $(-\infty,0]$ and $\mathbb{T}_{\boldsymbol{X}}$ in degrees $[0,\infty)$. So $\omega^0:\mathbb{T}_{\boldsymbol{X}}\rightarrow\mathbb{L}_{\boldsymbol{X}}[k]$ can be a quasi-isomorphism only if $k\leqslant 0$, and then $\mathbb{L}_{\boldsymbol{X}}$ lives in degrees $[k,0]$ and $\mathbb{T}_{\boldsymbol{X}}$ in degrees $[0,-k]$. If $k=0$ then $\boldsymbol{X}$ is a smooth classical $\mathbb{K}$-scheme, and if $k=-1$ then $\boldsymbol{X}$ is quasi-smooth.
A closed 2-form $\omega=[(\omega^0,\omega^1,\ldots)]$ of degree k on $\boldsymbol{X}$ is called a *k-shifted symplectic structure* if $[\omega^0]=\pi(\omega)$ is nondegenerate.

Although the details are complex, PTVV are following a simple recipe for translating some piece of geometry from smooth manifolds/smooth classical schemes to derived schemes:

(i) replace manifolds/smooth schemes X by derived schemes $\boldsymbol{X}$.

(ii) replace vector bundles $TX, T^*X, \Lambda^p T^*X, \ldots$ by complexes $\mathbb{T}_{\boldsymbol{X}}, \mathbb{L}_{\boldsymbol{X}}, \Lambda^p\mathbb{L}_{\boldsymbol{X}}, \ldots$.

(iii) replace sections of $TX, T^*X, \Lambda^p T^*X, \ldots$ by cohomology classes of the complexes $\mathbb{T}_{\boldsymbol{X}}, \mathbb{L}_{\boldsymbol{X}}, \Lambda^p\mathbb{L}_{\boldsymbol{X}}, \ldots$, in degree $k \in \mathbb{Z}$.

(iv) replace isomorphisms of vector bundles by quasi-isomorphisms of complexes.

Note that in (iii), we can specify the degree $k \in \mathbb{Z}$ of the cohomology class (e.g. $[\omega] \in H^k(\Lambda^p\mathbb{L}_{\boldsymbol{X}})$), which doesn't happen at the classical level.

Calabi–Yau moduli schemes and moduli stacks

PTVV prove that if Y is a Calabi–Yau m-fold over $\mathbb{K}$ and $\boldsymbol{\mathcal{M}}$ is a derived moduli scheme or stack of (complexes of) coherent sheaves on Y, then $\boldsymbol{\mathcal{M}}$ has a $(2-m)$-shifted symplectic structure ω.

This suggests applications — lots of interesting geometry concerns Calabi–Yau moduli schemes, e.g. Donaldson–Thomas theory.

We can understand the associated nondegenerate 2-form $[\omega^0]$ in terms of *Serre duality*. At a point $[E] \in \boldsymbol{\mathcal{M}}$, we have $h^i(\mathbb{T}_{\boldsymbol{\mathcal{M}}})|_{[E]} \cong \mathrm{Ext}^{i-1}(E,E)$ and $h^i(\mathbb{L}_{\boldsymbol{\mathcal{M}}})|_{[E]} \cong \mathrm{Ext}^{1-i}(E,E)^*$.

The Calabi–Yau condition gives $\mathrm{Ext}^i(E,E) \cong \mathrm{Ext}^{m-i}(E,E)^*$, which corresponds to $h^{i+1}(\mathbb{T}_{\boldsymbol{\mathcal{M}}})|_{[E]} \cong h^{i+1}(\mathbb{L}_{\boldsymbol{\mathcal{M}}}[2-m])|_{[E]}$. This is the cohomology at $[E]$ of the quasi-isomorphism $\omega^0 : \mathbb{T}_{\boldsymbol{\mathcal{M}}} \to \mathbb{L}_{\boldsymbol{\mathcal{M}}}[2-m]$.

Lagrangians and Lagrangian intersections

Let $(\boldsymbol{X},\omega)$ be a k-shifted symplectic derived scheme or stack. Then Pantev et al. define a notion of *Lagrangian* $\boldsymbol{L}$ in $(\boldsymbol{X},\omega)$, which is a morphism $\boldsymbol{i}:\boldsymbol{L}\to\boldsymbol{X}$ of derived schemes or stacks together with a homotopy $i^*(\omega)\sim 0$ satisfying a nondegeneracy condition, implying that $\mathbb{T}_{\boldsymbol{L}}\simeq\mathbb{L}_{\boldsymbol{L}/\boldsymbol{X}}[k-1]$.

If $\boldsymbol{L},\boldsymbol{M}$ are Lagrangians in $(\boldsymbol{X},\omega)$, then the fibre product $\boldsymbol{L}\times_{\boldsymbol{X}}\boldsymbol{M}$ has a natural $(k-1)$-shifted symplectic structure.

If (S,ω) is a classical smooth symplectic scheme, then it is a 0-shifted symplectic derived scheme in the sense of PTVV, and if $L,M\subset S$ are classical smooth Lagrangian subschemes, then they are Lagrangians in the sense of PTVV. Therefore the (derived) Lagrangian intersection $L\cap M=L\times_S M$ is a -1-shifted symplectic derived scheme.

Examples of Lagrangians

Let $(\boldsymbol{X},\omega)$ be k-shifted symplectic, and $\boldsymbol{i}_a:\boldsymbol{L}_a\to\boldsymbol{X}$ be Lagrangian in $\boldsymbol{X}$ for $a=1,\dots,d$. Then Ben-Bassat (arXiv:1309.0596) shows

$$\boldsymbol{L}_1\times_{\boldsymbol{X}}\boldsymbol{L}_2\times_{\boldsymbol{X}}\cdots\times_{\boldsymbol{X}}\boldsymbol{L}_d\longrightarrow(\boldsymbol{L}_1\times_{\boldsymbol{X}}\boldsymbol{L}_2)\times\cdots\times(\boldsymbol{L}_{d-1}\times_{\boldsymbol{X}}\boldsymbol{L}_d)\times(\boldsymbol{L}_d\times_{\boldsymbol{X}}\boldsymbol{L}_1)$$

is Lagrangian, where the r.h.s. is $(k-1)$-shifted symplectic by PTVV. This is relevant to defining 'Fukaya categories' of complex symplectic manifolds.

Let Y be a Calabi–Yau m-fold, so that the derived moduli stack $\boldsymbol{\mathcal{M}}$ of coherent sheaves (or complexes) on Y is $(2-m)$-shifted symplectic by PTVV, with symplectic form ω. We expect (Oren Ben-Bassat, work in progress) that

$$\boldsymbol{\mathcal{E}xact}\xrightarrow{\pi_1\times\pi_2\times\pi_3}(\boldsymbol{\mathcal{M}},\omega)\times(\boldsymbol{\mathcal{M}},-\omega)\times(\boldsymbol{\mathcal{M}},\omega)$$

is Lagrangian, where $\boldsymbol{\mathcal{E}xact}$ is the derived moduli stack of short exact sequences in $\mathrm{coh}(Y)$ (or distinguished triangles in $D^b\mathrm{coh}(Y)$). This is relevant to Cohomological Hall Algebras.

Summary of the story so far

- Derived schemes behave better than classical schemes in some ways – they are analogous to smooth schemes, or manifolds. So, we can extend stories in smooth geometry to derived schemes. This introduces an extra degree $k \in \mathbb{Z}$.
- PTVV define a version of ('k-shifted') symplectic geometry for derived schemes. This is a new geometric structure.
- 0-shifted symplectic derived schemes are just classical smooth symplectic schemes.
- Calabi–Yau m-fold moduli schemes and stacks are $(2-m)$-shifted symplectic. This gives a *new geometric structure* on Calabi–Yau moduli spaces – relevant to Donaldson–Thomas theory and its generalizations.
- One can go from k-shifted symplectic to $(k-1)$-shifted symplectic by taking intersections of Lagrangians.

3.2. A 'Darboux theorem' for shifted symplectic schemes

Theorem 3.1 (Brav, Bussi and Joyce arXiv:1305.6302)

Suppose $(\boldsymbol{X},\omega)$ is a k-shifted symplectic derived $\mathbb{K}$-scheme for $k<0$. If $k \not\equiv 2 \mod 4$, then each $x\in\boldsymbol{X}$ admits a Zariski open neighbourhood $\boldsymbol{Y}\subseteq\boldsymbol{X}$ with $\boldsymbol{Y}\simeq \mathrm{Spec}(A,\mathrm{d})$ for (A,d) an explicit cdga generated by graded variables x_j^{-i}, y_j^{k+i} for $0\leqslant i\leqslant -k/2$, and $\omega|_{\boldsymbol{Y}}=[(\omega^0,0,0,\ldots)]$ where x_j^l, y_j^l have degree l, and

$$\omega^0=\sum_{i=0}^{[-k/2]}\sum_{j=1}^{m_i}\mathrm{d}_{dR}y_j^{k+i}\mathrm{d}_{dR}x_j^{-i}.$$

Also the differential d in (A,d) is given by Poisson bracket with a Hamiltonian H in A of degree $k+1$.
If $k\equiv 2 \mod 4$, we have two statements, one étale local with ω^0 standard, and one Zariski local with the components of ω^0 in the degree $k/2$ variables depending on some invertible functions.

Sketch of the proof of Theorem 3.1

Suppose $(\boldsymbol{X},\omega)$ is a k-shifted symplectic derived $\mathbb{K}$-scheme for $k<0$, and $x\in\boldsymbol{X}$. Then $\mathbb{L}_{\boldsymbol{X}}$ lives in degrees $[k,0]$. We first show that we can build Zariski open $x\in\boldsymbol{Y}\subseteq\boldsymbol{X}$ with $\boldsymbol{Y}\simeq\mathrm{Spec}(A,\mathrm{d})$, for $A=\bigoplus_{i\leqslant 0}A^i,\mathrm{d}$ a cdga over $\mathbb{K}$ with A^0 a smooth $\mathbb{K}$-algebra, and such that A is freely generated over A^0 by graded variables x_j^{-i},y_j^{k+i} in degrees $-1,-2,\ldots,k$. We take $\dim A^0$ and the number of x_j^{-i},y_j^{k+i} to be minimal at x.

Using theorems about periodic cyclic cohomology, we show that on $Y\simeq\mathrm{Spec}(A,\mathrm{d})$ we can write $\omega|_Y=[(\omega^0,0,0,\ldots)]$, for ω^0 a 2-form of degree k with $\mathrm{d}\omega^0=\mathrm{d}_{dR}\omega^0=0$. Minimality at x implies ω^0 is strictly nondegenerate near x, so we can change variables to write $\omega^0=\sum_{i,j}\mathrm{d}_{dR}y_j^{k+i}\mathrm{d}_{dR}x_j^{-i}$. Finally, we show d in (A,d) is a symplectic vector field, which integrates to a Hamiltonian H.

The case of -1-shifted symplectic derived schemes

When $k=-1$ the Hamiltonian H in the theorem has degree 0. Then Theorem 3.1 reduces to:

Corollary 3.2

Suppose $(\boldsymbol{X},\omega)$ is a -1-shifted symplectic derived $\mathbb{K}$-scheme. Then $(\boldsymbol{X},\omega)$ is Zariski locally equivalent to a derived critical locus $\mathbf{Crit}(H:U\to\mathbb{A}^1)$, for U a smooth classical $\mathbb{K}$-scheme and $H:U\to\mathbb{A}^1$ a regular function. Hence, the underlying classical $\mathbb{K}$-scheme $X=t_0(\boldsymbol{X})$ is Zariski locally isomorphic to a classical critical locus $\mathrm{Crit}(H:U\to\mathbb{A}^1)$.

Combining this with results of Pantev et al. from §2 gives interesting consequences in classical algebraic geometry:

Corollary 3.3

Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a classical moduli $\mathbb{K}$-scheme of coherent sheaves, or complexes of coherent sheaves, on Y. Then $\mathcal{M}$ is Zariski locally isomorphic to the critical locus $\mathrm{Crit}(H : U \to \mathbb{A}^1)$ of a regular function on a smooth $\mathbb{K}$-scheme.

Here we note that $\mathcal{M} = t_0(\boldsymbol{\mathcal{M}})$ for $\boldsymbol{\mathcal{M}}$ the corresponding derived moduli scheme, which is -1-shifted symplectic by PTVV.
A complex analytic analogue of this for moduli of coherent sheaves was proved using gauge theory by Joyce and Song arXiv:0810.5645 (§1, key idea 3), and for moduli of complexes was claimed by Behrend and Getzler.
Note that the proof of the corollary is wholly algebro-geometric.

As intersections of algebraic Lagrangians are -1-shifted symplectic, we also deduce:

Corollary 3.4

Let (S, ω) be a classical smooth symplectic $\mathbb{K}$-scheme, and $L, M \subseteq S$ be smooth algebraic Lagrangians. Then the intersection $L \cap M$, as a $\mathbb{K}$-subscheme of S, is Zariski locally isomorphic to the critical locus $\mathrm{Crit}(H : U \to \mathbb{A}^1)$ of a regular function on a smooth $\mathbb{K}$-scheme.

In real or complex symplectic geometry, where the Darboux Theorem holds, the analogue of the corollary is easy to prove, but in classical algebraic symplectic geometry we do not have a Darboux Theorem, so the corollary is not obvious.

Outlook for generalizations of Donaldson–Thomas theory

We now know that 3-Calabi–Yau moduli spaces are locally modelled on critical loci, and we have nice geometric structures encoding this (-1-shifted symplectic structures).
There is some interesting geometry associated with critical loci:

- Perverse sheaves of vanishing cycles.
- Motivic Milnor fibres.
- Categories of matrix factorizations.

It seems natural to try and construct global structures on 3-Calabi–Yau moduli spaces, which are locally modelled on perverse vanishing cycles, motivic Milnor fibres, or matrix factorizations.
This leads to questions of *categorification* of Donaldson–Thomas theory, and *motivic Donaldson–Thomas invariants*.

3.3. Extension to shifted symplectic derived Artin stacks

In Ben-Bassat, Bussi, Brav and Joyce arXiv:1312.0090 we extend the material of §3.2 from (derived) schemes to (derived) Artin stacks. We call a derived stack $\boldsymbol{X}$ a *derived Artin stack* $\boldsymbol{X}$ if it is 1-geometric, and the associated classical (higher) stack $X = t_0(\boldsymbol{X})$ is 1-truncated, all in the sense of Toën and Vezzosi. Then the cotangent complex $\mathbb{L}_{\boldsymbol{X}}$ lives in degrees $(-\infty, 1]$, and $X = t_0(\boldsymbol{X})$ is a classical Artin stack (in particular, not a higher stack).
A derived Artin stack $\boldsymbol{X}$ admits a smooth atlas $\varphi : \boldsymbol{U} \to \boldsymbol{X}$ with $\boldsymbol{U}$ a derived scheme. If Y is a smooth projective scheme and $\boldsymbol{\mathcal{M}}$ is a derived moduli stack of coherent sheaves F on Y, or of complexes $F^\bullet$ in $D^b \mathrm{coh}(Y)$ with $\mathrm{Ext}^{<0}(F^\bullet, F^\bullet) = 0$, then $\boldsymbol{\mathcal{M}}$ is a derived Artin stack.

A 'Darboux Theorem' for atlases of derived stacks

Theorem 3.5 (Ben-Bassat, Bussi, Brav, Joyce, arXiv:1312.0090)

Let $(\boldsymbol{X},\omega_{\boldsymbol{X}})$ be a k-shifted symplectic derived Artin stack for $k<0$, and $p\in\boldsymbol{X}$. Then there exist 'standard form' affine derived schemes $\boldsymbol{U}=\operatorname{Spec}A$, $\boldsymbol{V}=\operatorname{Spec}B$, points $u\in\boldsymbol{U}$, $v\in\boldsymbol{V}$ with A,B minimal at u,v, morphisms $\varphi:\boldsymbol{U}\to\boldsymbol{X}$ and $\boldsymbol{i}:\boldsymbol{U}\to\boldsymbol{V}$ with $\varphi(u)=p$, $\boldsymbol{i}(u)=v$, such that φ is smooth of relative dimension $\dim H^1(\mathbb{L}_{\boldsymbol{X}}|_p)$, and $t_0(\boldsymbol{i}):t_0(\boldsymbol{U})\to t_0(\boldsymbol{V})$ is an isomorphism on classical schemes, and $\mathbb{L}_{\boldsymbol{U}/\boldsymbol{V}}\simeq\mathbb{T}_{\boldsymbol{U}/\boldsymbol{X}}[1-k]$, and a 'Darboux form' k-shifted symplectic form ω_B on $\boldsymbol{V}=\operatorname{Spec}B$ such that $\boldsymbol{i}^*(\omega_B)\sim\varphi^*(\omega_{\boldsymbol{X}})$ in k-shifted closed 2-forms on $\boldsymbol{U}$.

-1-shifted symplectic derived stacks

When $k=-1$, $(\boldsymbol{V},\omega_B)$ is a derived critical locus $\mathbf{Crit}(f:S\to\mathbb{A}^1)$ for S a smooth scheme. Then $t_0(\boldsymbol{V})\cong t_0(\boldsymbol{U})$ is the classical critical locus $\operatorname{Crit}(f:S\to\mathbb{A}^1)$, and $U=t_0(\boldsymbol{U})$ is a smooth atlas for the Artin stack $X=t_0(\boldsymbol{X})$. Thus we deduce:

Corollary 3.6

Let $(\boldsymbol{X},\omega_{\boldsymbol{X}})$ be a -1-shifted symplectic derived stack. Then the classical Artin stack $X=t_0(\boldsymbol{X})$ locally admits smooth atlases $\varphi:U\to X$ with $U=\operatorname{Crit}(f:S\to\mathbb{A}^1)$, for S a smooth scheme and f a regular function.

Corollary 3.7

Suppose Y is a Calabi–Yau 3-fold and $\mathcal{M}$ a classical moduli stack of coherent sheaves F on Y, or of complexes $F^\bullet$ in $D^b\operatorname{coh}(Y)$ with $\operatorname{Ext}^{<0}(F^\bullet,F^\bullet)=0$. Then $\mathcal{M}$ locally admits smooth atlases $\varphi:U\to X$ with $U=\operatorname{Crit}(f:S\to\mathbb{A}^1)$, for S a smooth scheme.

4. D-critical loci and perverse sheaves
References for §4

D. Joyce, *A classical model for derived critical loci*, J. Diff. Geom. 101 (2015), 289–367. arXiv:1304.4508.
C. Brav, V. Bussi and D. Joyce, *A Darboux theorem for derived schemes with shifted symplectic structure*, arXiv:1305.6302, 2013.
O. Ben-Bassat, C. Brav, V. Bussi, and D. Joyce, *A 'Darboux Theorem' for shifted symplectic structures on derived Artin stacks, with applications*, Geom. Top. 19 (2015), 1287–1359. arXiv:1312.0090.
C. Brav, V. Bussi, D. Dupont, D. Joyce, and B. Szendrői, *Symmetries and stabilization for sheaves of vanishing cycles*, Journal of Singularities 11 (2015), 85–151. arXiv:1211.3259.
V. Bussi, *Categorification of Lagrangian intersections on complex symplectic manifolds using perverse sheaves of vanishing cycles*, arXiv:1404.1329, 2014.

4.1. D-critical loci

We will define 'd-critical loci' and 'd-critical stacks', classical truncations of -1-shifted symplectic derived schemes and stacks.

Theorem (Joyce arXiv:1304.4508)

Let X be a classical $\mathbb{K}$-scheme. Then there exists a canonical sheaf $\mathcal{S}_X$ of $\mathbb{K}$-vector spaces on X, such that if $R \subseteq X$ is Zariski open and $i : R \hookrightarrow U$ is a closed embedding of R into a smooth $\mathbb{K}$-scheme U, and $I_{R,U} \subseteq \mathcal{O}_U$ is the ideal vanishing on $i(R)$, then

$$\mathcal{S}_X|_R \cong \mathrm{Ker}\left(\frac{\mathcal{O}_U}{I_{R,U}^2} \xrightarrow{\mathrm{d}} \frac{T^*U}{I_{R,U} \cdot T^*U}\right).$$

Also $\mathcal{S}_X$ splits naturally as $\mathcal{S}_X = \mathcal{S}_X^0 \oplus \mathbb{K}_X$, where $\mathbb{K}_X$ is the sheaf of locally constant functions $X \to \mathbb{K}$.

The meaning of the sheaves $\mathcal{S}_X, \mathcal{S}_X^0$

If $X = \mathrm{Crit}(f : U \to \mathbb{A}^1)$ then taking $R = X$, $i =$inclusion, we see that $f + I_{X,U}^2$ is a section of $\mathcal{S}_X$. Also $f|_{X^{\mathrm{red}}} : X^{\mathrm{red}} \to \mathbb{K}$ is locally constant, and if $f|_{X^{\mathrm{red}}} = 0$ then $f + I_{X,U}^2$ is a section of $\mathcal{S}_X^0$. Note that $f + I_{X,U} = f|_X$ in $\mathcal{O}_X = \mathcal{O}_U / I_{X,U}$. The theorem means that $f + I_{X,U}^2$ makes sense *intrinsically on* X, without reference to the embedding of X into U.

That is, if $X = \mathrm{Crit}(f : U \to \mathbb{A}^1)$ then we can remember f up to second order in the ideal $I_{X,U}$ as a piece of data on X, not on U.

Suppose $X = \mathrm{Crit}(f : U \to \mathbb{A}^1) = \mathrm{Crit}(g : V \to \mathbb{A}^1)$ is written as a critical locus in two different ways. Then $f + I_{X,U}^2$, $g + I_{X,V}^2$ are sections of $\mathcal{S}_X$, so we can ask whether $f + I_{X,U}^2 = g + I_{X,V}^2$. This gives a way to compare isomorphic critical loci in different smooth classical schemes.

The definition of d-critical loci

Definition (Joyce arXiv:1304.4508)

An (*algebraic*) *d-critical locus* (X, s) is a classical $\mathbb{K}$-scheme X and a global section $s \in H^0(\mathcal{S}_X^0)$ such that X may be covered by Zariski open $R \subseteq X$ with an isomorphism $i : R \to \mathrm{Crit}(f : U \to \mathbb{A}^1)$ identifying $s|_R$ with $f + I_{R,U}^2$, for f a regular function on a smooth $\mathbb{K}$-scheme U.

That is, a d-critical locus (X, s) is a $\mathbb{K}$-scheme X which may Zariski locally be written as a critical locus $\mathrm{Crit}(f : U \to \mathbb{A}^1)$, and the section s remembers f up to second order in the ideal $I_{X,U}$.

We also define *complex analytic d-critical loci*, with X a complex analytic space locally modelled on $\mathrm{Crit}(f : U \to \mathbb{C})$ for U a complex manifold and f holomorphic.

Orientations on d-critical loci

Theorem (Joyce arXiv:1304.4508)

Let (X,s) be an algebraic d-critical locus and $X^{\rm red}$ the reduced $\mathbb{K}$-subscheme of X. Then there is a natural line bundle $K_{X,s}$ on $X^{\rm red}$ called the **canonical bundle**, *such that if (X,s) is locally modelled on $\mathrm{Crit}(f:U\to\mathbb{A}^1)$ then $K_{X,s}$ is locally modelled on $K_U^{\otimes^2}|_{\mathrm{Crit}(f)^{\rm red}}$, for K_U the usual canonical bundle of U.*

Definition

Let (X,s) be a d-critical locus. An *orientation* on (X,s) is a choice of square root line bundle $K_{X,s}^{1/2}$ for $K_{X,s}$ on $X^{\rm red}$.

This is related to *orientation data* in Kontsevich–Soibelman 2008.

A truncation functor from -1-symplectic derived schemes

Theorem 4.1 (Brav, Bussi and Joyce arXiv:1305.6302)

Let $(\boldsymbol{X},\omega)$ be a -1-shifted symplectic derived $\mathbb{K}$-scheme. Then the classical $\mathbb{K}$-scheme $X=t_0(\boldsymbol{X})$ extends naturally to an algebraic d-critical locus (X,s). The canonical bundle of (X,s) satisfies $K_{X,s}\cong\det\mathbb{L}_{\boldsymbol{X}}|_{X^{\rm red}}$.

That is, we define a *truncation functor* from -1-shifted symplectic derived $\mathbb{K}$-schemes to algebraic d-critical loci. Examples show this functor is not full. Think of d-critical loci as *classical truncations* of -1-shifted symplectic derived $\mathbb{K}$-schemes.
An alternative semi-classical truncation, used in D–T theory, is *schemes with symmetric obstruction theory*. D-critical loci appear to be more useful, for both categorified and motivic D–T theory.

Corollaries 3.3–3.4 imply:

Corollary 4.2

Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a classical moduli $\mathbb{K}$-scheme of coherent sheaves, or complexes of coherent sheaves, on Y. Then $\mathcal{M}$ extends naturally to a d-critical locus $(\mathcal{M}, s)$. The canonical bundle satisfies $K_{\mathcal{M},s} \cong \det(\mathcal{E}^\bullet)|_{\mathcal{M}^{\rm red}}$, where $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$ is the (symmetric) obstruction theory on $\mathcal{M}$ defined by Thomas or Huybrechts and Thomas.

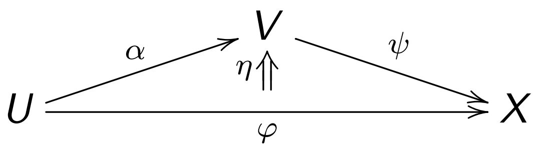
Corollary 4.3

Let (S, ω) be a classical smooth symplectic $\mathbb{K}$-scheme, and $L, M \subseteq S$ be smooth algebraic Lagrangians. Then $X = L \cap M$ extends naturally to a d-critical locus (X, s). The canonical bundle satisfies $K_{X,s} \cong K_L|_{X^{\rm red}} \otimes K_M|_{X^{\rm red}}$. Hence, choices of square roots $K_L^{1/2}, K_M^{1/2}$ give an orientation for (X, s).

Bussi extends Corollary 4.3 to complex Lagrangians in complex symplectic manifolds.

4.2. D-critical stacks

To generalize the d-critical loci in §4.1 to Artin stacks, we need a good notion of sheaves on Artin stacks. This is already well understood. Roughly, a sheaf $\mathcal{S}$ on an Artin stack X assigns a sheaf $\mathcal{S}(U, \varphi)$ on U (in the usual sense for schemes) for each smooth morphism $\varphi : U \to X$ with U a scheme, and a morphism $\mathcal{S}(\alpha, \eta) : \alpha^*(\mathcal{S}(V, \psi)) \to \mathcal{S}(U, \varphi)$ (often an isomorphism) for each 2-commutative diagram



(4.1)

with U, V schemes and φ, ψ smooth, such that $\mathcal{S}(\alpha, \eta)$ have the obvious associativity properties. So, we pass from stacks X to schemes U by working with smooth atlases $\varphi : U \to X$.

The definition of d-critical stacks

Generalizing d-critical loci to stacks is now straightforward. As above, on each scheme U we have a canonical sheaf $\mathcal{S}_U^0$. If $\alpha : U \to V$ is a morphism of schemes we have pullback morphisms $\alpha^\star : \alpha^{-1}(\mathcal{S}_V^0) \to \mathcal{S}_U^0$ with associativity properties.
So, for any classical Artin stack X, we define a sheaf $\mathcal{S}_X^0$ on X by $\mathcal{S}_X^0(U, \varphi) = \mathcal{S}_U^0$ for all smooth $\varphi : U \to X$ with U a scheme, and $\mathcal{S}_X^0(\alpha, \eta) = \alpha^\star$ for all diagrams (4.1).
A global section $s \in H^0(\mathcal{S}_X^0)$ assigns $s(U, \varphi) \in H^0(\mathcal{S}_U^0)$ for all smooth $\varphi : U \to X$ with $\alpha^\star[\alpha^{-1}(s(V, \psi))] = s(U, \varphi)$ for all diagrams (4.1). We call (X, s) a *d-critical stack* if $\bigl(U, s(U, \varphi)\bigr)$ is a d-critical locus for all smooth $\varphi : U \to X$.
That is, if X is a d-critical stack then any smooth atlas $\varphi : U \to X$ for X is a d-critical locus.

A truncation functor from −1-symplectic derived stacks

As for the scheme case in §4.1, we prove:

Theorem 4.4 (Ben-Bassat, Brav, Bussi, Joyce arXiv:1312.0090)

Let $(\boldsymbol{X}, \omega)$ be a -1-shifted symplectic derived Artin stack. Then the classical Artin stack $X = t_0(\boldsymbol{X})$ extends naturally to a d-critical stack (X, s), with canonical bundle $K_{X,s} \cong \det \mathbb{L}_{\boldsymbol{X}}|_{X^{\rm red}}$.

Corollary 4.5

Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a classical moduli stack of coherent sheaves F on Y, or complexes $F^\bullet$ in $D^b\operatorname{coh}(Y)$ with $\operatorname{Ext}^{<0}(F^\bullet, F^\bullet) = 0$. Then $\mathcal{M}$ extends naturally to a d-critical stack $(\mathcal{M}, s)$ with canonical bundle $K_{\mathcal{M},s} \cong \det(\mathcal{E}^\bullet)|_{\mathcal{M}^{\rm red}}$, where $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$ is the natural obstruction theory on $\mathcal{M}$.

4.3. Categorification using perverse sheaves

It's not easy to explain what perverse sheaves are. We can think of a perverse sheaf as a *system of coefficients for cohomology*. Let X be a complex manifold. The cohomology group $H^k(X;\mathbb{Q})$ is the sheaf cohomology group $H^k(X,\mathbb{Q}_X)$, where $\mathbb{Q}_X$ is the constant sheaf with fibre $\mathbb{Q}$. Working in complexes of sheaves of $\mathbb{Q}$-modules on X, consider the shifted sheaf $\mathbb{Q}_X[\dim_{\mathbb{C}} X]$. This is an example of a perverse sheaf. The shift means that Poincaré duality for X has the nice form $\mathbb{H}^i_{\mathrm{cs}}(\mathbb{Q}_X[\dim_{\mathbb{C}} X]) \cong \mathbb{H}^{-i}(\mathbb{Q}_X[\dim_{\mathbb{C}} X])^*$.
If instead X is a singular complex variety, rather than considering $H^*(X;\mathbb{Q})$, it can be helpful (e.g. in 'intersection cohomology', and to preserve nice properties like Poincaré duality) to consider cohomology $\mathbb{H}^*(X,\mathcal{P}^\bullet)$ with coefficients in a complex $\mathcal{P}^\bullet$ on X (a 'perverse sheaf') which treats the singularities of X in a special way.

Let U be a complex manifold, and $f : U \to \mathbb{C}$ a holomorphic function. Then one can define a perverse sheaf $\mathcal{PV}^\bullet_{U,f}$ on $\mathrm{Crit}\, f$ called the *perverse sheaf of vanishing cycles*, with nice properties. The *vanishing cohomology* $\mathbb{H}^\bullet(\mathcal{PV}^\bullet_{U,f})$ measures how $H^*(f^{-1}(c);\mathbb{Q})$ changes as c passes through critical values of f.
Kai Behrend observed that the pointwise Euler characteristic $\chi_{\mathcal{PV}^\bullet_{U,f}} : \mathrm{Crit}\, f \to \mathbb{Z}$ is the Behrend function of $\mathrm{Crit}\, f$, as used in classical Donaldson–Thomas theory.

Theorem 4.6 (Brav-Bussi-Dupont-Joyce-Szendrői arXiv:1211.3259)

Let (X,s) be an algebraic d-critical locus over $\mathbb{K}$, with an orientation $K^{1/2}_{X,s}$. Then we can construct a canonical perverse sheaf $P^\bullet_{X,s}$ on X, such that if (X,s) is locally modelled on $\mathrm{Crit}(f : U \to \mathbb{A}^1)$, then $P^\bullet_{X,s}$ is locally modelled on the perverse sheaf of vanishing cycles $\mathcal{PV}^\bullet_{U,f}$ of (U,f).
Similarly, we can construct a natural $\mathscr{D}$-module $D^\bullet_{X,s}$ on X, and when $\mathbb{K}=\mathbb{C}$ a natural mixed Hodge module $M^\bullet_{X,s}$ on X.

Sketch of the proof of Theorem 4.6

Roughly, we prove the theorem by taking a Zariski open cover $\{R_i : i \in I\}$ of X with $R_i \cong \mathrm{Crit}(f_i : U_i \to \mathbb{A}^1)$, and showing that $\mathcal{PV}^\bullet_{U_i,f_i}$ and $\mathcal{PV}^\bullet_{U_j,f_j}$ are canonically isomorphic on $R_i \cap R_j$, so we can glue the $\mathcal{PV}^\bullet_{U_i,f_i}$ to get a global perverse sheaf $P^\bullet_{X,s}$ on X.

In fact things are more complicated: the (local) isomorphisms $\mathcal{PV}^\bullet_{U_i,f_i} \cong \mathcal{PV}^\bullet_{U_j,f_j}$ are only canonical *up to sign*. To make them canonical, we use the orientation $K^{1/2}_{X,s}$ to define natural principal $\mathbb{Z}_2$-bundles Q_i on R_i, such that $\mathcal{PV}^\bullet_{U_i,f_i} \otimes_{\mathbb{Z}_2} Q_i \cong \mathcal{PV}^\bullet_{U_j,f_j} \otimes_{\mathbb{Z}_2} Q_j$ is canonical, and then we glue the $\mathcal{PV}^\bullet_{U_i,f_i} \otimes_{\mathbb{Z}_2} Q_i$ to get $P^\bullet_{X,s}$.

Theorem 4.6 and Corollary 4.2 imply:

Corollary 4.7

Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a classical moduli $\mathbb{K}$-scheme of coherent sheaves, or complexes of coherent sheaves, on Y, with (symmetric) obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$. Suppose we are given a square root $\det(\mathcal{E}^\bullet)^{1/2}$ for $\det(\mathcal{E}^\bullet)$ (i.e. ***orientation data****, K–S). Then we have a natural perverse sheaf $P^\bullet_{\mathcal{M},s}$ on $\mathcal{M}$.*

(Compare Kiem and Li arXiv:1212.6444).

The *hypercohomology* $\mathbb{H}^*(P^\bullet_{\mathcal{M},s})$ is a finite-dimensional graded vector space (if $\mathcal{M}$ is of finite type). The pointwise Euler characteristic $\chi(P^\bullet_{\mathcal{M},s})$ is the *Behrend function* $\nu_{\mathcal{M}}$ of $\mathcal{M}$. Thus

$$\textstyle\sum_{i\in\mathbb{Z}}(-1)^i \dim \mathbb{H}^i(P^\bullet_{\mathcal{M},s}) = \chi(\mathcal{M}, \nu_{\mathcal{M}}).$$

Now by Behrend 2005, the Donaldson–Thomas invariant of $\mathcal{M}$ is $DT(\mathcal{M}) = \chi(\mathcal{M}, \nu_{\mathcal{M}})$. So, $\mathbb{H}^*(P^\bullet_{\mathcal{M},s})$ is a graded vector space with dimension $DT(\mathcal{M})$, that is, a *categorification* of $DT(\mathcal{M})$.

Categorifying Lagrangian intersections

Theorem 4.6 and Corollary 4.3 imply:

Corollary 4.8

Let (S, ω) be a classical smooth symplectic $\mathbb{K}$-scheme of dimension $2n$, and $L, M \subseteq S$ be smooth algebraic Lagrangians, with square roots $K_L^{1/2}, K_M^{1/2}$ of their canonical bundles. Then we have a natural perverse sheaf $P_{L,M}^\bullet$ on $X = L \cap M$.

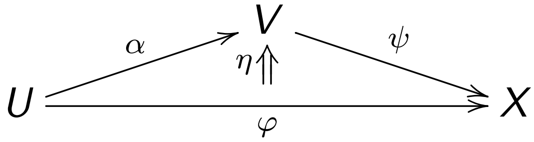
Bussi extends this to complex Lagrangians in complex symplectic manifolds. This is related to Behrend and Fantechi 2009. We think of the hypercohomology $\mathbb{H}^*(P_{L,M}^\bullet)$ as being morally related to the *Lagrangian Floer cohomology* $HF^*(L, M)$ by

$$\mathbb{H}^i(P_{L,M}^\bullet) \approx HF^{i+n}(L, M).$$

We are working on defining 'Fukaya categories' for algebraic/complex symplectic manifolds using these ideas (§6.2(B)).

Extension to Artin stacks

Let (X, s) be a d-critical stack, with an orientation $K_{X,s}^{1/2}$. Then for any smooth $\varphi : U \to X$ with U a scheme, $\big(U, s(U, \varphi)\big)$ is an oriented d-critical locus, so as above, Theorem 4.6 constructs a perverse sheaf $P_{U,\varphi}^\bullet$ on U. Given a diagram



with U, V schemes and φ, ψ smooth, we can construct a natural isomorphism $P_{\alpha,\eta}^\bullet : \alpha^*(P_{V,\psi}^\bullet)[\dim \varphi - \dim \psi] \to P_{U,\varphi}^\bullet$.
All this data $P_{U,\varphi}^\bullet, P_{\alpha,\eta}^\bullet$ is equivalent to a perverse sheaf on X.

Thus we prove:

Theorem 4.9 (Ben-Bassat, Brav, Bussi, Joyce arXiv:1312.0090)

Let (X, s) be a d-critical stack, with an orientation $K_{X,s}^{1/2}$. Then we can construct a canonical perverse sheaf $P_{X,s}^\bullet$ on X.

Corollary 4.10

Suppose Y is a Calabi–Yau 3-fold and $\mathcal{M}$ a classical moduli stack of coherent sheaves F on Y, or of complexes $F^\bullet$ in $D^b \operatorname{coh}(Y)$ with $\operatorname{Ext}^{<0}(F^\bullet, F^\bullet) = 0$, with (symmetric) obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$. Suppose we are given a square root $\det(\mathcal{E}^\bullet)^{1/2}$ for $\det(\mathcal{E}^\bullet)$. Then we construct a natural perverse sheaf $P_{\mathcal{M},s}^\bullet$ on $\mathcal{M}$.

The *hypercohomology* $\mathbb{H}^*(P_{\mathcal{M},s}^\bullet)$ is a categorification of the Donaldson–Thomas theory of Y.