Shifted Symplectic Derived Algebraic Geometry and generalizations of Donaldson–Thomas Theory

Lecture 3 of 3: Motivic invariants, D-T type invariants for Calabi–Yau 4-folds, and future work

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These slides, and references, etc., available at
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Plan of talk:

5. Motivic invariants
References for §5

V. Bussi, D. Joyce and S. Meinhardt, *On motivic vanishing cycles of critical loci*, arXiv:1305.6428, 2013.
O. Ben-Bassat, C. Brav, V. Bussi, and D. Joyce, *A 'Darboux Theorem' for shifted symplectic structures on derived Artin stacks, with applications*, Geom. Top. 19 (2015), 1287–1359. arXiv:1312.0090.
M. Kontsevich and Y. Soibelman, *'Stability structures, motivic Donaldson–Thomas invariants and cluster transformations'*, arXiv:0811.2435, 2008.
M. Kontsevich and Y. Soibelman, *'Motivic Donaldson–Thomas invariants: summary of results'*, arXiv:0910.4315, 2009.

Motivic invariants

Let $\mathbb{K}$ be a field, and R a commutative ring. A *motivic invariant of* $\mathbb{K}$*-schemes* I assigns $I(S) \in R$ for each finite type $\mathbb{K}$-scheme S, such that $I(S) = I(T)$ if $S \cong T$, $I(\mathrm{Spec}\,\mathbb{K}) = 1$, $I(S \times T) = I(S)I(T)$, and if $T \subset S$ is a closed $\mathbb{K}$-subscheme then $I(S) = I(T) + I(S \setminus T)$.
Examples include the *Euler characteristic* χ with $R = \mathbb{Z}$, *virtual Poincaré polynomials*, and *virtual Hodge polynomials*.
As in §1.1, Behrend (2005) showed classical Donaldson–Thomas invariants $DT^\alpha(\tau)$ can be written as *weighted Euler characteristics* $DT^\alpha(\tau) = \chi(\mathfrak{M}^\alpha_{\mathrm{st}}(\tau), \nu)$ for ν a constructible function on $\mathfrak{M}^\alpha_{\mathrm{st}}(\tau)$, the Behrend function. Thus D–T invariants are motivic invariants.
Kontsevich and Soibelman 2009 explained (partially conjecturally) how to extend D–T invariants from Euler characteristics to more-or-less any motivic invariant of $\mathbb{K}$-schemes, for suitable $\mathbb{K}$ (e.g. $\mathbb{K} = \mathbb{C}$), and explained a wall-crossing formula for their invariants under change of stability condition.

Motives of d-critical loci

By similar (but easier) arguments to those used to construct the perverse sheaves $P^\bullet_{X,s}$ in §4.3, we prove:

Theorem 5.1 (Bussi, Joyce and Meinhardt arXiv:1305.6428)

Let (X,s) be a finite type algebraic d-critical locus over $\mathbb{K}$, with an orientation $K^{1/2}_{X,s}$. Then we can construct a natural motive $MF_{X,s}$ in a certain ring of $\hat\mu$-equivariant motives $\bar{\mathcal{M}}^{\hat\mu}_X$ on X, such that if (X,s) is locally modelled on $\mathrm{Crit}(f:U\to\mathbb{A}^1)$, then $MF_{X,s}$ is locally modelled on $\mathbb{L}^{-\dim U/2}([X]-MF^{\mathrm{mot}}_{U,f})$, where $MF^{\mathrm{mot}}_{U,f}$ is the **motivic Milnor fibre** *or* **motivic nearby cycle** *of f.*

Here $\hat\mu$ is roughly the group of roots of unity, and $\bar{\mathcal{M}}^{\hat\mu}_X$ is the ring R in which our motivic invariants take values, and $\mathbb{L}=I(\mathbb{A}^1)$ is the 'Tate motive', the motivic invariant of $\mathbb{A}^1$, where we require $\mathbb{L}^{-1/2}$ to exist in $\bar{\mathcal{M}}^{\hat\mu}_X$. We are using a theory of 'vanishing cycles' for motivic invariants due to Denef–Loeser, Looijenga, ...

Relation to Kontsevich–Soibelman's motivic D–T invariants

Theorem 5.1 and Corollary 4.2 imply:

Corollary 5.2

Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a finite type classical moduli $\mathbb{K}$-scheme of (complexes of) coherent sheaves on Y, with (symmetric) obstruction theory $\phi:\mathcal{E}^\bullet\to\mathbb{L}_{\mathcal{M}}$. Suppose we are given a square root $\det(\mathcal{E}^\bullet)^{1/2}$ for $\det(\mathcal{E}^\bullet)$ (i.e. **orientation data**, *K–S). Then we have a natural motive $MF^\bullet_{\mathcal{M},s}$ on $\mathcal{M}$.*

This motive $MF^\bullet_{\mathcal{M},s}$ is essentially the motivic Donaldson–Thomas invariant of $\mathcal{M}$ defined (partially conjecturally) by Kontsevich and Soibelman, arXiv:0811.2435. K–S work with motivic Milnor fibres of formal power series at each point of $\mathcal{M}$. Our results show the formal power series can be taken to be a regular function, and clarify how the motivic Milnor fibres vary in families over $\mathcal{M}$.

Extension to Artin stacks

We can also generalize Theorem 5.1 to d-critical stacks:

Theorem 5.3 (Ben-Bassat, Brav, Bussi, Joyce)

Let (X, s) be an oriented d-critical stack, of finite type and locally a global quotient. Then we can construct a natural motive $MF_{X,s}$ in a certain ring of $\hat{\mu}$-equivariant motives $\bar{\mathcal{M}}_X^{\mathrm{st},\hat{\mu}}$ on X, such that if $\varphi : U \to X$ is smooth and U is a scheme then

$$\varphi^*(MF_{X,s}) = \mathbb{L}^{\dim\varphi/2} \odot MF_{U,s(U,\varphi)},$$

where $MF_{U,s(U,\varphi)}$ for the scheme case is as in BJM above.

For CY3 moduli stacks, these $MF_{X,s}$ are basically Kontsevich–Soibelman's motivic Donaldson–Thomas invariants.

6. D–T type invariants for Calabi–Yau 4-folds
References for §6

D. Borisov and D. Joyce, *Virtual fundamental classes for moduli spaces of sheaves on Calabi–Yau four-folds*, Geometry and Topology 21 (2017), 3231–3311. arXiv:1504.00690.
S.K. Donaldson and R.P. Thomas, *Gauge Theory in higher dimensions*, §3 in S.A. Huggett et al., editors, *The Geometric Universe*, OUP, 1998.
Y. Cao, *Donaldson–Thomas theory for Calabi–Yau four-folds*, arXiv:1309.4230, 2013.
Y. Cao and N.C. Leung, *Donaldson–Thomas theory for Calabi–Yau four-folds*, arXiv:1407.7659, 2014.
Y. Cao and N.C. Leung, *Orientability for gauge theories on Calabi–Yau manifolds*, arXiv:1502.01141, 2015.

6.1. Introduction

If Y is a Calabi–Yau 3-fold (say over $\mathbb{C}$), then the *Donaldson-Thomas invariants* $DT^\alpha(\tau)$ in $\mathbb{Z}$ or $\mathbb{Q}$ 'count' τ-(semi)stable coherent sheaves on Y with Chern character $\alpha \in H^{\mathrm{even}}(Y, \mathbb{Q})$, for τ a (say Gieseker) stability condition. The $DT^\alpha(\tau)$ are unchanged under continuous deformations of Y, and transform by a wall-crossing formula under change of stability condition τ.

We have τ-(semi)stable moduli schemes $\mathcal{M}^\alpha_{\mathrm{st}}(\tau) \subseteq \mathcal{M}^\alpha_{\mathrm{ss}}(\tau)$, where $\mathcal{M}^\alpha_{\mathrm{ss}}(\tau)$ is proper, and $\mathcal{M}^\alpha_{\mathrm{st}}(\tau)$ has a symmetric obstruction theory.

The easy case (Thomas 1998) is when $\mathcal{M}^\alpha_{\mathrm{ss}}(\tau) = \mathcal{M}^\alpha_{\mathrm{st}}(\tau)$. Then $DT^\alpha(\tau) \in \mathbb{Z}$ is the *virtual cycle* (which has dimension zero) of the proper scheme with obstruction theory $\mathcal{M}^\alpha_{\mathrm{st}}(\tau)$.

Holomorphic Donaldson invariants?

In joint work with Dennis Borisov and Yalong Cao, I am developing a similar story for Calabi–Yau 4-folds. We want to define invariants 'counting' τ-(semi)stable coherent sheaves on Calabi–Yau 4-folds. If CY3 Donaldson–Thomas invariants are 'holomorphic Casson invariants', as in Thomas 1998, these should be thought of as 'holomorphic Donaldson invariants'.

The idea for doing this goes back to Donaldson–Thomas 1998, using gauge theory: one wants to 'count' moduli spaces of Spin(7)-instantons on a Calabi–Yau 4-fold (or more generally a Spin(7)-manifold). However, it has not gone very far, as compactifying such higher-dimensional gauge-theoretic moduli spaces in a nice way is too difficult. (See Cao arXiv:1309.4230 and Cao and Leung arXiv:1407.7659 for a gauge-theoretic approach.)

Virtual cycles using algebraic geometry?

Rather than using gauge theory, we stay within algebraic geometry, so we get compactness of moduli spaces more-or-less for free. So, suppose Y is a Calabi–Yau 4-fold, and $\alpha \in H^{\mathrm{even}}(Y,\mathbb{Q})$ such that $\mathcal{M}_{\mathrm{ss}}^\alpha(\tau)=\mathcal{M}_{\mathrm{st}}^\alpha(\tau)$ (the easy case).
There is a natural obstruction theory $\phi:\mathcal{E}^\bullet\to\mathbb{L}_{\mathcal{M}}$ on $\mathcal{M}_{\mathrm{st}}^\alpha(\tau)$, but $\mathcal{E}^\bullet$ is perfect in $[-2,0]$ not $[-1,0]$, so **the usual Behrend–Fantechi virtual cycles do not work**.
Instead, we will use a **completely new method** to define a virtual cycle, which is special to the Calabi–Yau 4-fold case, and for the moment works only over $\mathbb{C}$. It uses heavy machinery from Derived Algebraic Geometry — the 'shifted symplectic derived schemes' of Pantev–Toën–Vaquié–Vezzosi (PTVV) — and Derived Differential Geometry — 'derived smooth manifolds'.

Virtual cycles using 'derived smooth manifolds'

Let $\boldsymbol{\mathcal{M}}_{\mathrm{st}}^\alpha(\tau)$ be the derived moduli scheme corresponding to the classical moduli scheme $\mathcal{M}_{\mathrm{st}}^\alpha(\tau)$. Then as in §3, PTVV show that $\boldsymbol{\mathcal{M}}_{\mathrm{st}}^\alpha(\tau)$ has a '−2-shifted symplectic structure' ω, a geometric structure which roughly encodes Serre duality of sheaves on Y.
Using our Darboux Theorem, we show that given any −2-shifted symplectic derived $\mathbb{C}$-scheme $(\boldsymbol{X},\omega)$, we can construct a 'derived smooth manifold' X_{dm} with the same underlying topological space. The virtual dimension is $\mathrm{vdim}_{\mathbb{R}} X_{\mathrm{dm}}=\frac{1}{2}\mathrm{vdim}_{\mathbb{R}}\boldsymbol{X}$, which is half what one would expect. Roughly, this is because X_{dm} is the base of a 'real Lagrangian fibration' $\pi:\boldsymbol{X}\to X_{\mathrm{dm}}$ of the −2-shifted symplectic derived scheme $\boldsymbol{X}$. If $\boldsymbol{X}$ is proper, so that X_{dm} is compact, and we can find an orientation on X_{dm}, then X_{dm} has a deformation-invariant virtual cycle, in bordism or homology. Using this, we can define our new Donaldson–Thomas type invariants of Calabi–Yau 4-folds.

6.2. Virtual cycles for −2-shifted symplectic derived schemes

Here is the first part of what we prove:

Theorem 6.1 (Borisov–Joyce, arXiv:1504.00690)

Let $(\boldsymbol{X},\omega)$ be a −2-shifted symplectic derived scheme over $\mathbb{C}$. Then one can construct a **d-manifold**, *or* **m-Kuranishi space**, *or* **Spivak derived manifold** *(all forms of* **derived smooth manifolds**, *more-or-less equivalent) $X_{\rm dm}$ which has the same underlying topological space X as $(\boldsymbol{X},\omega)$, with the complex analytic topology.*
The construction involves arbitrary choices, but $X_{\rm dm}$ is unique up to bordisms which fix the topological space X.
The (real) virtual dimension of $X_{\rm dm}$ is

$$\mathrm{vdim}_{\mathbb{R}} X_{\rm dm} = \mathrm{vdim}_{\mathbb{C}} \boldsymbol{X} = \tfrac{1}{2}\,\mathrm{vdim}_{\mathbb{R}} \boldsymbol{X},$$

which is half what one would have expected.

Derived manifolds and bordisms

I haven't time to explain derived smooth manifolds properly – see `people.maths.ox.ac.uk/~joyce/dmanifolds.html`, and papers arXiv:1206.4207, arXiv:1208.4948 on d-manifolds, and arXiv:1409.6908 on (m-)Kuranishi spaces. Some useful facts:

- Moduli spaces of solutions of nonlinear elliptic equations on compact manifolds have the structure of derived manifolds.
- A derived manifold $X_{\rm dm}$ is locally modelled by a 'Kuranishi neighbourhood' (V,E,s) of a real manifold V, real vector bundle $E\to V$ and smooth section $s:V\to E$, where the topological space of $X_{\rm dm}$ is locally homeomorphic to $s^{-1}(0)\subset V$. Think of $X_{\rm dm}$ as locally the (homotopy) fibre product $V\times_{s,E,0}V$.
- Any (compact) derived manifold $\boldsymbol{X}$ can be perturbed to a (compact) ordinary manifold $\tilde X$, which is unique up to bordism. In a Kuranishi neighbourhood (V,E,s), perturb s to a generic, transverse $\tilde s:V\to E$, so that $\tilde s^{-1}(0)\subset V$ is a manifold.

Orientations of −2-shifted symplectic derived schemes

To lift this to a virtual cycle over $\mathbb{Z}$, we need to include orientations of $(\boldsymbol{X},\omega)$ and $X_{\rm dm}$.
Recall that if $(\boldsymbol{X},\omega)$ is a −1-shifted symplectic derived scheme (the Calabi–Yau 3 case), an *orientation* of $(\boldsymbol{X},\omega)$ is a square root line bundle $\det(\mathbb{L}_{\boldsymbol{X}})^{1/2}$. These were introduced by Kontsevich and Soibelman, and are essential for motivic and categorified D–T theory. Here is the Calabi–Yau 4 analogue:

Definition

Let $(\boldsymbol{X},\omega)$ be a −2-shifted symplectic derived scheme. There is a natural isomorphism $\iota:\det(\mathbb{L}_{\boldsymbol{X}})^{\otimes^2}\longrightarrow\mathcal{O}_{\boldsymbol{X}}$. An *orientation* of $(\boldsymbol{X},\omega)$ is an isomorphism $\alpha:\det(\mathbb{L}_{\boldsymbol{X}})\longrightarrow\mathcal{O}_{\boldsymbol{X}}$ with $\alpha\otimes\alpha=\iota$.

This is simpler, one categorical level down from the CY3 case.

Lemma 6.2

In Theorem 6.1, there is a natural 1-1 *correspondence between orientations on* $(\boldsymbol{X},\omega)$ *and orientations on the d-manifold* $X_{\rm dm}$.

Theorem 6.3

Suppose $\boldsymbol{X}_{\rm dm}$ *is a compact, oriented derived manifold, with* $\mathrm{vdim}\,\boldsymbol{X}_{\rm dm}=k$. *Then we can can define a* **virtual cycle** $[\boldsymbol{X}_{\rm dm}]_{\rm virt}$ *in both the bordism group* $B_k(*)$, *and in the Steenrod homology* $H_k^{\rm St}(X_{\rm dm};\mathbb{Z})$. *If* $X_{\rm dm}$ *is a nice topological space (e.g. a Euclidean Neighbourhood Retract) then* $H_k^{\rm St}(X_{\rm dm};\mathbb{Z})\cong H_k(X_{\rm dm};\mathbb{Z})$.

Combining Theorems 6.1, 6.3 and Lemma 6.2 gives:

Corollary 6.4

Let $(\boldsymbol{X},\omega)$ *be a proper, oriented* −2*-shifted symplectic derived* $\mathbb{C}$*-scheme, with* $\mathrm{vdim}_{\mathbb{C}}\,\boldsymbol{X}=k$. *Then we can construct a virtual cycle* $[\boldsymbol{X}]_{\rm virt}$ *in bordism* $B_k(*)$, *and in the homology* $H_k(X;\mathbb{Z})$ *of the complex analytic topological space* X *of* $\boldsymbol{X}$.

Note that the virtual cycle has real virtual dimension $k=\mathrm{vdim}_{\mathbb{C}}\,\boldsymbol{X}=\frac{1}{2}\,\mathrm{vdim}_{\mathbb{R}}\,\boldsymbol{X}$, which is half what one would expect.

Sketch proof of Theorem 6.1

Let $(\boldsymbol{X},\omega)$ be a -2-shifted symplectic derived $\mathbb{C}$-scheme. Then the 'Darboux Theorem', Theorem 3.1, gives local models for $(\boldsymbol{X},\omega)$ in the Zariski topology. In the -2-shifted case, the local models reduce to the following data:

- A smooth $\mathbb{C}$-scheme U
- A vector bundle $E \to U$
- A section $s \in H^0(E)$
- A nondegenerate quadratic form Q on E with $Q(s,s)=0$.

The underlying topological space of $\boldsymbol{X}$ is $\{x \in U : s(x)=0\}$. The virtual dimension of $\boldsymbol{X}$ is $\mathrm{vdim}_{\mathbb{C}}\,\boldsymbol{X} = 2\dim_{\mathbb{C}} U - \mathrm{rank}_{\mathbb{C}}\,E$. The cotangent complex $\mathbb{L}_{\boldsymbol{X}}|_X$ of $\boldsymbol{X}$ is

$$\big[\underset{-2}{TU}|_{s^{-1}(0)} \xrightarrow{Q\circ\mathrm{d}s} \underset{-1}{E^*}|_{s^{-1}(0)} \xrightarrow{\mathrm{d}s} \underset{0}{T^*U}|_{s^{-1}(0)}\big].$$

This is the local model for (derived) moduli schemes of (simple) coherent sheaves F on a Calabi–Yau 4-fold Y. Think of U as an étale open neighbourhood of 0 in $\mathrm{Ext}^1(F,F)$, and $E \to U$ as a trivial vector bundle with fibre $\mathrm{Ext}^2(F,F)$, and Q as the nondegenerate quadratic form on $\mathrm{Ext}^2(F,F)$

$$\mathrm{Ext}^2(F,F)\times\mathrm{Ext}^2(F,F) \xrightarrow{\wedge} \mathrm{Ext}^4(F,F) \xrightarrow{\text{Serre duality}} \mathrm{Ext}^0(F,F)^* \xrightarrow{\mathrm{id}_F^*} \mathbb{K},$$

and s as a Kuranishi map $s : \mathrm{Ext}^1(E,E) \supseteq U \to \mathrm{Ext}^2(E,E)$. The special thing the theorem tells us is that we can choose U,E,s,Q such that $Q(s,s)=0$, rather than just $Q(s,s)=0$ modulo s^3, for instance.

The local model for X_{dm}

Here is how to build the derived manifold X_{dm} locally: regard $E \to U$ as a real vector bundle over the real manifold U. Choose a splitting $E = E_+ \oplus E_-$, where $Q|_{E_+}$ is real and positive definite, and $E_- = iE_+$ so that $Q|_{E_-}$ is real and negative definite. Write $s = s_+ \oplus s_-$ with $s_\pm \in C^\infty(E_\pm)$. Then X_{dm} is locally the derived fibre product $U \times_{0,E_+,s_+} U$, given by the 'Kuranishi neighbourhood' (U, E_+, s_+). It has real virtual dimension

$$\dim_{\mathbb{R}} U - \mathrm{rank}_{\mathbb{R}} E_+ = 2\dim_{\mathbb{C}} U - \mathrm{rank}_{\mathbb{C}} E = \mathrm{vdim}_{\mathbb{C}} \boldsymbol{X}.$$

Observe that $Q(s,s) = 0$ implies that $|s_+|^2 = |s_-|^2$, where norms $|\,.\,|$ on E_+, E_- are defined using $\pm \mathrm{Re}\, Q$. Hence as sets we have

$$\{x \in U : s(x) = 0\} = \{x \in U : s_+(x) = 0\} \subseteq U.$$

This is why $\boldsymbol{X}$ and X_{dm} have the same topological space X. The difficult bit is to show we can choose compatible splittings $E = E_+ \oplus E_-$ on an open cover of $\boldsymbol{X}$, and glue the local models to make a global derived manifold X_{dm}.

6.3. D–T style invariants for Calabi–Yau 4-folds

Suppose Y is a Calabi–Yau 4-fold over $\mathbb{C}$, and $\alpha \in H^{\mathrm{even}}(Y,\mathbb{Q})$ such that $\mathcal{M}_{\mathrm{ss}}^\alpha(\tau) = \mathcal{M}_{\mathrm{st}}^\alpha(\tau)$. Write $\boldsymbol{\mathcal{M}}_{\mathrm{st}}^\alpha(\tau)$ for the corresponding derived moduli scheme. Then $\boldsymbol{\mathcal{M}}_{\mathrm{st}}^\alpha(\tau)$ has a -2-shifted symplectic structure by PTVV. Suppose we can choose an orientation. (Work in progress by Cao–Gross–Joyce–Upmeier will show that this is possible, and how to choose orientations canonically.) Then Theorem 6.1 constructs a compact, oriented derived manifold $\mathcal{M}_{\mathrm{st}}^\alpha(\tau)_{\mathrm{dm}}$ with the same topological space, of dimension

$$\mathrm{vdim}_{\mathbb{R}}\, \mathcal{M}_{\mathrm{st}}^\alpha(\tau)_{\mathrm{dm}} = \mathrm{vdim}_{\mathbb{C}}\, \boldsymbol{\mathcal{M}}_{\mathrm{st}}^\alpha(\tau) = 2 - \deg(\alpha \cup \bar\alpha \cup \mathrm{td}(TY))_8 = d.$$

The derived manifold has a virtual cycle $[\mathcal{M}_{\mathrm{st}}^\alpha(\tau)_{\mathrm{dm}}]_{\mathrm{vir}}$ in bordism, or in homology $H_d(\mathcal{M}_{\mathrm{st}}^\alpha(\tau);\mathbb{Z})$. If $d=0$ this virtual cycle is an integer, and we define $DT_4^\alpha(\tau) = [\mathcal{M}_{\mathrm{st}}^\alpha(\tau)_{\mathrm{dm}}]_{\mathrm{vir}} \in \mathbb{Z}$.

If $d > 0$ then as for Donaldson invariants, we hope to find cohomology classes on $\mathcal{M}^\alpha_{\rm st}(\tau)$ using the Chern characters of the universal sheaf $\mathcal{E} \to \mathcal{M}^\alpha_{\rm st}(\tau) \times Y$, and make integer invariants by integrating products of these classes over $[\mathcal{M}^\alpha_{\rm st}(\tau)_{\rm dm}]_{\rm vir}$.

These invariants will have the nice property of being unchanged under continuous deformations of the complex structure of the Calabi–Yau 4-fold Y. There are still lots of interesting open questions – computation in examples such as Hilbert schemes of points, wall-crossing formulae, use for curve-counting, and so on.

Motivation from gauge theory

We can explain using gauge theory why one should pass from the -2-shifted symplectic derived scheme $\boldsymbol{\mathcal{M}}^\alpha_{\rm st}(\tau)$ to the derived manifold $\mathcal{M}^\alpha_{\rm st}(\tau)_{\rm dm}$, which was part of our original motivation. Consider a moduli space $\mathcal{M}$ of stable rank r holomorphic vector bundles $F \to Y$ over Y, with $c_1(F) = 0$ for simplicity. By the Hitchin–Kobayashi correspondence, each F has a unique connection ∇_F with curvature R_F satisfying the Hermitian–Einstein equations

$$R_F^{2,0} = 0, \qquad R_F^{1,1} \wedge \omega^3 = 0. \tag{6.1}$$

These equations are overdetermined ($13r^2$ equations plus r^2 gauge rescalings on $8r^2$ unknowns), which corresponds to the fact that $\boldsymbol{\mathcal{M}}^\alpha_{\rm st}(\tau)$ does not have cotangent complex in $[-1, 0]$ (is not 'quasi-smooth'), and so does not have a virtual cycle.

In the CY4 case, there is a splitting $R_F^{2,0}=R_F^{2,0\,+}\oplus R_F^{2,0\,-}$ into real 'self-dual' and 'anti-self-dual' components. So we can impose instead the weaker 'Spin(7) instanton equations'

$$R_F^{2,0\,+}=0, \qquad R_F^{1,1}\wedge\omega^3=0. \tag{6.2}$$

These equations are determined ($7r^2$ equations plus r^2 gauge rescalings on $8r^2$ unknowns), and elliptic, and form a moduli space $\mathcal{M}_{\rm st}^\alpha(\tau)_{\rm dm}$ with a virtual cycle, at least if compact and oriented. Since (6.2) is a subset of (6.1), one would expect the moduli space $\mathcal{M}_{\rm HE}$ of solutions of (6.1) to be a subset of the moduli space $\mathcal{M}_{{\rm Spin}(7)}$ of solutions of (6.2). But using L^2 norms of components of R_F, one can show that (for suitable Chern characters α) we have $\mathcal{M}_{\rm HE}=\mathcal{M}_{{\rm Spin}(7)}$ as sets, they differ only as non-reduced spaces. Thus it is reasonable to expect a derived scheme $\mathcal{M}_{\rm HE}=\boldsymbol{\mathcal{M}}_{\rm st}^\alpha(\tau)$ and a derived manifold $\mathcal{M}_{{\rm Spin}(7)}=\mathcal{M}_{\rm st}^\alpha(\tau)_{\rm dm}$ with the same underlying topological space.

7. Future work (conjectural from here onwards)
7.1. A conjecture on perverse sheaf morphisms

Let $(\boldsymbol{X},\omega_{\boldsymbol{X}})$ be a -1-shifted symplectic derived scheme, and $\boldsymbol{i}:\boldsymbol{L}\to\boldsymbol{X}$ a Lagrangian, in the sense of PTVV.
Choose an orientation $K_{X,s}^{1/2}$ for $(\boldsymbol{X},\omega_{\boldsymbol{X}})$. There is then a notion of relative orientation for $\boldsymbol{i}:\boldsymbol{L}\to\boldsymbol{X}$, choose one of these.
We get a perverse sheaf $P^\bullet_{\boldsymbol{X},\omega_{\boldsymbol{X}}}$ on $\boldsymbol{X}$, by Theorem 4.6.

Conjecture 7.1 (Work in progress with Lino Amorim, should be a theorem soon. See Amorim–Ben-Bassat arXiv:1601.01536)

There is a natural morphism in $D^b_c(\boldsymbol{L})$

$$\mu_{\boldsymbol{L}}:\mathbb{Q}_{\boldsymbol{L}}[\mathrm{vdim}\,\boldsymbol{L}]\longrightarrow \boldsymbol{i}^!(P^\bullet_{\boldsymbol{X},\omega_{\boldsymbol{X}}}), \tag{7.1}$$

with given local models in 'Darboux form' presentations for $\boldsymbol{X},\boldsymbol{L}$. These $\mu_{\boldsymbol{L}}$ should satisfy a package of properties under products, composition of Lagrangian correspondences, Verdier duality, etc.

This conjecture has important consequences.

We already know local models for $\boldsymbol{i} : \boldsymbol{L} \to \boldsymbol{X}$ and $\mu_{\boldsymbol{L}}$ in (7.1) (Joyce–Safronov arXiv:1506.04024). What makes the conjecture difficult is that local models are not enough: $\mu_{\boldsymbol{L}}$ is a morphism of complexes, not of (perverse) sheaves, and such morphisms do not glue like sheaves. For instance, one could imagine $\mu_{\boldsymbol{L}}$ to be globally nonzero, but zero on the sets of an open cover of $\boldsymbol{L}$.
So to construct $\mu_{\boldsymbol{L}}$, we have to do a gluing problem in an ∞-category. Lino Amorim and I are writing a paper on this.

7.2. Applications of Conjecture 7.1
(A)Cohomological Hall Algebras

Let Y be a Calabi–Yau 3-fold, and $\boldsymbol{\mathcal{M}}$ the derived moduli stack of coherent sheaves (or suitable complexes) on Y, with its -1-shifted symplectic structure ω. Then Theorem 4.4 makes the classical stack $\mathcal{M}$ into a d-critical stack $(\mathcal{M}, s)$. Suppose we have 'orientation data' for Y, i.e. an orientation $K^{1/2}_{\mathcal{M},s}$, with compatibility condition on exact sequences. Then we have a perverse sheaf $P^\bullet_{\mathcal{M},s}$, with hypercohomology $\mathbb{H}^*(P^\bullet_{\mathcal{M},s})$.
We would like to define an associative multiplication on $\mathbb{H}^*(P^\bullet_{\mathcal{M},s})$, making it into a *Cohomological Hall Algebra*, in the style of Kontsevich and Soibelman (arXiv:1006.2706).

Let $\boldsymbol{\mathcal{E}xact}$ be the derived stack of short exact sequences $0 \to F_1 \to F_2 \to F_3 \to 0$ in $\mathrm{coh}(Y)$ (or distinguished triangles in $D^b\,\mathrm{coh}(Y)$), with projections $\pi_1, \pi_2, \pi_3 : \boldsymbol{\mathcal{E}xact} \to \boldsymbol{\mathcal{M}}$.

Claim (Oren Ben-Bassat, work in progress?)

$\pi_1 \times \pi_2 \times \pi_3 : \boldsymbol{\mathcal{E}xact} \to (\boldsymbol{\mathcal{M}}, \omega) \times (\boldsymbol{\mathcal{M}}, -\omega) \times (\boldsymbol{\mathcal{M}}, \omega)$ is Lagrangian in -1-shifted symplectic.

Then apply the stack version of Conjecture 7.1 and manipulate using Verdier duality of perverse sheaves to get the COHA multiplication.

(B) 'Fukaya categories' of complex symplectic manifolds

Let (S, ω) be a complex symplectic manifold, with $\dim_{\mathbb{C}} S = 2n$, and $L, M \subset S$ be complex Lagrangians (not supposed compact or closed). The intersection $L \cap M$, as a complex analytic space, has a d-critical structure s (Vittoria Bussi, arXiv:1404.1329). Given square roots of canonical bundles $K_L^{1/2}, K_M^{1/2}$, we get an orientation on $(L \cap M, s)$, and so a perverse sheaf $P^\bullet_{L,M}$ on $L \cap M$.
I claim that we should think of the shifted hypercohomology $\mathbb{H}^{*-n}(P^\bullet_{L,M})$ as a substitute for the Lagrangian Floer cohomology $HF^*(L, M)$ in symplectic geometry. But $HF^*(L, M)$ is the morphisms in the derived Fukaya category $D^b\mathrm{Fuk}(S, \omega)$.

Problem 7.2

Given a complex symplectic manifold (S,ω), build a 'Fukaya category' with objects $(L, K_L^{1/2})$ for L a complex Lagrangian, and graded morphisms $\mathbb{H}^{*-n}(P^\bullet_{L,M})$.
Extend to **derived** *Lagrangians L in (S,ω).*
Work out the 'right' way to form a 'derived Fukaya category' for (S,ω) out of this, as a (Calabi–Yau?) triangulated category.
Show that (derived) Lagrangian correspondences induce functors between these derived Fukaya categories.

Question 7.3

Can we include complex coisotropic submanifolds as objects?
Maybe using $\mathscr{D}$-modules?

Conjecture 7.1 is what we need to define composition of morphisms in this 'Fukaya category', as follows. If L, M, N are Lagrangians in (S,ω), then $M\cap L, N\cap M, L\cap N$ are -1-shifted symplectic / d-critical loci, and $L\cap M\cap N$ is Lagrangian in the product $(M\cap L)\times(N\cap M)\times(L\cap N)$. Applying the Conjecture to $L\cap M\cap N$ and rearranging gives a morphism of constructible complexes

$$\mu_{L,M,N} : P^\bullet_{L,M} \overset{L}{\otimes} P^\bullet_{M,N}[n] \longrightarrow P^\bullet_{L,N}.$$

Taking hypercohomology gives the multiplication $\mathrm{Hom}^*(L,M)\times\mathrm{Hom}^*(M,N)\to\mathrm{Hom}^*(L,N)$.

Relation to deformation quantization and DQ-modules

Kashiwara and Schapira (Astérisque 345) develop a theory of *deformation quantization modules*, or *DQ-modules*, on a complex symplectic manifold (S,ω), which are roughly symplectic versions of $\mathscr{D}$-modules. *Holonomic* DQ-modules are supported on (singular) Lagrangians. If L is a closed, embedded complex Lagrangian in (S,ω) with $K_L^{1/2}$, D'Agnolo and Schapira construct a simple holonomic DQ-module $D_L^\bullet$ supported on L. For Lagrangians L,M, Kashiwara and Schapira show that $R\mathscr{H}om(D_L^\bullet,D_M^\bullet)$ is a perverse sheaf over $\mathbb{C}((\hbar))$ supported on $X=L\cap M$. Schapira (private communication) explained that this perverse sheaf should be isomorphic to the perverse sheaf $P_{L,M}^\bullet$ we construct, over base ring $A=\mathbb{C}((\hbar))$.

Relation to deformation quantization and DQ-modules

All this looks very similar to our 'complex Fukaya category' picture, but there are some puzzling differences:

- Our perverse sheaf picture works over (nearly) any base ring A, e.g. $A=\mathbb{Z},\mathbb{Q}$. DQ-modules work only over $A=\mathbb{C}((\hbar))$. Is our picture related to 'microlocal perverse sheaves'?
- We have natural monodromy and Verdier duality operators on our perverse sheaves. Does $\hbar$ encode the monodromy?
- Our objects live on $i:L\to S$, where i need not be an embedding, and L can be derived, with classical singularities. Holonomic DQ-modules live on embedded Lagrangians $L\subset S$. They can be singular, but the singularities allowed look very different to those in our picture.

I would like to understand the relation between the theories better.

(C) Perverse sheaves and DT4 virtual cycles

Let $(\boldsymbol{X},\omega)$ be an oriented -2-shifted symplectic derived scheme over $\mathbb{C}$, e.g. a Calabi–Yau 4-fold derived moduli scheme. Regard the point $*$ as an oriented -1-shifted symplectic derived scheme. Its perverse sheaf is the constant sheaf $\mathbb{Q}_*$. Then $\boldsymbol{\pi}:\boldsymbol{X}\to *$ is Lagrangian in $(*,\omega)$, so Conjecture 7.1 gives a morphism

$$\mu_{\boldsymbol{X}}:\mathbb{Q}_{\boldsymbol{X}}[\mathrm{vdim}\,\boldsymbol{X}]\longrightarrow \boldsymbol{\pi}^!(\mathbb{Q}_*)=\mathcal{D}_{\boldsymbol{X}}(\mathbb{Q}_{\boldsymbol{X}}).$$

Taking hypercohomology induces a linear map

$$H^{\mathrm{vdim}\,\boldsymbol{X}}_{\mathrm{cs}}(X,\mathbb{Q})\longrightarrow \mathbb{Q}.$$

If $\boldsymbol{X}$ is compact, this should be contraction with a class $[\boldsymbol{X}]_{\mathrm{virt}}\in H_{\mathrm{vdim}\,\boldsymbol{X}}(X,\mathbb{Q})$. I expect this to be the virtual cycle in Theorem 6.3. Note that this also works over other fields $\mathbb{K}\neq\mathbb{C}$. The perverse sheaf picture does not obviously explain why $[\boldsymbol{X}]_{\mathrm{virt}}$ should be deformation-invariant.

7.3. Other problems
(D) Gluing matrix factorization categories

Suppose $f:U\to\mathbb{A}^1$ is a regular function on a smooth scheme U. The *matrix factorization category* $\mathrm{MF}(U,f)$ is a $\mathbb{Z}_2$-periodic triangulated category. It depends only on U,f in a neighbourhood of $\mathrm{Crit}(f)$, and we can think of it as a sheaf of triangulated categories on $\mathrm{Crit}(f)$. By the Darboux Theorem, -1-symplectic derived schemes $(X,\omega_{\boldsymbol{X}})$ are locally modelled on $\mathrm{Crit}(f:U\to\mathbb{A}^1)$.

Problem 7.4

Given a -1-shifted symplectic derived scheme $(X,\omega_{\boldsymbol{X}})$ with extra data (orientation and 'spin structure'?), construct a sheaf of $\mathbb{Z}_2$-periodic triangulated categories $\mathrm{MF}_{\boldsymbol{X},\omega_{\boldsymbol{X}}}$ on $\boldsymbol{X}$, such that if $(\boldsymbol{X},\omega_{\boldsymbol{X}})$ is locally modelled on $\mathrm{Crit}(f:U\to\mathbb{A}^1)$, then $\mathrm{MF}_{\boldsymbol{X},\omega_{\boldsymbol{X}}}$ is locally modelled on $\mathrm{MF}(U,f)$.

Although d-critical loci (X, s) are also locally modelled on $\mathrm{Crit}(f : U \to \mathbb{A}^1)$, I do not expect the analogue for d-critical loci to work; $\mathrm{MF}_{\boldsymbol{X},\omega_{\boldsymbol{X}}}$ will encode derived data in $(\boldsymbol{X}, \omega_{\boldsymbol{X}})$ which is forgotten by the d-critical locus (X, s).
I expect that a Lagrangian $\boldsymbol{i} : \boldsymbol{L} \to \boldsymbol{X}$ (plus extra data) should define an object (global section of sheaf of objects) in $\mathrm{MF}_{\boldsymbol{X},\omega_{\boldsymbol{X}}}$, with nice properties.
It is conceivable that one could actually *define* $\mathrm{MF}_{\boldsymbol{X},\omega_{\boldsymbol{X}}}$ as a derived 'Fukaya category' of Lagrangians $\boldsymbol{i} : \boldsymbol{L} \to \boldsymbol{X}$ in $(\boldsymbol{X}, \omega_{\boldsymbol{X}})$.

(E) Kapustin–Rozansky 2-categories for complex symplectic manifolds

Given a complex symplectic manifold (S, ω), Kapustin and Rozansky conjecture the existence of an interesting 2-category, with objects complex Lagrangians L with $K_L^{1/2}$, such that $\mathrm{Hom}(L, M)$ is a $\mathbb{Z}_2$-periodic triangulated category (or sheaf of such on $L \cap M$), and if $L \cap M$ is locally modelled on $\mathrm{Crit}(f : U \to \mathbb{A}^1)$ then $\mathrm{Hom}(L, M)$ is locally modelled on $\mathrm{MF}(U, f)$.
A lot of this K–R Conjecture would follow by combining §7.2(B), Fukaya categories and §7.3(D), Gluing matrix factorization categories above.
Seeing what the rest of the Kapustin–Rozansky Conjecture requires should tell us some interesting properties to expect of $\mathrm{MF}_{\boldsymbol{X},\omega_{\boldsymbol{X}}}$.

(F) Categorifying Cohomological Hall Algebras?

Question 7.5

Do Cohomological Hall algebras of Calabi–Yau 3-folds Y admit a categorification using matrix factorization categories, in a similar way to the Kapustin–Rozansky conjectured categorification of the 'Fukaya categories' of complex symplectic manifolds?

Let $\mathcal{M}$ be the derived moduli stack of coherent sheaves on Y, with its -1-shifted symplectic structure ω, and discrete extra data (orientation and 'spin structure'). One would expect to build such a categorification by writing $\mathcal{M}$ as a critical locus locally in the smooth topology, and then 'gluing' the associated matrix factorization categories.
Compare Kontsevich and Soibelman arXiv:1006.2706, §8.1, 'Categorification of critical COHA'.