Counting semistable coherent sheaves on surfaces

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(See also arXiv:2005.05637 with Jacob Gross and Yuuji Tanaka.)

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These slides available at
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1. Introduction

An *enumerative invariant theory* in Algebraic or Differential Geometry is the study of invariants $I_\alpha(\tau)$ which 'count' τ-semistable objects E with fixed topological invariants $[\![E]\!]=\alpha$ in some geometric problem, usually by means of a virtual class $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ for the moduli space $\mathcal{M}_\alpha^{\rm ss}(\tau)$ of τ-semistable objects in some homology theory, with $I_\alpha(\tau)=\int_{[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}}\mu_\alpha$ for some natural cohomology class μ_α. We call the theory $\mathbb{C}$-*linear* if the objects E live in a $\mathbb{C}$-linear additive category $\mathcal{A}$. For example:

- Invariants counting semistable vector bundles on curves.
- Mochizuki-style invariants counting coherent sheaves on surfaces. (Think of as algebraic Donaldson invariants.)
- Donaldson–Thomas invariants of Calabi–Yau or Fano 3-folds.
- Donaldson–Thomas type invariants of Calabi–Yau 4-folds.
- Invariants counting representations of quivers Q.
- $\mathrm{U}(m)$ Donaldson invariants of 4-manifolds.

I have proved that many such theories in Algebraic Geometry, in which either the moduli spaces are automatically smooth (e.g. coherent sheaves on curves, quiver representations), or the invariants are defined using Behrend–Fantechi obstruction theories and virtual classes, share a common universal structure.

I expect this picture also to extend to Calabi–Yau 4-fold invariants defined using Borisov–Joyce / Oh–Thomas virtual classes, provided these virtual classes have a package of properties.

Here is an outline of this structure:

(a) We form two moduli stacks $\mathcal{M}, \mathcal{M}^{\mathrm{pl}}$ of all objects E in $\mathcal{A}$, where $\mathcal{M}$ is the usual moduli stack, and $\mathcal{M}^{\mathrm{pl}}$ the 'projective linear' moduli stack of objects E modulo 'projective isomorphisms', i.e. quotient by $\lambda\,\mathrm{id}_E$ for $\lambda \in \mathbb{G}_m$.

(b) We are given a quotient $K_0(\mathcal{A}) \twoheadrightarrow K(\mathcal{A})$, where $K(\mathcal{A})$ is the lattice of topological invariants $[\![E]\!]$ of E (e.g. fixed Chern classes). We split $\mathcal{M} = \coprod_{\alpha \in K(\mathcal{A})} \mathcal{M}_\alpha$, $\mathcal{M}^{\mathrm{pl}} = \coprod_{\alpha \in K(\mathcal{A})} \mathcal{M}^{\mathrm{pl}}_\alpha$.

(c) There is a symmetric biadditive *Euler form* $\chi : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$.

(d) We can form the homology $H_*(\mathcal{M}), H_*(\mathcal{M}^{\rm pl})$ over $\mathbb{Q}$, with $H_*(\mathcal{M}) = \bigoplus_{\alpha\in K(\mathcal{A})} H_*(\mathcal{M}_\alpha)$, $H_*(\mathcal{M}^{\rm pl}) = \bigoplus_{\alpha\in K(\mathcal{A})} H_*(\mathcal{M}^{\rm pl}_\alpha)$. Define shifted versions $\hat{H}_*(\mathcal{M}), \check{H}_*(\mathcal{M}^{\rm pl})$ by $\hat{H}_n(\mathcal{M}_\alpha) = H_{n-\chi(\alpha,\alpha)}(\mathcal{M}_\alpha)$, $\check{H}_n(\mathcal{M}^{\rm pl}_\alpha) = H_{n+2-\chi(\alpha,\alpha)}(\mathcal{M}^{\rm pl}_\alpha)$. Then previous work by me makes $\hat{H}_*(\mathcal{M})$ into a *graded vertex algebra*, and $\check{H}_*(\mathcal{M}^{\rm pl})$ into a *graded Lie algebra*.

(e) There is a notion of *stability condition* τ on $\mathcal{A}$. When $\mathcal{A} = \mathrm{coh}(X)$, this can be Gieseker stability for a polarization on X. For each $\alpha \in K(\mathcal{A})$ we can form moduli spaces $\mathcal{M}^{\rm st}_\alpha(\tau) \subseteq \mathcal{M}^{\rm ss}_\alpha(\tau)$ of τ-(semi)stable objects in class α. Here $\mathcal{M}^{\rm st}_\alpha(\tau)$ is a substack of $\mathcal{M}^{\rm pl}_\alpha$, and is a $\mathbb{C}$-scheme with perfect obstruction theory. Also $\mathcal{M}^{\rm ss}_\alpha(\tau)$ is proper. Thus, if $\mathcal{M}^{\rm st}_\alpha(\tau) = \mathcal{M}^{\rm ss}_\alpha(\tau)$ we have a virtual class $[\mathcal{M}^{\rm ss}_\alpha(\tau)]_{\rm virt}$, which we regard as an element of $H_*(\mathcal{M}^{\rm pl}_\alpha)$. The virtual dimension is $\mathrm{vdim}_{\mathbb{R}}[\mathcal{M}^{\rm ss}_\alpha(\tau)]_{\rm virt} = 2 - \chi(\alpha,\alpha)$, so $[\mathcal{M}^{\rm ss}_\alpha(\tau)]_{\rm virt}$ lies in $\check{H}_0(\mathcal{M}^{\rm pl}_\alpha) \subset \check{H}_0(\mathcal{M}^{\rm pl})$, which is a Lie algebra by (d).

(f) For many theories, there is a problem defining the invariants $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ when $\mathcal{M}_\alpha^{\rm st}(\tau)\neq\mathcal{M}_\alpha^{\rm ss}(\tau)$, i.e. when the moduli spaces $\mathcal{M}_\alpha^{\rm ss}(\tau)$ contain *strictly τ-semistable points*.
I give a systematic way to define $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ in homology over $\mathbb{Q}$ (not $\mathbb{Z}$) in these cases, using auxiliary pair invariants. (This method is well known, e.g. in Joyce–Song D–T theory.)
I prove the $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ are independent of the choices used in the pair invariant method.

(g) If $\tau,\tilde\tau$ are stability conditions and $\alpha\in K(\mathcal{A})$, I prove a wall crossing formula

$$[\mathcal{M}_\alpha^{\rm ss}(\tilde\tau)]_{\rm virt}=\sum_{\alpha_1+\cdots+\alpha_n=\alpha}\tilde U(\alpha_1,\ldots,\alpha_n;\tau,\tilde\tau)\cdot\bigl[\bigl[\ldots\bigl[[\mathcal{M}_{\alpha_1}^{\rm ss}(\tau)]_{\rm virt},[\mathcal{M}_{\alpha_2}^{\rm ss}(\tau)]_{\rm virt}\bigr],\ldots\bigr],[\mathcal{M}_{\alpha_n}^{\rm ss}(\tau)]_{\rm virt}\bigr],\quad(1.1)$$

where $\tilde U(-)$ are combinatorial coefficients defined in my previous work on wall-crossing formulae for motivic invariants, and $[\,,\,]$ is the Lie bracket on $\check H_0(\mathcal{M}^{\rm pl})$ from (d).

(h) In some theories the natural obstruction theory on $\mathcal{M}_\alpha^{\rm st}(\tau)=\mathcal{M}_\alpha^{\rm ss}(\tau)$ has a trivial summand $\mathbb{C}^{o_\alpha}$ in its obstruction sheaf for $o_\alpha>0$, and so the virtual class $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ is zero. In these cases one defines a *reduced* obstruction theory on $\mathcal{M}_\alpha^{\rm st}(\tau)$ by deleting the $\mathbb{C}^{o_\alpha}$ factor, and obtains *reduced* virtual classes $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm red}$. For example, this holds for coherent sheaves on surfaces X with geometric genus $p_g>0$, with $o_\alpha=p_g$ when $\mathrm{rank}\,\alpha>0$.
My theory extends to 'reduced' invariants, allowing o_α to depend on $\alpha\in K(\mathcal{A})$ with $o_\alpha+o_\beta\geqslant o_{\alpha+\beta}$, giving invariants $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm red}$ in $\check H_{2o_\alpha}(\mathcal{M}_\alpha^{\rm pl})$. Generalizing (1.1), they satisfy the wall crossing formula

$$[\mathcal{M}_\alpha^{\rm ss}(\tilde\tau)]_{\rm red}=\sum_{\substack{\alpha_1+\cdots+\alpha_n=\alpha:\\ o_{\alpha_1}+\cdots+o_{\alpha_n}=o_\alpha}}\tilde U(\alpha_1,\ldots,\alpha_n;\tau,\tilde\tau)\cdot\bigl[\bigl[\ldots\bigl[[\mathcal{M}_{\alpha_1}^{\rm ss}(\tau)]_{\rm red},[\mathcal{M}_{\alpha_2}^{\rm ss}(\tau)]_{\rm red}\bigr],\ldots\bigr],[\mathcal{M}_{\alpha_n}^{\rm ss}(\tau)]_{\rm red}\bigr]. \quad (1.2)$$

If $o_\alpha=o>0$ for all α this reduces to $[\mathcal{M}_\alpha^{\rm ss}(\tilde\tau)]_{\rm red}=[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm red}$, that is, the invariants are independent of stability condition.

2. Vertex and Lie algebras on homology of moduli stacks

Let $\mathcal{A}$ be a $\mathbb{C}$-linear abelian or triangulated category from Algebraic Geometry or Representation Theory, e.g. $\mathcal{A} = \mathrm{coh}(X)$ or $D^b \mathrm{coh}(X)$ for X a smooth projective $\mathbb{C}$-scheme, or $\mathcal{A} = \mathrm{mod}\text{-}\mathbb{C}Q$ or $D^b \mathrm{mod}\text{-}\mathbb{C}Q$. Write $\mathcal{M}$ for the moduli stack of objects in $\mathcal{A}$, which is an Artin $\mathbb{C}$-stack in the abelian case, and a higher $\mathbb{C}$-stack in the triangulated case. There is a morphism $\Phi : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ acting by $([E],[F]) \to [E \oplus F]$ on $\mathbb{C}$-points.

Now $\mathbb{G}_m$ acts on objects E in $\mathcal{A}$ with $\lambda \in \mathbb{G}_m$ acting as $\lambda\,\mathrm{id}_E : E \to E$. This induces an action $\Psi : [*/\mathbb{G}_m] \times \mathcal{M} \to \mathcal{M}$ of the group stack $[*/\mathbb{G}_m]$ on $\mathcal{M}$. We write $\mathcal{M}^{\mathrm{pl}} = \mathcal{M}/[*/\mathbb{G}_m]$ for the quotient, called the 'projective linear' moduli stack. There is a morphism $\mathcal{M} \to \mathcal{M}^{\mathrm{pl}}$ which is a $[*/\mathbb{G}_m]$-fibration on $\mathcal{M} \setminus \{[0]\}$.

As I explained at previous Simons conferences, I can give $H_*(\mathcal{M})$ the structure of a *graded vertex algebra*, and use this to give $H_*(\mathcal{M}^{\mathrm{pl}})$ the structure of a *graded Lie algebra*. To save time, today I will only explain the Lie algebra side.

Writing the vertex and Lie algebras explicitly

In good cases we can write down $\hat{H}_*(\mathcal{M})$ and $\check{H}_*(\mathcal{M}^{\rm pl})$ with their algebraic structures completely explicitly. This will be important for our enumerative invariant programme, in which we write invariants $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm inv}$ as elements of $\check{H}_*(\mathcal{M}^{\rm pl})$. It is helpful to take $\mathcal{M},\mathcal{M}^{\rm pl}$ to be (higher) moduli stacks of objects in $D^b\,{\rm coh}(X)$, not ${\rm coh}(X)$.

Theorem 2.1 (Simons PhD student Jacob Gross arXiv:1907.03269)

Let X be a smooth projective $\mathbb{C}$-scheme which is a curve, surface, toric variety, or a few other cases. Write $\mathcal{M}$ for the moduli stack of objects in $D^b\,{\rm coh}(X)$ and $K^0_{\rm sst}(X)$ for the **semi-topological K-theory** *of X (equal to ${\rm Image}(K^0({\rm coh}(X))\to K^0_{\rm top}(X))$ for X a surface). Then $\mathcal{M}=\coprod_{\alpha\in K^0_{\rm sst}(X)}\mathcal{M}_\alpha$ with $\mathcal{M}_\alpha$ connected, and*

$$\begin{aligned} H_*(\mathcal{M}_\alpha,\mathbb{Q})\cong{}&{\rm Sym}^*\bigl(H^{\rm even}(X,\mathbb{Q})\otimes_{\mathbb{Q}}t^2\mathbb{Q}[t^2]\bigr)\otimes_{\mathbb{Q}}\\ &{\textstyle\bigwedge^*}\bigl(H^{\rm odd}(X,\mathbb{Q})\otimes_{\mathbb{Q}}t\mathbb{Q}[t^2]\bigr). \end{aligned} \tag{2.1}$$

A similar equation holds for cohomology $H^*(\mathcal{M}_\alpha,\mathbb{Q})$.

Definition

Let $X,\mathcal{M},\mathcal{M}_\alpha$ be as in Theorem 2.1, and write $\mathcal{U}_\alpha^\bullet\to X\times\mathcal{M}_\alpha$ for the universal complex. Write $m=\dim_{\mathbb{C}}X$ and $b^k=b^k(X)$, and choose bases $(e_j^k)_{j=1}^{b^k}$ for $H_k(X,\mathbb{Q})$ with $e_1^0=1$ and $e_1^{2m}=[X]$. For $l>k/2$ define $S_{jkl}\in H^{2l-k}(\mathcal{M}_\alpha)$ by $S_{jkl}=\mathrm{ch}_l(\mathcal{U}_\alpha^\bullet)\backslash e_j^k$. Regard S_{jkl} as of degree $2l-k$, and as an even (or odd) variable if k is even (or odd). Then Theorem 2.1 implies that $H^*(\mathcal{M}_\alpha)$ is the graded polynomial superalgebra

$$H^*(\mathcal{M}_\alpha)\cong\mathbb{Q}[S_{jkl}:0\leqslant k\leqslant 2m,\ 1\leqslant j\leqslant b^k,\ l>k/2].\tag{2.2}$$

We also give a dual description of homology $H_*(\mathcal{M}_\alpha)$ by

$$H_*(\mathcal{M}_\alpha)\cong e^\alpha\otimes\mathbb{Q}[s_{jkl}:0\leqslant k\leqslant 2m,\ 1\leqslant j\leqslant b^k,\ l>k/2],\tag{2.3}$$

where e^α is a formal symbol to remember α, and

$$\Bigl(\prod_{j,k,l}S_{jkl}^{m_{jkl}}\Bigr)\cdot\Bigl(e^\alpha\prod_{j,k,l}s_{jkl}^{m'_{jkl}}\Bigr)=\begin{cases}\pm\prod_{j,k,l}m_{jkl}!, & m_{jkl}=m'_{jkl}\text{ all }j,k,l,\\ 0, & \text{otherwise.}\end{cases}$$

This pairing has the property that if $\Phi:\mathcal{M}\times\mathcal{M}\to\mathcal{M}$ maps $([E^\bullet],[F^\bullet])\mapsto[E^\bullet\oplus F^\bullet]$ then

$$H_*(\Phi)\bigl(e^\alpha P(s_{jkl})\boxtimes e^\beta Q(s_{jkl})\bigr)=e^{\alpha+\beta}P(s_{jkl})Q(s_{jkl})$$

for polynomials P,Q. Also $-\cap S_{jkl}$ acts as $\frac{\partial}{\partial s_{jkl}}$.

It will be convenient to restrict to sheaves of *positive rank*. Write $\mathcal{M}_{\mathrm{rk}>0}=\coprod_{\alpha\in K^0_{\mathrm{sst}}(X):\mathrm{rk}\,\alpha>0}\mathcal{M}_\alpha$, and similarly for $\mathcal{M}^{\mathrm{pl}}_{\mathrm{rk}>0}$. Then $\Pi_{\mathrm{rk}>0}:\mathcal{M}_{\mathrm{rk}>0}\to\mathcal{M}^{\mathrm{pl}}_{\mathrm{rk}>0}$ induces a surjective morphism $H_*(\mathcal{M}_{\mathrm{rk}>0})\to H_*(\mathcal{M}^{\mathrm{pl}}_{\mathrm{rk}>0})$. It turns out this induces an isomorphism from $\mathrm{Ker}(-\cap S_{101})$ to $H_*(\mathcal{M}^{\mathrm{pl}}_{\mathrm{rk}>0})$, where $\mathrm{Ker}(-\cap S_{101})$ is functions independent of s_{101}. Thus we identify

$$H_*(\mathcal{M}^{\mathrm{pl}}_{\mathrm{rk}>0})\cong\bigoplus_{\alpha\in K^0_{\mathrm{sst}}(X):\mathrm{rk}\,\alpha>0}\begin{aligned}[t]&e^\alpha\otimes\mathbb{Q}[s_{jkl}:0\leqslant k\leqslant 2m,\ 1\leqslant j\leqslant b^k,\\&l>k/2,\ (j,k,l)\neq(1,0,1)].\end{aligned}\tag{2.4}$$

In the representation (2.4), with $(\mathrm{N}_{jk}^{j'k'})$ the matrix of the symmetrized Mukai pairing, we may write the Lie bracket on $\check{H}_*(\mathcal{M}^{\mathrm{pl}}_{\mathrm{rk}>0})$ as

$$\begin{aligned}
&\bigl[e^{\alpha}u(s_{jkl}),e^{\beta}v(s'_{j'k'l'})\bigr]_{\mathrm{rk}>0}=\mathrm{Res}_z\Bigl[(-1)^{\chi(\alpha,\beta)}z^{\chi(\alpha,\beta)+\chi(\beta,\alpha)}e^{\alpha+\beta}\cdot\\
&\Bigl\{\exp\Bigl(z\frac{\mathrm{rk}\,\beta}{\mathrm{rk}(\alpha+\beta)}\Bigl(\sum_{j,k}\alpha_{jk}s_{jk(1+k/2)}+\sum_{j,k,l}s_{jk(l+1)}\frac{\partial}{\partial s_{jkl}}\Bigr)\Bigr)\circ\\
&\exp\Bigl(-z\frac{\mathrm{rk}\,\alpha}{\mathrm{rk}(\alpha+\beta)}\Bigl(\sum_{j',k'}\beta_{j'k'}s'_{j'k'(1+k'/2)}+\sum_{j',k',l'}s'_{j'k'(l'+1)}\frac{\partial}{\partial s'_{j'k'l'}}\Bigr)\Bigr)\circ\\
&\exp\Bigl(-\sum_{j,k,j',k',\;l>k/2}(-1)^{l}(l-k/2-1)!\,z^{k/2-l}\mathrm{N}_{jk}^{j'k'}\beta_{j'k'}\frac{\partial}{\partial s_{jkl}}\\
&-\sum_{j,k,j',k',\;l'>k'/2}(-1)^{k/2}(l'-k'/2-1)!\,z^{k'/2-l'}\mathrm{N}_{jk}^{j'k'}\alpha_{jk}\frac{\partial}{\partial s'_{j'k'l'}}\\
&-\sum_{\substack{j,k,j',k',\\ l>k/2,\;l'>k'/2}}(-1)^{l}(l+l'-(k+k')/2-1)!\,z^{(k+k')/2-l-l'}\cdot\mathrm{N}_{jk}^{j'k'}\frac{\partial^2}{\partial s_{jkl}\partial s'_{j'k'l'}}\Bigr)\\
&\bigl(u(s_{jkl})\cdot v(s'_{j'k'l'})\bigr)\Bigr\}\Bigr|_{s'_{jkl}=s_{jkl}}\Bigr].
\end{aligned}\tag{2.5}$$

3. Counting coherent sheaves on surfaces

Now restrict to X a complex projective surface, with geometric genus p_g, and to classes $\alpha\in K^0_{\rm sst}(X)$ with $\mathrm{rk}\,\alpha>0$. Let $(\tau,T,\leqslant)$ be either Gieseker or μ-stability on $\mathrm{coh}(X)$ with respect to a real Kähler class $\omega\in$ Käh(X). Then my theory defines invariants

$$[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm inv}\in H_{2+2p_g-2\chi(\alpha,\alpha)}(\mathcal{M}_\alpha^{\rm pl},\mathbb{Q})\cong e^\alpha\mathbb{Q}[s_{jkl},\ (j,k,l)\neq(1,0,1)],$$

which are reduced if $p_g>0$, and are virtual classes $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ if $\mathcal{M}_\alpha^{\rm st}(\tau)=\mathcal{M}_\alpha^{\rm ss}(\tau)$. We may write $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm inv}=e^\alpha P_\alpha(s_{jkl})$, for $P_\alpha(s_{jkl})$ a $\mathbb{Q}$-polynomial in the infinitely many graded variables s_{jkl}, homogeneous of degree $2+2p_g-2\chi(\alpha,\alpha)$. When $p_g=0$ these satisfy the wall-crossing formula (1.1) under change of stability condition, using the Lie bracket (2.5). When $p_g>0$ they are independent of stability condition. Our mission, should we choose to accept it, is to compute the polynomials $P_\alpha(s_{jkl})$ (or better, generating functions encoding the $P_\alpha(s_{jkl})$) as explicitly as possible.

Relation to other invariants in the literature

There is a big literature on computing invariants of $\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$. Essentially all of these are integrals $\int_{[\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)]_{\mathrm{virt}}}\mu$ of particular universal cohomology classes $\mu\in H^*(\mathcal{M}_\alpha)$ over the virtual class $[\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)]_{\mathrm{virt}}$. It is usually easy to write μ explicitly as a polynomial $Q(S_{jkl})$ in the generating variables S_{jkl} in $H^*(\mathcal{M}_\alpha)$. Then

$$\int_{[\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)]_{\mathrm{virt}}}\mu=\Big(Q\big(\tfrac{\partial}{\partial s_{jkl}}\big)\cdot P_\alpha(s_{jkl})\Big)\Big|_{s_{jkl}=0}\in\mathbb{Q}.$$

Thus, if we can compute the $P_\alpha(s_{jkl})$, we know all the other invariants as well. This applies to virtual Euler characteristics, virtual χ_y-genera, Donaldson invariants, K-theoretic Donaldson invariants, Segre integrals, and Verlinde integrals.

Example

Donaldson invariants are defined when $\mathrm{rk}\,\alpha=2$ as integrals $\int_{[\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)]_{\mathrm{virt}}}Q(S_{102},S_{j22}:j=1,\ldots,b^2)$ of polynomials Q in $S_{102}\in H^4(\mathcal{M}_\alpha)$ and $S_{j22}\in H^2(\mathcal{M}_\alpha)$. So they are determined by taking $P_\alpha(s_{jkl})$ and setting $s_{jkl}=0$ if $(j,k,l)\neq(1,0,2)$ or $(j,2,2)$.

3.1. Hilbert schemes of points

Write $\alpha \in K^0_{\rm sst}(X)$ as $(r,\beta,k)=(\mathrm{rk}\,\alpha, c_1(\alpha),\mathrm{ch}_2(\alpha))$. The Hilbert scheme $\mathrm{Hilb}^n(X)$ has a fundamental class $[\mathrm{Hilb}^n(X)]_{\rm fund}$ which we regard as lying in $H_{4n}(\mathcal{M}_{(1,0,-n)})=e^{(1,0,-n)}\mathbb{Q}[s_{jkl}]$. Define a generating function

$$\mathrm{Hilb}(X,q)=\sum_{n\geqslant 0}\frac{[\mathrm{Hilb}^n(X)]_{\rm fund}}{e^{(1,0,-n)}}q^n\in\mathbb{Q}[s_{jkl}][[q]].$$

By weaponizing Ellingsrud–Göttsche–Lehn I show that

$$\mathrm{Hilb}(X,q)=1+q(\cdots), \tag{3.1}$$

$$\begin{aligned}&\frac{\partial}{\partial q}\mathrm{Hilb}(X,q)=\\ &\int_X \mathrm{Res}_z\Bigl\{z^{-1}\exp\Bigl[-\sum_{\substack{j,k,j',k',\\ l'>k'/2:\ l'\geqslant(k+k')/2}}\frac{z^{(k+k')/2-l'}}{(l'-(k+k')/2)!}\mu_{jk}^{j'k'}\epsilon_{jk}\boxtimes s_{j'k'l'}\Bigr]\\ &\circ\exp\Bigl[-z^2\,\epsilon_{14}\boxtimes q\frac{\partial}{\partial q}+\sum_{j,k,\ l>k/2}(l-1)!z^l\epsilon_{jk}\boxtimes\frac{\partial}{\partial s_{jkl}}\Bigr]\cdot\mathrm{Hilb}(X,q)\Bigr\},\end{aligned} \tag{3.2}$$

where $(\epsilon_{jk})_{j=1}^{b^k}$ is the basis of $H^k(X,\mathbb{Q})$ dual to $(e_{jk})_{j=1}^{b^k}$, and $(\mu_{jk}^{j'k'})$ is the inverse Mukai pairing. These determine $\mathrm{Hilb}(X,q)$ uniquely.

Then by solving (3.1)–(3.2) explicitly I prove:

Theorem 3.1

Writing $\boldsymbol{u} = (u_1, u_2, \ldots)$, *there exist formal functions* $A(q, \boldsymbol{u})$, $B(q, \boldsymbol{u})$, $C(q, \boldsymbol{u})$, $D(q, \boldsymbol{u})$ *defined uniquely as the solutions to p.d.e.s, such that for any complex projective surface* X *we have*

$$\begin{aligned}\mathrm{Hilb}(X, q) = \exp\Big[\int_X \Big(& A(qe^{-\dot{r}_0}, \boldsymbol{r}) + c_1(X) \cup B(qe^{-\dot{r}_0}, \boldsymbol{r}) \\ & + c_1(X)^2 \cup C(qe^{-\dot{r}_0}, \boldsymbol{r}) + \mathrm{td}_2(X) \cup D(qe^{-\dot{r}_0}, \boldsymbol{r}) \Big) \Big].\end{aligned} \tag{3.3}$$

Here $\boldsymbol{r} = (r_1, r_2, \ldots)$ *with, in* $H^*(X, \mathbb{Q}) \otimes \mathbb{Q}[s_{jkl}]$

$$\dot{r}_0 = \sum_{j,k,j',k':k>0} \lambda_{jk}^{j'k'} \epsilon_{jk} \boxtimes s_{j'k'2}, \tag{3.4}$$

$$r_l = \frac{1}{l!} \sum_{j,k,j',k'} \lambda_{jk}^{j'k'} \epsilon_{jk} \boxtimes s_{j'k'(l+2)}, \quad l = 1, 2, \ldots, \tag{3.5}$$

where $(\lambda_{jk}^{j'k'})$ *is the inverse matrix of* $(\alpha, \beta) \mapsto \int_X \alpha \cup \beta$ *on* $H^*(X)$. *Later we also use functions of* $\boldsymbol{v} = (v_1, v_2, \ldots)$ *with* $v_l = \frac{1}{l!} s_{14(l+2)}$.

I can compute $A(q,\boldsymbol{u}),\ldots,D(q,\boldsymbol{u})$ up to some order using Mathematica.

If $b^1(X)=0$ then any rank 1 sheaf moduli space $\mathcal{M}^{\rm ss}_{(1,\beta,k)}(\tau)$ is isomorphic to $\mathrm{Hilb}^n(X)$ for some n. If $b^1(X)>0$ then $\mathcal{M}^{\rm ss}_{(1,\beta,k)}(\tau)$ is isomorphic to $\mathrm{Pic}^0(X)\times\mathrm{Hilb}^n(X)$. Thus Theorem 3.1 gives generating functions for all rank 1 invariants $[\mathcal{M}^{\rm ss}_{(1,\beta,k)}(\tau)]_{\rm inv}$.

Note that (3.3) has the general form

$$\sum_k \frac{[\mathcal{M}^{\rm ss}_{(r,\beta,k)}(\tau)]_{\rm inv}}{e^{(r,\beta,k)}}q^{\mathrm{const}-k}=\exp\biggl[\int_X F_r(\beta,c_1(X),\mathrm{td}_2(X),q,\dot r_0,\boldsymbol{r},\boldsymbol{v})\biggr]$$

for some universal function F_r depending on the rank, where $q,\dot r_0,\boldsymbol{r},\boldsymbol{v}$ as are in Theorem 3.1. We will see a similar equation later, though also including a sum over Seiberg–Witten invariants.

3.2. Constructing invariants by induction on rank

There is a method to compute invariants $[\mathcal{M}^{\rm ss}_{(r,\beta,k)}(\tau)]_{\rm inv}$ by induction on the rank $r=1,2,\ldots$ starting from rank 1 data. This is due to Mochizuki 2009 in the algebraic case, and is the analogue of the construction of Donaldson invariants from Seiberg–Witten invariants. Fix a line bundle $L\to X$, and define an auxiliary abelian category $\mathcal{A}$ with objects (V,E,ϕ), where V is a finite-dimensional $\mathbb{C}$-vector space, $E\in\mathrm{coh}(X)$, and $\phi:V\otimes_{\mathbb{C}}L\to E$ is a morphism. Write the class of (E,V,ϕ) as $[\![E,V,\phi]\!]=((r,\beta,k),d)$ where $[\![E]\!]=(r,\beta,k)$ and $\dim_{\mathbb{C}}V=d$. Starting from τ on $\mathrm{coh}(X)$ we define a 1-parameter family of stability conditions $\acute\tau_t$ on $\mathcal{A}$ for $t\in[0,\infty)$. Thus we get semistable moduli stacks $\mathcal{M}^{\rm ss}_{((r,\beta,k),d)}(\acute\tau_t)$ of objects in $\mathcal{A}$. My theory defines 'pair invariants' $[\mathcal{M}^{\rm ss}_{((r,\beta,k),d)}(\acute\tau_t)]_{\rm inv}$ (at least when $r>0$ and $d=0,1$) satisfying a wall-crossing formula under change of stability condition $\acute\tau_t$.

It turns out that:

- When $d=0$, $\mathcal{M}^{\rm ss}_{((r,\beta,k),0)}(\acute\tau_t)=\mathcal{M}^{\rm ss}_{(r,\beta,k)}(\tau)$. Thus the sheaf invariants $[\mathcal{M}^{\rm ss}_{(r,\beta,k)}(\tau)]_{\rm inv}$ are pair invariants with $d=0$.
- If $r=1$, $\mathcal{M}^{\rm ss}_{((1,\beta,k),1)}(\acute\tau_t)$ is independent of t and may be written using Seiberg–Witten invariants and Hilbert schemes.
- If $r>1$, $d=1$ and $t\gg 0$ then $\mathcal{M}^{\rm ss}_{((r,\beta,k),1)}(\acute\tau_t)=\emptyset$, so $[\mathcal{M}^{\rm ss}_{((r,\beta,k),d)}(\acute\tau_t)]_{\rm inv}=0$. Thus wall-crossing from $t\gg 0$ to $t=0$ gives a WCF of the general form

$$[\mathcal{M}^{\rm ss}_{((r,\beta,k),1)}(\acute\tau_0)]_{\rm inv}=\text{sum of repeated Lie brackets of}$$
$$[\mathcal{M}^{\rm ss}_{((1,\beta',k'),1)}(\acute\tau_0)]_{\rm inv}\text{ and }[\mathcal{M}^{\rm ss}_{(r'',\beta'',k'')}(\tau)]_{\rm inv}\text{ for }r''<r.$$

- If $L=\mathcal{O}_X(-N)$ for $N\gg 0$ we can recover $[\mathcal{M}^{\rm ss}_{(r,\beta,k)}(\tau)]_{\rm inv}$ from $[\mathcal{M}^{\rm ss}_{((r,\beta,k),1)}(\acute\tau_0)]_{\rm inv}$.
- By induction we may now compute $[\mathcal{M}^{\rm ss}_{(r,\beta,k)}(\tau)]_{\rm inv}\Rightarrow [\mathcal{M}^{\rm ss}_{((r+1,\beta,k),1)}(\acute\tau_0)]_{\rm inv}\Rightarrow[\mathcal{M}^{\rm ss}_{(r+1,\beta,k)}(\tau)]_{\rm inv}\Rightarrow\cdots$.
- Thus, we can compute $[\mathcal{M}^{\rm ss}_{(r,\beta,k)}(\tau)]_{\rm inv}$ for $r>1$ in terms of classes of $\mathrm{Hilb}^n(X)$, $\mathrm{Pic}^0(X)$ and Seiberg–Witten invariants.

I can carry this programme out explicitly, at least up to a certain point. I work with generating functions in $\mathbb{Q}[s_{jkl},\text{other vars}][[q^{\frac{1}{2r}}]]$

$$\sum_k \frac{[\mathcal{M}^{\mathrm{ss}}_{(r,\beta,k)}(\tau)]_{\mathrm{inv}}}{e^{(r,\beta,k)}} q^{\mathrm{const}-k}, \quad \sum_k \frac{[\mathcal{M}^{\mathrm{ss}}_{((r,\beta,k),1)}(\acute{\tau}_t)]_{\mathrm{inv}}}{e^{((r,\beta,k),1)}} q^{\mathrm{const}-k}.$$

I take τ to be μ*-stability* rather than Gieseker stability, as then the combinatorial coefficients in the WCF are independent of $k = \mathrm{ch}_2(\alpha)$, so I can do the WCF for entire generating functions at once.

The difficulty in pushing the calculation through for higher ranks – and getting a comprehensible answer – is that the Lie bracket in the WCF (similar to (2.5) but with extra terms) involves a residue and some horribly complicated exponentiated differential operators.

Going from rank r to rank $r+1$ involves three steps:

(i) Apply differential operator in z, involving $L = \mathcal{O}_X(-N)$.
(ii) Take residue in z.
(iii) Take limit $N \to \infty$ in $\mathcal{O}_X(-N)$ (lower bound for N depends on k) and recover $r+1$ sheaf invariant from $r+1$ pair invariant.

So apparently, we would expect that the generating function for rank r invariants $[\mathcal{M}^{\rm ss}_{(r,\beta,k)}(\tau)]_{\rm inv}$ will involve $r-1$ residues and $r-1$ limits, giving a complicated and unattractive answer.

The next part is still work in progress. I believe there is a way to make the limit 'cancel out' with the residue in the inductive step from rank r to rank $r+1$, so that for each rank $r\geqslant 1$ we have a generating function of the same general form, without residues.

An important idea in the proof is that in the residue in z, we change variables from z to another variable y with

$$z(x,y)=\bigl(-\tfrac{r}{r+1}\bigr)^{-\frac{1}{2}}q^{\frac{1}{2(r+1)}}y^{\frac{r}{r+1}}\bigl(1-y^{-2}\bigr)^{-\frac{1}{2}},$$

such that invariants being independent of $N\gg 0$ for $L=\mathcal{O}_X(-N)$ imply that parts of the expression are not Laurent series in y but *Laurent polynomials*, and then taking the residue in y sets $y=1$. This gives a cool way for *algebraic numbers* to appear in the generating function. Parts of the expression must be Laurent series of algebraic functions of y, as polynomials in the power series are Laurent polynomials. Setting $y=1$ gives an algebraic number.

3.3. What I hope to prove

I expect that when $p_g > 0$, for $r \geqslant 1$ there should be a formula like

$$\Omega_{-\beta/r}\left(\frac{[\mathcal{M}^{\mathrm{ss}}_{(r,\beta,k)}(\tau)]_{\mathrm{fd}}}{e^{(r,\beta,k)}}\right) = \left[q^{\frac{1}{2r}\,\mathrm{vdim}\,\mathcal{M}^{\mathrm{ss}}_{(r,\beta,k)}(\tau)_{\mathrm{fd}}}\right] \tag{3.6}$$

$$\left(\sum_{\substack{\gamma_1,\ldots,\gamma_{r-1}\\ \in H^2(X,\mathbb{Z})^{1,1}:\\ \mathfrak{s}_{\gamma_a}\in\mathrm{SW}(X),\\ a=1,\ldots,r-1}} \begin{array}{l} r^{2-\int_X \mathrm{td}_2(X)} \displaystyle\prod_{a=1}^{r-1} \mathrm{SW}(\mathfrak{s}_{\gamma_a}) \cdot e^{\frac{2\pi i}{r}\sum_{a=1}^{r-1} a\int_X \beta\cup\gamma_a} \cdot \\ \exp\left[\displaystyle\int_X A_r(\gamma_1,\ldots,\gamma_{r-1},c_1(X),\mathrm{td}_2(X),q^{\frac{1}{2r}},\dot{r}_0,\boldsymbol{r},\boldsymbol{v})\right] \end{array}\right).$$

Here $[\mathcal{M}^{\mathrm{ss}}_{(r,\beta,k)}(\tau)]_{\mathrm{fd}}$ is the 'fixed determinant' invariant, equal to $[\mathcal{M}^{\mathrm{ss}}_{(r,\beta,k)}(\tau)]_{\mathrm{inv}}$ when $b^1(X)=0$, and $\Omega_{-\beta/r}$ is an explicit change of variables in $\mathbb{Q}[s_{jkl}]$ which mimics $E\mapsto E\otimes L$ for L a 'line bundle' with $c_1(L)=-\beta/r$ (though $-\beta/r$ need not be integral), and A_r is a universal function, and $\mathrm{SW}(\mathfrak{s}_{\gamma_a})\in\mathbb{Q}$ are Seiberg–Witten invariants. Note that most of (3.6) is independent of $\beta\in H^2(X,\mathbb{Z})^{1,1}$.

I can also say a lot about the function A_r, especially when $\gamma_i = 0$ or $q = 0$. The general shape of (3.6) is related to many conjectures and theorems by Lothar Göttsche, Martijn Kool, and other authors. For simplicity take $b^1(X) = 0$, so we have variables s_{jkl} for $k = 0, 2, 4$ only. Now $A_r(\cdots)$ involves r_l in (3.5) which is a sum of $\epsilon_{jk} \boxtimes s_{j'k'(l+2)}$ with $k + k' = 4$. The operation $\int_X$ in $\int_X A_r(\cdots)$ selects products of ϵ_{jk} in which the degrees k sum to 4. Because of this, the $\int_X A_r(\gamma_1, \ldots, \gamma_{r-1}, c_1(X), \mathrm{td}_2(X), q^{\frac{1}{2r}}, \dot{r}_0, \boldsymbol{r}, \boldsymbol{v})$ in (3.6) involves terms which are:

- At most linear in s_{10l} for $l \geqslant 1$.
- At most quadratic in s_{j2l} for $j = 1, \ldots, b^2$ and $l \geqslant 2$.
- Arbitrary power series in s_{14l} for $l \geqslant 3$

The way many formulae in the literature get nice generating functions is to (sometimes) first twist by a Hirzebruch genus of $\mathcal{M}^{\mathrm{ss}}_{(r,\beta,k)}(\tau)$, and then set $s_{14l} = 0$ for $l \geqslant 3$, and exploit the fact that $\int_X A_r(\cdots)$ has simple dependence on s_{10l}, s_{j2l}.