Derived Differential Geometry and moduli spaces in differential geometry

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based on arXiv:1208.4948, arXiv:1206.4207, arXiv:1409.6908, arXiv:1510.07444, and work in progress.
Also see website
`people.maths.ox.ac.uk/~joyce/dmanifolds.html.`
These slides available at
`people.maths.ox.ac.uk/~joyce/talks.html.`

1. Derived Differential Geometry

Derived Differential Geometry (DDG) is the study of *derived smooth manifolds* and *derived smooth orbifolds*, where 'derived' is in the sense of the Derived Algebraic Geometry (DAG) of Jacob Lurie and Toën–Vezzosi. Derived manifolds include ordinary smooth manifolds, but also many singular objects.

Derived manifolds and orbifolds form higher categories – 2-categories **dMan**, **dOrb** or **mKur**, **Kur** in my set-up, and ∞-categories in the set-ups of Spivak–Borisov–Noel.

Many interesting moduli spaces over $\mathbb{R}$ or $\mathbb{C}$ in both algebraic and differential geometry are naturally derived manifolds or derived orbifolds, including those used to define Donaldson, Donaldson–Thomas, Gromov–Witten and Seiberg–Witten invariants, Floer theories, and Fukaya categories.

A compact, oriented derived manifold or orbifold $\boldsymbol{X}$ has a *virtual class* in homology (or a *virtual chain* if $\partial\boldsymbol{X} \neq \emptyset$), which can be used to define these enumerative invariants, Floer theories, ...

Different definitions of derived manifolds and orbifolds

There are several versions of 'derived manifolds' and 'derived orbifolds' in the literature, in order of increasing simplicity:

- Spivak's ∞-category $\mathbf{DerMan_{Spi}}$ of derived manifolds (2008).
- Borisov–Noël's ∞-category $\mathbf{DerMan_{BN}}$ (2011,2012).
- My d-manifolds and d-orbifolds (2010–2016), which form strict 2-categories $\mathbf{dMan}, \mathbf{dOrb}$.
- My μ-Kuranishi spaces, m-Kuranishi spaces and Kuranishi spaces (2014), which form a category $\mathbf{mKur}$ and weak 2-categories $\mathbf{mKur}, \mathbf{Kur}$.
  Here μ-, m-Kuranishi spaces are types of derived manifold, and Kuranishi spaces a type of derived orbifold.

In fact the Kuranishi space approach is motivated by earlier work by Fukaya, Oh, Ohta and Ono in symplectic geometry (1999,2009–) whose 'Kuranishi spaces' are really a prototype kind of derived orbifold, from before the invention of DAG.

Derived Differential Geometry
The category of μ-Kuranishi spaces
Making moduli spaces into derived manifolds/orbifolds

Relation between these definitions

- Borisov–Noel (2011) prove an equivalence of ∞-categories $\mathbf{DerMan_{Spi}} \simeq \mathbf{DerMan_{BN}}$.
- Borisov (2012) gives a 2-functor $\pi_2(\mathbf{DerMan_{BN}}) \to \mathbf{dMan}$ which is nearly an equivalence of 2-categories (e.g. it is a 1-1 correspondence on equivalence classes of objects), where $\pi_2(\mathbf{DerMan_{BN}})$ is the 2-category truncation of $\mathbf{DerMan_{BN}}$.
- I prove (2016) equivalences of 2-categories $\mathbf{dMan} \simeq \mathbf{mKur}$, $\mathbf{dOrb} \simeq \mathbf{Kur}$ and of categories $\mathrm{Ho}(\mathbf{dMan}) \simeq \mathrm{Ho}(\mathbf{mKur}) \simeq \boldsymbol{\mu}\mathbf{Kur}$, where $\mathrm{Ho}(\cdots)$ is the homotopy category.

Thus all these notions of derived manifold are more-or-less equivalent. Kuranishi spaces are easiest. There is a philosophical difference between $\mathbf{DerMan_{Spi}}, \mathbf{DerMan_{BN}}$ (locally modelled on $X \times_Z Y$ for smooth maps of manifolds $g : X \to Z$, $h : Y \to Z$) and $\mathbf{dMan}, \boldsymbol{\mu}\mathbf{Kur}, \mathbf{mKur}$ (locally modelled on $s^{-1}(0)$ for E a vector bundle over a manifold V with $s : V \to E$ a smooth section).

Two ways to define ordinary manifolds

Definition 1.1

A *manifold* of dimension n is a Hausdorff, second countable topological space X with a sheaf $\mathcal{O}_X$ of $\mathbb{R}$-algebras (or C^∞-rings) locally isomorphic to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$, where $\mathcal{O}_{\mathbb{R}^n}$ is the sheaf of smooth functions $f : \mathbb{R}^n \to \mathbb{R}$.

Definition 1.2

A *manifold* of dimension n is a Hausdorff, second countable topological space X equipped with an atlas of charts $\{(V_i, \psi_i) : i \in I\}$, where $V_i \subseteq \mathbb{R}^n$ is open, and $\psi_i : V_i \to X$ is a homeomorphism with an open subset $\mathrm{Im}\,\psi_i$ of X for all $i \in I$, and $\psi_j^{-1} \circ \psi_i : \psi_i^{-1}(\mathrm{Im}\,\psi_j) \to \psi_j^{-1}(\mathrm{Im}\,\psi_i)$ is a diffeomorphism of open subsets of $\mathbb{R}^n$ for all $i, j \in I$.

If you define derived manifolds by generalizing Definition 1.1, you get something like d-manifolds; if you generalize Definition 1.2, you get something like (μ- or m-)Kuranishi spaces.

2. The definition of μ-Kuranishi spaces

I define 2-categories **mKur** of *m-Kuranishi spaces*, the ‘manifold’ version, and **Kur** of *Kuranishi spaces*, the ‘orbifold’ version, with **mKur** $\subset$ **Kur**. I also define a simplified ordinary category $\boldsymbol{\mu}$**Kur** of ‘μ-Kuranishi spaces’, with $\boldsymbol{\mu}$**Kur** $\simeq \mathrm{Ho}($**mKur**$)$. For simplicity, today I will only explain μ-Kuranishi spaces.

Definition 2.1

Let X be a topological space. A *μ-Kuranishi neighbourhood* on X is a quadruple (V, E, s, ψ) such that:

(a) V is a smooth manifold.
(b) $\pi : E \to V$ is a vector bundle over V, the *obstruction bundle*.
(c) $s \in C^\infty(E)$ is a smooth section of E, the *Kuranishi section*.
(d) ψ is a homeomorphism from $s^{-1}(0)$ to an open subset $\mathrm{Im}\,\psi$ in X, where $\mathrm{Im}\,\psi$ is called the *footprint* of (V, E, s, ψ).

If $S \subseteq X$ is open, we call (V, E, s, ψ) a *μ-Kuranishi neighbourhood over S* if $S \subseteq \mathrm{Im}\,\psi \subseteq X$.

Definition 2.2

Let X be a topological space, (V_i, E_i, s_i, ψ_i), (V_j, E_j, s_j, ψ_j) be μ-Kuranishi neighbourhoods on X, and $S \subseteq \mathrm{Im}\,\psi_i \cap \mathrm{Im}\,\psi_j \subseteq X$ be an open set. Consider triples $(V_{ij}, \phi_{ij}, \hat\phi_{ij})$ satisfying:

(a) V_{ij} is an open neighbourhood of $\psi_i^{-1}(S)$ in V_i.

(b) $\phi_{ij} : V_{ij} \to V_j$ is smooth, with $\psi_i = \psi_j \circ \phi_{ij}$ on $s_i^{-1}(0) \cap V_{ij}$.

(c) $\hat\phi_{ij} : E_i|_{V_{ij}} \to \phi_{ij}^*(E_j)$ is a morphism of vector bundles on V_{ij}, with $\hat\phi_{ij}(s_i|_{V_{ij}}) = \phi_{ij}^*(s_j) + O(s_i^2)$.

Define an equivalence relation $\sim$ on such triples $(V_{ij}, \phi_{ij}, \hat\phi_{ij})$ by $(V_{ij}, \phi_{ij}, \hat\phi_{ij}) \sim (V'_{ij}, \phi'_{ij}, \hat\phi'_{ij})$ if there are open $\psi_i^{-1}(S) \subseteq \dot V_{ij} \subseteq V_{ij} \cap V'_{ij}$ and a morphism $\Lambda : E_i|_{\dot V_{ij}} \to \phi_{ij}^*(TV_j)|_{\dot V_{ij}}$ of vector bundles on $\dot V_{ij}$ satisfying $\phi'_{ij} = \phi_{ij} + \Lambda \circ s_i + O(s_i^2)$ and $\hat\phi'_{ij} = \hat\phi_{ij} + \phi_{ij}^*(\mathrm{d}s_j) \circ \Lambda + O(s_i)$ on $\dot V_{ij}$. We write $[V_{ij}, \phi_{ij}, \hat\phi_{ij}]$ for the $\sim$-equivalence class of $(V_{ij}, \phi_{ij}, \hat\phi_{ij})$, and call $[V_{ij}, \phi_{ij}, \hat\phi_{ij}] : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ a *morphism of μ-Kuranishi neighbourhoods over S*.

Given morphisms $[V_{ij}, \phi_{ij}, \hat\phi_{ij}] : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$, $[V_{jk}, \phi_{jk}, \hat\phi_{jk}] : (V_j, E_j, s_j, \psi_j) \to (V_k, E_k, s_k, \psi_k)$ of μ-Kuranishi neighbourhoods over $S \subseteq X$, the *composition* is

$$[V_{jk}, \phi_{jk}, \hat\phi_{jk}] \circ [V_{ij}, \phi_{ij}, \hat\phi_{ij}] = \left[\phi_{ij}^{-1}(V_{jk}), \phi_{jk} \circ \phi_{ij}|_{\ldots}, \phi_{ij}^{-1}(\hat\phi_{jk}) \circ \hat\phi_{ij}|_{\ldots}\right] : (V_i, E_i, s_i, \psi_i) \longrightarrow (V_k, E_k, s_k, \psi_k).$$

Then μ-Kuranishi neighbourhoods over $S \subseteq X$ form a category $\mathrm{mKur}_S(X)$. We call $[V_{ij}, \phi_{ij}, \hat\phi_{ij}]$ a *coordinate change over S* if it is an isomorphism in $\mathrm{mKur}_S(X)$. We have:

Theorem 2.3

A morphism $[V_{ij}, \phi_{ij}, \hat\phi_{ij}] : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ is a coordinate change over S if and only if for all $x \in S$ with $v_i = \psi_i^{-1}(x)$ and $v_j = \psi_j^{-1}(x)$, the following sequence is exact:

$$0 \longrightarrow T_{v_i}V_i \xrightarrow{\mathrm{d}s_i|_{v_i} \oplus T_{v_i}\phi_{ij}} E_i|_{v_i} \oplus T_{v_j}V_j \xrightarrow{\hat\phi_{ij}|_{v_i} \oplus -\mathrm{d}s_j|_{v_j}} E_j|_{v_j} \longrightarrow 0.$$

The sheaf property of morphisms

Theorem 2.4

Let (V_i, E_i, s_i, ψ_i), (V_j, E_j, s_j, ψ_j) be μ-Kuranishi neighbourhoods on X. For each open $S \subseteq \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$, write $\mathcal{H}om\big((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j)\big)(S)$ for the set of morphisms $\Phi_{ij} : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ over S, and for all open $T \subseteq S \subseteq \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$ define

$$\rho_{ST} : \mathcal{H}om\big((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j)\big)(S) \longrightarrow \mathcal{H}om\big((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j)\big)(T) \text{ by } \rho_{ST} : \Phi_{ij} \longmapsto \Phi_{ij}|_T.$$

Then $\mathcal{H}om\big((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j)\big)$ is a sheaf of sets on $\operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$. Similarly, coordinate changes from (V_i, E_i, s_i, ψ_i) to (V_j, E_j, s_j, ψ_j) are a subsheaf of $\mathcal{H}om\big((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j)\big)$.

This is not obvious. It means we can glue (iso)morphisms of μ-Kuranishi neighbourhoods over the sets of an open cover. In the 2-category version for **mKur**, **Kur**, we get a stack instead of a sheaf.

Definition 2.5

Let X be a Hausdorff, second countable topological space, and $n \in \mathbb{Z}$. A *μ-Kuranishi structure $\mathcal{K}$ on X of virtual dimension n* is data $\mathcal{K} = \big(I, (V_i, E_i, s_i, \psi_i)_{i \in I}, \Phi_{ij,\ i,j \in I}\big)$, where:

(a) I is an indexing set.

(b) (V_i, E_i, s_i, ψ_i) is a μ-Kuranishi neighbourhood on X for each $i \in I$, with $\dim V_i - \operatorname{rank} E_i = n$.

(c) $\Phi_{ij} = [V_{ij}, \phi_{ij}, \hat\phi_{ij}] : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ is a coordinate change over $S = \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$ for all $i, j \in I$.

(d) $\bigcup_{i \in I} \operatorname{Im}\psi_i = X$.

(e) $\Phi_{ii} = \operatorname{id}_{(V_i, E_i, s_i, \psi_i)}$ for all $i \in I$.

(f) $\Phi_{jk} \circ \Phi_{ij} = \Phi_{ik}$ for all $i, j, k \in I$ over $S = \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j \cap \operatorname{Im}\psi_k$.

We call $\boldsymbol{X} = (X, \mathcal{K})$ a *μ-Kuranishi space*, of *virtual dimension* $\operatorname{vdim}\boldsymbol{X} = n$.

Definition 2.6

Let $f:X\rightarrow Y$ be a continuous map of topological spaces, (V_i,E_i,s_i,ψ_i), (W_j,F_j,t_j,χ_j) be μ-Kuranishi neighbourhoods on X,Y, and $S\subseteq \mathrm{Im}\,\psi_i\cap f^{-1}(\mathrm{Im}\,\chi_j)\subseteq X$ be an open set. Consider triples $(V_{ij},f_{ij},\hat f_{ij})$ satisfying:

(a) V_{ij} is an open neighbourhood of $\psi_i^{-1}(S)$ in V_i.

(b) $f_{ij}:V_{ij}\rightarrow W_j$ is smooth, with $f\circ\psi_i=\chi_j\circ f_{ij}$ on $s_i^{-1}(0)\cap V_{ij}$.

(c) $\hat f_{ij}:E_i\vert_{V_{ij}}\rightarrow f_{ij}^*(F_j)$ is a morphism of vector bundles on V_{ij}, with $\hat f_{ij}(s_i\vert_{V_{ij}})=f_{ij}^*(t_j)+O(s_i^2)$.

Define an equivalence relation $\sim$ by $(V_{ij},f_{ij},\hat f_{ij})\sim(V'_{ij},f'_{ij},\hat f'_{ij})$ if there are open $\psi_i^{-1}(S)\subseteq\dot V_{ij}\subseteq V_{ij}\cap V'_{ij}$ and $\Lambda:E_i\vert_{\dot V_{ij}}\rightarrow f_{ij}^*(TW_j)\vert_{\dot V_{ij}}$ with $f'_{ij}=f_{ij}+\Lambda\cdot s_i+O(s_i^2)$ and $\hat f'_{ij}=\hat f_{ij}+\Lambda\cdot f_{ij}^*(\mathrm{d}t_j)+O(s_i)$. We write $[V_{ij},f_{ij},\hat f_{ij}]$ for the $\sim$-equivalence class of $(V_{ij},f_{ij},\hat f_{ij})$, and call $[V_{ij},f_{ij},\hat f_{ij}]:(V_i,E_i,s_i,\psi_i)\rightarrow(W_j,F_j,t_j,\chi_j)$ a *morphism over* S,f.

When $Y=X$ and $f=\mathrm{id}_X$, this recovers the notion of morphisms of μ-Kuranishi neighbourhoods on X. We have the obvious notion of compositions of morphisms of μ-Kuranishi neighbourhoods over $f:X\rightarrow Y$ and $g:Y\rightarrow Z$.

Here is the generalization of Theorem 2.4:

Theorem 2.7

Let (V_i,E_i,s_i,ψ_i), (W_j,F_j,t_j,χ_j) be μ-Kuranishi neighbourhoods on X,Y, and $f:X\rightarrow Y$ be continuous. Then morphisms from (V_i,E_i,s_i,ψ_i) to (W_j,F_j,t_j,χ_j) over f form a sheaf $\mathcal{H}om_f\big((V_i,E_i,s_i,\psi_i),(W_j,F_j,t_j,\chi_j)\big)$ on $\mathrm{Im}\,\psi_i\cap f^{-1}(\mathrm{Im}\,\chi_j)$.

This will be essential for defining compositions of morphisms of μ-Kuranishi spaces. The lack of such a sheaf property in the FOOO theory is why FOOO Kuranishi spaces are not a category.

Definition 2.8

Let $\boldsymbol{X}=(X,\mathcal{K})$ with $\mathcal{K}=\bigl(I,(V_i,E_i,s_i,\psi_i)_{i\in I},\Phi_{ii',\,i,i'\in I}\bigr)$ and $\boldsymbol{Y}=(Y,\mathcal{L})$ with $\mathcal{L}=\bigl(J,(W_j,F_j,t_j,\chi_j)_{j\in J},\Psi_{jj',\,j,j'\in J}\bigr)$ be μ-Kuranishi spaces. A *morphism* $\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ is $\boldsymbol{f}=\bigl(f,\boldsymbol{f}_{ij,\,i\in I,\,j\in J}\bigr)$, where $f:X\rightarrow Y$ is a continuous map, and $\boldsymbol{f}_{ij}=[V_{ij},f_{ij},\hat f_{ij}]:(V_i,E_i,s_i,\psi_i)\rightarrow(W_j,F_j,t_j,\chi_j)$ is a morphism of μ-Kuranishi neighbourhoods over $S=\mathrm{Im}\,\psi_i\cap f^{-1}(\mathrm{Im}\,\chi_j)$ and f for all $i\in I$, $j\in J$, satisfying the conditions:

(a) If $i,i'\in I$ and $j\in J$ then $\boldsymbol{f}_{i'j}\circ\Phi_{ii'}|_S=\boldsymbol{f}_{ij}|_S$ over $S=\mathrm{Im}\,\psi_i\cap\mathrm{Im}\,\psi_{i'}\cap f^{-1}(\mathrm{Im}\,\chi_j)$ and f.

(b) If $i\in I$ and $j,j'\in J$ then $\Psi_{jj'}\circ\boldsymbol{f}_{ij}|_S=\boldsymbol{f}_{ij'}|_S$ over $S=\mathrm{Im}\,\psi_i\cap f^{-1}(\mathrm{Im}\,\chi_j\cap\mathrm{Im}\,\chi_{j'})$ and f.

When $\boldsymbol{Y}=\boldsymbol{X}$, so that $J=I$, define the identity morphism $\mathbf{id}_{\boldsymbol{X}}:\boldsymbol{X}\rightarrow\boldsymbol{X}$ by $\mathbf{id}_{\boldsymbol{X}}=\bigl(\mathrm{id}_X,\Phi_{ij,\,i,j\in I}\bigr)$.

Composition of morphisms in $\boldsymbol{\mu}$**Kur**

Let $\boldsymbol{X}=(X,\mathcal{I})$ with $\mathcal{I}=\bigl(I,(U_i,D_i,r_i,\phi_i)_{i\in I},\Phi_{ii',\,i,i'\in I}\bigr)$ and $\boldsymbol{Y}=(Y,\mathcal{J})$ with $\mathcal{J}=\bigl(J,(V_j,E_j,s_j,\psi_j)_{j\in J},\Psi_{jj',\,j,j'\in J}\bigr)$ and $\boldsymbol{Z}=(Z,\mathcal{K})$ with $\mathcal{K}=\bigl(K,(W_k,F_k,t_k,\xi_k)_{k\in K},\Xi_{kk',\,k,k'\in K}\bigr)$ be μ-Kuranishi spaces, and $\boldsymbol{f}=(f,\boldsymbol{f}_{ij}):\boldsymbol{X}\rightarrow\boldsymbol{Y}$, $\boldsymbol{g}=(g,\boldsymbol{g}_{jk}):\boldsymbol{Y}\rightarrow\boldsymbol{Z}$ be morphisms. Consider the problem of how to define the composition $\boldsymbol{g}\circ\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$.

For all $i\in I$ and $k\in K$, $\boldsymbol{g}\circ\boldsymbol{f}$ must contain a morphism $(\boldsymbol{g}\circ\boldsymbol{f})_{ik}:(U_i,D_i,r_i,\phi_i)\rightarrow(W_k,F_k,t_k,\xi_k)$ defined over $S_{ik}=\mathrm{Im}\,\phi_i\cap(g\circ f)^{-1}(\mathrm{Im}\,\xi_k)$ and $g\circ f$.

For each $j\in J$, we have a morphism $\boldsymbol{g}_{jk}\circ\boldsymbol{f}_{ij}:(U_i,D_i,r_i,\phi_i)\rightarrow(W_k,F_k,t_k,\xi_k)$, but it is defined over $S_{ijk}=\mathrm{Im}\,\phi_i\cap f^{-1}(\mathrm{Im}\,\psi_j)\cap(g\circ f)^{-1}(\mathrm{Im}\,\xi_k)$ and $g\circ f$, not over the whole of $S_{ik}=\mathrm{Im}\,\phi_i\cap(g\circ f)^{-1}(\mathrm{Im}\,\xi_k)$.

Composition of morphisms in $\boldsymbol{\mu}$**Kur**

The solution is to use the sheaf property of morphisms, Theorem 2.7. The sets S_{ijk} for $j \in J$ form an open cover of S_{ik}. Using Definition 2.8(a),(b) we can show that $\boldsymbol{g}_{jk} \circ \boldsymbol{f}_{ij}|_{S_{ijk} \cap S_{ij'k}} = \boldsymbol{g}_{j'k} \circ \boldsymbol{f}_{ij'}|_{S_{ijk} \cap S_{ij'k}}$. Therefore by Theorem 2.7 there is a unique morphism of μ-Kuranishi neighbourhoods $(\boldsymbol{g} \circ \boldsymbol{f})_{ik} : (U_i, D_i, r_i, \phi_i) \to (W_k, F_k, t_k, \xi_k)$ defined over S_{ik} and $g \circ f$ with $(\boldsymbol{g} \circ \boldsymbol{f})_{ik}|_{S_{ijk}} = \boldsymbol{g}_{jk} \circ \boldsymbol{f}_{ij}$ for all $j \in J$. We show that $\boldsymbol{g} \circ \boldsymbol{f} := \big(g \circ f, (\boldsymbol{g} \circ \boldsymbol{f})_{ik,\ i\in I,\ k \in K}\big)$ is a morphism $\boldsymbol{g} \circ \boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Z}$ of μ-Kuranishi spaces, which we call *composition*.
Composition is associative, and makes μ-Kuranishi spaces into a category $\boldsymbol{\mu}$**Kur**.

Comparison with m-Kuranishi spaces and Kuranishi spaces

To define **mKur** and **Kur** instead of $\boldsymbol{\mu}$**Kur**, we must work with 2-categories throughout. So we define (m-)Kuranishi neighbourhoods (V_i, E_i, s_i, ψ_i) or $(V_i, E_i, \Gamma_i, s_i, \psi_i)$ on X, and 1-morphisms Φ_{ij} between them, and 2-morphisms $\Lambda_{ij} : \Phi_{ij} \Rightarrow \Phi'_{ij}$ between 1-morphisms. A Kuranishi structure $\mathcal{K}$ on X assigns an open cover of X by (m-)Kuranishi neighbourhoods, with coordinate changes Φ_{ij} on double overlaps $\mathrm{Im}\,\psi_i \cap \mathrm{Im}\,\psi_j$, and 2-morphisms $\Lambda_{ijk} : \Phi_{jk} \circ \Phi_{ij} \Rightarrow \Phi_{ik}$ on triple overlaps $\mathrm{Im}\,\psi_i \cap \mathrm{Im}\,\psi_j \cap \mathrm{Im}\,\psi_k$. This leads to an awful lot of notation.

We need 2-categories to do orbifolds/Kuranishi spaces properly (the analogues of Theorems 2.4 and 2.7 are false in the orbifold case if we try to work with ordinary categories). Also, some important constructions such as fibre products of (m-)Kuranishi spaces need the 2-category structure, and don't work in $\boldsymbol{\mu}$**Kur**.

Differential geometry of (m-)Kuranishi spaces

Manifolds and orbifolds include into m-Kuranishi spaces and Kuranishi spaces, in a diagram of 2-categories

$$\begin{array}{ccc} \mathbf{Man} & \xrightarrow{\subset} & \mathbf{mKur} \\ \downarrow \subset & & \subset \downarrow \\ \mathbf{Orb} & \xrightarrow{\subset} & \mathbf{Kur}. \end{array}$$

Much of the differential geometry of ordinary manifolds extends nicely to Kuranishi spaces. There are good notions of dimension, orientation, submersions, immersions, embeddings, transversality and fibre products, gluing by equivalences on open covers. There are also good notions of (m-)Kuranishi space with boundary and corners, forming 2-categories $\mathbf{mKur} \subset \mathbf{mKur^b} \subset \mathbf{mKur^c}$ and $\mathbf{Kur} \subset \mathbf{Kur^b} \subset \mathbf{Kur^c}$. Some results are stronger than the classical case. For example, if $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$ and $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ are 1-morphisms in $\mathbf{Kur}$ with $\boldsymbol{Z}$ a manifold or orbifold then a fibre product $\boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ exists in $\mathbf{Kur}$, without further transversality conditions.

Virtual classes and virtual chains

If X is a compact, oriented manifold (or orbifold) of dimension k, it has a fundamental class $[X] \in H_k(X;\mathbb{Z})$ (or $[X] \in H_k(X;\mathbb{Q})$). In the same way, if $\boldsymbol{X}$ is a compact, oriented derived manifold (or derived orbifold) of virtual dimension k, it has a *virtual class* $[\boldsymbol{X}]_{\rm virt} \in H_k(X;\mathbb{Z})$ (or $[\boldsymbol{X}]_{\rm virt} \in H_k(X;\mathbb{Q})$). Technically we need to use the Steenrod homology $H_k^{\rm St}(X;\mathbb{Z})$ or Čech homology $\check{H}_k(X;\mathbb{Q})$ here, but they agree with ordinary homology if X is a nice topological space (e.g. a Euclidean Neighbourhood Retract). These virtual classes (also *virtual chains*, for derived manifolds / orbifolds with corners) are very important in applications of DDG. They have deformation-invariance properties under bordism of derived manifolds/orbifolds. They are used in Symplectic Geometry to define Gromov–Witten invariants, and could be used to define other enumerative invariants (Donaldson, Seiberg–Witten, Donaldson–Thomas), Floer theories, and Fukaya categories.

3. Making moduli spaces into derived manifolds/orbifolds

Many classes of moduli spaces $\mathcal{M}$ in Differential Geometry, and in Algebraic Geometry over $\mathbb{C}$, are known to have the stucture of derived manifolds or derived orbifolds. For example:

Theorem 3.1

Let $\mathcal{V}$ be a Banach manifold, $\mathcal{E} \to \mathcal{V}$ a Banach vector bundle, and $s : \mathcal{V} \to \mathcal{E}$ a smooth Fredholm section, with constant Fredholm index $n \in \mathbb{Z}$. Then there is a canonical derived manifold $\boldsymbol{X}$ with topological space $X = s^{-1}(0)$ and $\mathrm{vdim}\, \boldsymbol{X} = n$.

Nonlinear elliptic equations, when written as maps between suitable Hölder or Sobolev spaces, are the zeroes of Fredholm sections of a Banach vector bundle over a Banach manifold. Thus we have:

Corollary 3.2

Let $\mathcal{M}$ be a moduli space of solutions of a nonlinear elliptic equation on a compact manifold, with fixed topological invariants. Then $\mathcal{M}$ extends to a derived manifold $\boldsymbol{\mathcal{M}}$.

linearization of the elliptic p.d.e. at x, given by the A–S Index Theorem.

Truncation functors from other structures

Theorem 3.3

Suppose X is a Hausdorff, second countable topological space equipped with any of the following geometric structures, each of constant virtual dimension $n \in \mathbb{Z}$:

(a) *A $\mathbb{C}$-scheme or Deligne–Mumford $\mathbb{C}$-stack with perfect obstruction theory in the sense of Behrend and Fantechi (where X is the underlying complex analytic space).*
(b) *A quasi-smooth derived $\mathbb{C}$-scheme or D–M $\mathbb{C}$-stack.*
(c) *An M-polyfold or polyfold Fredholm structure in the sense of Hofer, Wysocki and Zehnder.*
(d) *A Kuranishi structure in the sense of Fukaya–Oh–Ohta–Ono.*
(e) *A Kuranishi atlas in the sense of McDuff and Wehrheim.*

Then X may be given the structure of a derived manifold/orbifold, natural up to equivalence in **mKur**, **Kur**, *with $\mathrm{vdim}\, \boldsymbol{X} = n$. We can also allow corners in (c)–(e), with $\boldsymbol{X} \in$* $\mathbf{mKur^c}, \mathbf{Kur^c}$.

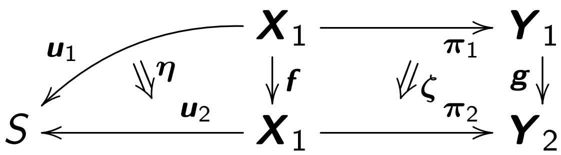
The method of universal families

In future work, I hope to use DDG to develop new proofs of the existence of natural derived manifold/orbifold structures on large classes of differential-geometric moduli spaces. The key notion is that of *universal family* of objects over a derived orbifold $\boldsymbol{Y}$.

We illustrate this for moduli spaces of J-holomorphic curves. Let (S,ω) be a symplectic manifold, and J an almost complex structure on S. Suppose we want to construct the moduli space $\mathcal{M}$ of J-holomorphic maps $u:\Sigma\to S$, where (Σ,j) is a nonsingular genus g Riemann surface, and $[u(\Sigma)]=\beta\in H_2(S;\mathbb{Z})$.

• Define a *family of curves* to be a quintuple $(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{\pi},\boldsymbol{u},j)$, where $\boldsymbol{X},\boldsymbol{Y}$ are derived orbifolds with $\mathrm{vdim}\,\boldsymbol{X}=\mathrm{vdim}\,\boldsymbol{Y}+2$, $\boldsymbol{\pi}:\boldsymbol{X}\to\boldsymbol{Y}$ is a proper, representable submersion in **Kur** with $\boldsymbol{\pi}^{-1}(y)$ a genus g surface for all $y\in\boldsymbol{Y}$, $\boldsymbol{u}:\boldsymbol{X}\to S$ is a 1-morphism with $[\boldsymbol{u}(\boldsymbol{\pi}^{-1}(y))]=\beta$ for all $y\in\boldsymbol{Y}$, and $j:\mathbb{T}_{\boldsymbol{\pi}}\to\mathbb{T}_{\boldsymbol{\pi}}$ is bundle linear with $j^2=-\mathrm{id}$ and $\boldsymbol{u}^*(J)\circ\mathrm{d}\boldsymbol{u}=\mathrm{d}\boldsymbol{u}\circ j$, for $\mathbb{T}_{\boldsymbol{\pi}}$ the relative tangent bundle of $\boldsymbol{\pi}$, a rank 2 vector bundle over $\boldsymbol{X}$.

• A 1-*morphism* $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{\eta},\boldsymbol{\zeta}):(\boldsymbol{X}_1,\boldsymbol{Y}_1,\boldsymbol{\pi}_1,\boldsymbol{u}_1,j_1)\to(\boldsymbol{X}_2,\boldsymbol{Y}_2,\boldsymbol{\pi}_2,\boldsymbol{u}_2,j_2)$ is a diagram with square 2-Cartesian: such that $H^0(\mathrm{d}\boldsymbol{f})$ identifies j_1,j_2.



• If $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{\eta},\boldsymbol{\zeta}),(\boldsymbol{f}',\boldsymbol{g}',\boldsymbol{\eta}',\boldsymbol{\zeta}'):(\boldsymbol{X}_1,\boldsymbol{Y}_1,\boldsymbol{\pi}_1,\boldsymbol{u}_1,j_1)\to(\boldsymbol{X}_2,\boldsymbol{Y}_2,\boldsymbol{\pi}_2,\boldsymbol{u}_2,j_2)$ are 1-morphisms, a 2-*morphism* $(\boldsymbol{\alpha},\boldsymbol{\beta}):(\boldsymbol{f},\boldsymbol{g},\boldsymbol{\eta},\boldsymbol{\zeta})\Rightarrow(\boldsymbol{f}',\boldsymbol{g}',\boldsymbol{\eta}',\boldsymbol{\zeta}')$ is 2-morphisms $\boldsymbol{\alpha}:\boldsymbol{f}\Rightarrow\boldsymbol{f}'$ and $\boldsymbol{\beta}:\boldsymbol{g}\Rightarrow\boldsymbol{g}'$ in **Kur** such that $\boldsymbol{\eta}=\boldsymbol{\eta}'\odot(\mathbf{id}_{\boldsymbol{u}_2}*\boldsymbol{\alpha})$ and $(\mathbf{id}_{\boldsymbol{\pi}_2}*\boldsymbol{\alpha})\odot\boldsymbol{\zeta}=\boldsymbol{\zeta}'\odot(\boldsymbol{\beta}*\mathbf{id}_{\boldsymbol{\pi}_1})$.

• A family $(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{\pi},\boldsymbol{u},j)$ is called *universal* if for every other family $(\boldsymbol{X}',\boldsymbol{Y}',\boldsymbol{\pi}',\boldsymbol{u}',j')$, there exists a 1-morphism $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{\eta},\boldsymbol{\zeta}):(\boldsymbol{X}',\boldsymbol{Y}',\boldsymbol{\pi}',\boldsymbol{u}',j')\to(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{\pi},\boldsymbol{u},j)$, unique up to 2-isomorphism.

If a universal family $(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{\pi},\boldsymbol{u},j)$ exists, then $\boldsymbol{Y}$ is the moduli space of J-holomorphic maps, as a derived orbifold. It is automatically unique up to equivalence in **Kur**.

I believe I will be able to prove universal families exist in large classes of moduli problems, by some standard arguments.

Some advantages of the universal families approach:

- Many current presentations of moduli spaces (e.g. FOOO, HWZ) are long, complicated ad hoc constructions. The effort is mostly in the definition. It is unclear how natural they are. Our definition makes the naturality clear. We have a short, easy definition (universal families), followed by a difficult theorem (universal families exist).
- The definition involves only finite-dimensional families of smooth objects – no Hölder or Sobolev spaces, etc. (though these will be used in proof of existence of universal families). This enables us to sidestep some technical issues in current approaches, e.g. sc-smoothness in polyfolds.
- In our approach, the existence of natural morphisms between moduli spaces (e.g. 'forgetful morphisms' in Symplectic Geometry forgetting a marked point) is essentially trivial.