The d-orbifold programme. Lecture 2 of 5: The 2-categories of d-manifolds and d-orbifolds.

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Some references (to be revised, especially concerning corners):
For d-manifolds and d-orbifolds, see arXiv:1206.4207 (survey), arXiv:1208.4948, and preliminary version of book available at `http://people.maths.ox.ac.uk/~joyce/dmanifolds.html`.
For C^∞ geometry, see arXiv:1104.4951 (survey), and arXiv:1001.0023.

Disclaimer

Much of this lecture series is work in progress, and not yet written up — it exists only in my head.
What there is you can find on my web page at `http://people.maths.ox.ac.uk/~joyce/dmanifolds.html`, including a preliminary version (768 pages) of a book [book] on d-manifolds and d-orbifolds and their differential geometry.
I am intending to rewrite the book from the beginning, not because there is very much wrong with it, but because it turns out (irritatingly) that I need a more general notion of d-orbifold with corners to do moduli spaces of J-holomorphic curves properly.

1. Introduction

I will tell you about new classes of geometric objects I call *d-manifolds* and *d-orbifolds* — 'derived' smooth manifolds, in the sense of Derived Algebraic Geometry. Some properties:

- D-manifolds form a *strict 2-category* **dMan**. That is, we have objects $\mathbf{X}$, the d-manifolds, 1-morphisms $\mathbf{f}, \mathbf{g} : \mathbf{X} \to \mathbf{Y}$, the smooth maps, and also 2-morphisms $\eta : \mathbf{f} \Rightarrow \mathbf{g}$.
- Smooth manifolds embed into d-manifolds as a full (2)-subcategory. So, d-manifolds generalize manifolds.
- There are also 2-categories $\mathbf{dMan^b}$, $\mathbf{dMan^c}$ of d-manifolds *with boundary* and *with corners*, and orbifold versions $\mathbf{dOrb}, \mathbf{dOrb^b}, \mathbf{dOrb^c}$ of these, *d-orbifolds*.
- Much of differential geometry extends nicely to d-manifolds: submersions, immersions, orientations, submanifolds, transverse fibre products, cotangent bundles, ...

Origins in derived algebraic geometry

D-manifolds are based on ideas from *derived algebraic geometry*. First consider an algebro-geometric version of what we want to do. A good algebraic analogue of smooth manifolds are *complex algebraic manifolds*, that is, separated smooth $\mathbb{C}$-schemes S of pure dimension. These form a full subcategory $\mathbf{AlgMan}_{\mathbb{C}}$ in the category $\mathbf{Sch}_{\mathbb{C}}$ of $\mathbb{C}$-schemes, and can roughly be characterized as the (sufficiently nice) objects S in $\mathbf{Sch}_{\mathbb{C}}$ whose cotangent complex $\mathbb{L}_S$ is a vector bundle (i.e. perfect in the interval $[0, 0]$).

To make a derived version of this, we first define an ∞-category $\mathbf{DerSch}_{\mathbb{C}}$ of *derived* $\mathbb{C}$*-schemes*, and then define the ∞-category $\mathbf{DerAlgMan}_{\mathbb{C}}$ of *derived complex algebraic manifolds* to be the full ∞-subcategory of objects $\mathbf{S}$ in $\mathbf{DerSch}_{\mathbb{C}}$ which are *quasi-smooth* (have cotangent complex $\mathbb{L}_S$ perfect in the interval $[-0, 0]$), and satisfy some other niceness conditions (separated, etc.).

Derived algebraic geometry in the C^∞ world

Thus, we have 'classical' categories $\mathbf{AlgMan}_{\mathbb{C}} \subset \mathbf{Sch}_{\mathbb{C}}$, and related 'derived' ∞-categories $\mathbf{DerAlgMan}_{\mathbb{C}} \subset \mathbf{DerSch}_{\mathbb{C}}$.
David Spivak (arXiv:0810.5175, Duke Math. J.), a student of Jacob Lurie, defined an ∞-category **DerMan** of 'derived smooth manifolds' using a similar structure: he considered 'classical' categories $\mathbf{Man} \subset \mathbf{C^\infty Sch}$ and related 'derived' ∞-categories $\mathbf{DerMan} \subset \mathbf{DerC^\infty Sch}$. Here $\mathbf{C^\infty Sch}$ is C^∞*-schemes*, and $\mathbf{DerC^\infty Sch}$ *derived* C^∞*-schemes*. That is, before we can 'derive', we must first embed **Man** into a larger category of C^∞-schemes, singular generalizations of manifolds.
My set-up is a simplification of Spivak's. I consider 'classical' categories $\mathbf{Man} \subset \mathbf{C^\infty Sch}$ and related 'derived' 2-categories $\mathbf{dMan} \subset \mathbf{dSpa}$, where **dMan** is *d-manifolds*, and **dSpa** *d-spaces*. Here $\mathbf{dMan}, \mathbf{dSpa}$ are roughly 2-category truncations of Spivak's $\mathbf{DerMan}, \mathbf{DerC^\infty Sch}$ — see Borisov arXiv:1212.1153.

2. D-manifolds without boundary

I will concentrate today on d-manifolds *without boundary*. Lecture 4 will explain how to include boundaries and corners.
We begin by discussing C^∞-algebraic geometry, C^∞-rings, and C^∞-schemes. Algebraic geometry (based on algebra and polynomials) has excellent tools for studying singular spaces – the theory of schemes.
In contrast, conventional differential geometry (based on smooth real functions and calculus) deals well with nonsingular spaces – manifolds – but poorly with singular spaces.
There is a little-known theory of schemes in differential geometry, C^∞*-schemes*, going back to Lawvere, Dubuc, Moerdijk and Reyes, … in synthetic differential geometry in the 1960s-1980s.
C^∞-schemes are essentially *algebraic* objects, on which smooth real functions and calculus make sense.

2.1. C^∞-rings

Let X be a manifold, and write $C^\infty(X)$ for the smooth functions $c : X \to \mathbb{R}$. Then $C^\infty(X)$ is an $\mathbb{R}$-*algebra*: we can add smooth functions $(c, d) \mapsto c + d$, and multiply them $(c, d) \mapsto cd$, and multiply by $\lambda \in \mathbb{R}$.

But there are many more operations on $C^\infty(X)$ than this, e.g. if $c : X \to \mathbb{R}$ is smooth then $\exp(c) : X \to \mathbb{R}$ is smooth, giving $\exp : C^\infty(X) \to C^\infty(X)$, which is algebraically independent of addition and multiplication.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth. Define $\Phi_f : C^\infty(X)^n \to C^\infty(X)$ by $\Phi_f(c_1, \ldots, c_n)(x) = f\bigl(c_1(x), \ldots, c_n(x)\bigr)$ for all $x \in X$. Then addition comes from $f : \mathbb{R}^2 \to \mathbb{R}$, $f : (x, y) \mapsto x + y$, multiplication from $(x, y) \mapsto xy$, etc. This huge collection of algebraic operations Φ_f make $C^\infty(X)$ into an algebraic object called a C^∞-*ring*.

Definition

A C^∞-*ring* is a set $\mathfrak{C}$ together with n-fold operations $\Phi_f : \mathfrak{C}^n \to \mathfrak{C}$ for all smooth maps $f : \mathbb{R}^n \to \mathbb{R}$, $n \geqslant 0$, satisfying:

Let $m, n \geqslant 0$, and $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$ and $g : \mathbb{R}^m \to \mathbb{R}$ be smooth functions. Define $h : \mathbb{R}^n \to \mathbb{R}$ by

$$h(x_1, \ldots, x_n) = g(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1 \ldots, x_n)),$$

for $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Then for all $c_1, \ldots, c_n$ in $\mathfrak{C}$ we have

$$\Phi_h(c_1, \ldots, c_n) = \Phi_g(\Phi_{f_1}(c_1, \ldots, c_n), \ldots, \Phi_{f_m}(c_1, \ldots, c_n)).$$

Also defining $\pi_j : (x_1, \ldots, x_n) \mapsto x_j$ for $j = 1, \ldots, n$ we have $\Phi_{\pi_j} : (c_1, \ldots, c_n) \mapsto c_j$.

A *morphism* of C^∞-rings is $\phi : \mathfrak{C} \to \mathfrak{D}$ with $\Phi_f \circ \phi^n = \phi \circ \Phi_f : \mathfrak{C}^n \to \mathfrak{D}$ for all smooth $f : \mathbb{R}^n \to \mathbb{R}$. Write $\mathbf{C^\infty Rings}$ for the category of C^∞-rings.

Examples of C^∞-rings

Then $C^\infty(X)$ is a C^∞-ring for any manifold X, and from $C^\infty(X)$ we can recover X up to canonical isomorphism.
If $f : X \to Y$ is smooth then $f^* : C^\infty(Y) \to C^\infty(X)$ is a morphism of C^∞-rings; conversely, if $\phi : C^\infty(Y) \to C^\infty(X)$ is a morphism of C^∞-rings then $\phi = f^*$ for some unique smooth $f : X \to Y$. This gives a *full and faithful functor* $F : \mathbf{Man} \to \mathbf{C^\infty Rings}^{\mathrm{op}}$ by $F : X \mapsto C^\infty(X)$, $F : f \mapsto f^*$.
Thus, we can think of manifolds as examples of C^∞-rings, and C^∞-rings as generalizations of manifolds. But there are many more C^∞-rings than manifolds. For example, $C^0(X)$ is a C^∞-ring for any topological space X.
Any C^∞-ring $\mathfrak{C}$ has a *cotangent module* $\Omega_{\mathfrak{C}}$. If $\mathfrak{C} = C^\infty(X)$ for X a manifold, then $\Omega_{\mathfrak{C}} = C^\infty(T^*X)$.

2.2. C^∞-schemes

We can now develop the whole machinery of scheme theory in algebraic geometry, replacing rings or algebras by C^∞-rings throughout — see my arXiv:1104.4951, arXiv:1001.0023.
A *C^∞-ringed space* $\underline{X} = (X, \mathcal{O}_X)$ is a topological space X with a sheaf of C^∞-rings $\mathcal{O}_X$. Write $\mathbf{C^\infty RS}$ for the category of C^∞-ringed spaces.
The *global sections functor* $\Gamma : \mathbf{C^\infty RS} \to \mathbf{C^\infty Rings}^{\mathrm{op}}$ maps $\Gamma : (X, \mathcal{O}_X) \mapsto \mathcal{O}_X(X)$. It has a right adjoint, the *spectrum functor* $\mathrm{Spec} : \mathbf{C^\infty Rings}^{\mathrm{op}} \to \mathbf{C^\infty RS}$. That is, for each C^∞-ring $\mathfrak{C}$ we construct a C^∞-ringed space $\mathrm{Spec}\,\mathfrak{C}$. Points $x \in \mathrm{Spec}\,\mathfrak{C}$ are $\mathbb{R}$-algebra morphisms $x : \mathfrak{C} \to \mathbb{R}$ (this implies x is a C^∞-ring morphism). We don't use prime ideals.
On the subcategory of *fair* C^∞-rings, Spec is full and faithful.

A C^∞-ringed space $\underline{X}$ is called an *affine C^∞-scheme* if $\underline{X} \cong \operatorname{Spec} \mathfrak{C}$ for some C^∞-ring $\mathfrak{C}$. We call $\underline{X}$ a *C^∞-scheme* if X can be covered by open subsets U with $(U, \mathcal{O}_X|_U)$ an affine C^∞-scheme. Write $\mathbf{C^\infty Sch}$ for the full subcategory of C^∞-schemes in $\mathbf{C^\infty RS}$.

If X is a manifold, define a C^∞-scheme $\underline{X} = (X, \mathcal{O}_X)$ by $\mathcal{O}_X(U) = C^\infty(U)$ for all open $U \subseteq X$. Then $\underline{X} \cong \operatorname{Spec} C^\infty(X)$. This defines a full and faithful embedding $\mathbf{Man} \hookrightarrow \mathbf{C^\infty Sch}$. So we can regard manifolds as examples of C^∞-schemes.

All *fibre products* exist in $\mathbf{C^\infty Sch}$. In manifolds $\mathbf{Man}$, fibre products $X \times_{g,Z,h} Y$ need exist only if $g : X \to Z$ and $h : Y \to Z$ are transverse. When g, h are not transverse, the fibre product $X \times_{g,Z,h} Y$ exists in $\mathbf{C^\infty Sch}$, but may not be a manifold.

We also define *vector bundles* and *quasicoherent sheaves* on a C^∞-scheme $\underline{X}$, and write $\operatorname{qcoh}(\underline{X})$ for the abelian category of quasicoherent sheaves. A C^∞-scheme $\underline{X}$ has a well-behaved *cotangent sheaf* $T^*\underline{X}$.

Differences with ordinary Algebraic Geometry

- The topology on C^∞-schemes is finer than the Zariski topology on schemes – affine schemes are always Hausdorff. No need to introduce the étale topology.
- Can find smooth functions supported on (almost) any open set.
- (Almost) any open cover has a subordinate partition of unity.
- Our C^∞-rings $\mathfrak{C}$ are generally *not noetherian* as $\mathbb{R}$-algebras. So ideals I in $\mathfrak{C}$ may not be finitely generated, even in $C^\infty(\mathbb{R}^n)$.

2.3. Differential graded C^∞-rings

We can define derived $\mathbb{C}$-schemes by replacing $\mathbb{C}$-algebras A by *dg $\mathbb{C}$-algebras* $A^\bullet$ in the definition of $\mathbb{C}$-scheme — commutative differential graded $\mathbb{C}$-algebras in degrees $\leqslant 0$, of the form $\cdots \to A^{-2} \xrightarrow{\mathrm{d}} A^{-1} \xrightarrow{\mathrm{d}} A^0$, where A^0 is an ordinary $\mathbb{C}$-algebra. The corresponding 'classical' $\mathbb{C}$-algebra is $H^0(A^\bullet) = A^0/\mathrm{d}[A^{-1}]$.
There is a parallel notion of *dg C^∞-ring* $\mathfrak{C}^\bullet$, of the form $\cdots \to \mathfrak{C}^{-2} \xrightarrow{\mathrm{d}} \mathfrak{C}^{-1} \xrightarrow{\mathrm{d}} \mathfrak{C}^0$, where $\mathfrak{C}^0$ is an ordinary C^∞-ring, and $\mathfrak{C}^{-1}, \mathfrak{C}^{-2}, \ldots$ are modules over $\mathfrak{C}^0$. The corresponding 'classical' C^∞-ring is $H^0(\mathfrak{C}^\bullet) = \mathfrak{C}^0/\mathrm{d}[\mathfrak{C}^{-1}]$.
One could use dg C^∞-rings to define 'derived C^∞-schemes'; an alternative is to use *simplicial C^∞-rings*, see Spivak arXiv:0810.5175, Borisov–Noel arXiv:1112.0033, and Borisov arXiv:1212.1153.

Square zero dg C^∞-rings

My d-spaces are a 2-category truncation of derived C^∞-schemes. To define them, I use a special class of dg C^∞-rings called *square zero dg C^∞-rings*, which form a 2-category $\mathbf{SZC^\infty Rings}$.
A dg C^∞-ring $\mathfrak{C}^\bullet$ is *square zero* if $\mathfrak{C}^i = 0$ for $i < -1$ and $\mathfrak{C}^{-1} \cdot \mathrm{d}[\mathfrak{C}^{-1}] = 0$. Then $\mathfrak{C}$ is $\mathfrak{C}^{-1} \xrightarrow{\mathrm{d}} \mathfrak{C}^0$, and $\mathrm{d}[\mathfrak{C}^{-1}]$ is a square zero ideal in the (ordinary) C^∞-ring $\mathfrak{C}^0$, and $\mathfrak{C}^{-1}$ is a module over the 'classical' C^∞-ring $H^0(\mathfrak{C}^\bullet) = \mathfrak{C}^0/\mathrm{d}[\mathfrak{C}^{-1}]$.
A 1-*morphism* $\alpha^\bullet : \mathfrak{C}^\bullet \to \mathfrak{D}^\bullet$ in $\mathbf{SZC^\infty Rings}$ is maps $\alpha^0 : \mathfrak{C}^0 \to \mathfrak{D}^0$, $\alpha^{-1} : \mathfrak{C}^{-1} \to \mathfrak{D}^{-1}$ preserving all the structure. Then $H^0(\alpha^\bullet) : H^0(\mathfrak{C}) \to H^0(\mathfrak{D})$ is a morphism of C^∞-rings.
For 1-morphisms $\alpha^\bullet, \beta^\bullet : \mathfrak{C}^\bullet \to \mathfrak{D}^\bullet$ a 2-*morphism* $\eta : \alpha^\bullet \Rightarrow \beta^\bullet$ is a linear $\eta : \mathfrak{C}^0 \to \mathfrak{D}^{-1}$ with $\beta^0 = \alpha^0 + \mathrm{d} \circ \eta$ and $\beta^{-1} = \alpha^{-1} + \eta \circ \mathrm{d}$.
There is an embedding of (2-)categories $\mathbf{C^\infty Rings} \subset \mathbf{SZC^\infty Rings}$ as the (2-)subcategory of $\mathfrak{C}^\bullet$ with $\mathfrak{C}^{-1} = 0$.

Cotangent complexes in the 2-category setting

Let $\mathfrak{C}^\bullet$ be a square zero dg C^∞-ring. Define the *cotangent complex* $\mathbb{L}_{\mathfrak{C}}^{-1} \xrightarrow{\mathrm{d}_{\mathfrak{C}}} \mathbb{L}_{\mathfrak{C}}^{0}$ to be the 2-term complex of $H^0(\mathfrak{C}^\bullet)$-modules

$$\mathfrak{C}^{-1} \xrightarrow{\quad \mathrm{d}_{\mathrm{DR}}\circ \mathrm{d} \quad} \Omega_{\mathfrak{C}^0} \otimes_{\mathfrak{C}^0} H^0(\mathfrak{C}^\bullet),$$

regarded as an element of the 2-category of 2-term complexes of $H^0(\mathfrak{C}^\bullet)$-modules. Let $\alpha^\bullet, \beta^\bullet : \mathfrak{C}^\bullet \to \mathfrak{D}^\bullet$ be 1-morphisms and $\eta : \alpha^\bullet \Rightarrow \beta^\bullet$ a 2-morphism in $\mathbf{SZC^\infty Rings}$. Then $H^0(\alpha^\bullet) = H^0(\beta^\bullet)$, so we may regard $\mathfrak{D}^{-1}$ as an $H^0(\mathfrak{C}^\bullet)$-module. And $\eta : \mathfrak{C}^0 \to \mathfrak{D}^{-1}$ is a derivation, so it factors through an $H^0(\mathfrak{C}^\bullet)$-linear map $\hat\eta : \Omega_{\mathfrak{C}^0} \otimes_{\mathfrak{C}^0} H^0(\mathfrak{C}^\bullet) \to \mathfrak{D}^{-1}$. We have a diagram

$$\begin{array}{ccc} \mathbb{L}_{\mathfrak{C}}^{-1} & \xrightarrow{\quad \mathrm{d}_{\mathfrak{C}} \quad} & \mathbb{L}_{\mathfrak{C}}^{0} \\ {\scriptstyle \mathbb{L}_\alpha^{-1}}\downarrow\downarrow{\scriptstyle \mathbb{L}_\beta^{-1}} & \swarrow{\scriptstyle \hat\eta} & {\scriptstyle \mathbb{L}_\alpha^{0}}\downarrow\downarrow{\scriptstyle \mathbb{L}_\beta^{0}} \\ \mathbb{L}_{\mathfrak{D}}^{-1} & \xrightarrow{\quad \mathrm{d}_{\mathfrak{D}} \quad} & \mathbb{L}_{\mathfrak{D}}^{0}. \end{array}$$

So 1-morphisms induce morphisms, and 2-morphisms homotopies, of virtual cotangent modules.

Examples of square zero dg C^∞-rings

Let V be a manifold, $E \to V$ a vector bundle, and $s : V \to E$ a smooth section. Then we call (V, E, s) a *Kuranishi neighbourhood* (compare Kuranishi spaces); for d-orbifolds, we take V an orbifold. Associate a square zero dg C^∞-ring $\mathfrak{C}^{-1} \xrightarrow{\mathrm{d}} \mathfrak{C}^0$ to (V, E, s) by

$$\mathfrak{C}^0 = C^\infty(V)/I_s^2, \qquad \mathfrak{C}^{-1} = C^\infty(E^*)/I_s \cdot C^\infty(E^*),$$
$$\mathrm{d}(\epsilon + I_s \cdot C^\infty(E^*)) = \epsilon(s) + I_s^2,$$

where $I_s = C^\infty(E^*) \cdot s \subset C^\infty(V)$ is the ideal generated by s. The d-manifold $\mathbf{X}$ associated to (V, E, s) is $\mathrm{Spec}\, \mathfrak{C}^\bullet$. It only knows about functions on V up to $O(s^2)$, and sections of E up to $O(s)$.

2.4. D-spaces

A *d-space* $\mathbf{X}$ is a topological space X with a sheaf of square zero dg-C^∞-rings $\mathcal{O}_{\mathbf{X}}^\bullet = \mathcal{O}_X^{-1} \xrightarrow{\mathrm{d}} \mathcal{O}_{\mathbf{X}}^0$, such that $\underline{X} = (X, H^0(\mathcal{O}_{\mathbf{X}}^\bullet))$ and $(X, \mathcal{O}_{\mathbf{X}}^0)$ are C^∞-schemes, and $\mathcal{O}_X^{-1}$ is quasicoherent over $\underline{X}$. We call $\underline{X}$ the *underlying classical* C^∞*-scheme*.
D-spaces form a strict 2-category **dSpa**, with 1-morphisms and 2-morphisms defined using sheaves of 1-morphisms and 2-morphisms in **SZC**$^\infty$**Rings** in the obvious way.
All fibre products exist in **dSpa**.
C^∞-schemes include into d-spaces as those $\mathbf{X}$ with $\mathcal{O}_X^{-1} = 0$.
Thus we have inclusions of (2-)categories **Man** $\subset$ **C**$^\infty$**Sch** $\subset$ **dSpa**, so manifolds are examples of d-spaces.
The *cotangent complex* $\mathbb{L}_{\mathbf{X}}^\bullet$ of $\mathbf{X}$ is the sheaf of cotangent complexes of $\mathcal{O}_{\mathbf{X}}^\bullet$, a 2-term complex $\mathbb{L}_{\mathbf{X}}^{-1} \xrightarrow{\mathrm{d}_{\mathbf{X}}} \mathbb{L}_{\mathbf{X}}^0$ of quasicoherent sheaves on $\underline{X}$. Such complexes form a 2-category $\mathrm{qcoh}^{[-1,0]}(\underline{X})$.

2.5. D-manifolds

A *d-manifold* $\mathbf{X}$ of *virtual dimension* $n \in \mathbb{Z}$ is a d-space $\mathbf{X}$ whose topological space X is Hausdorff and second countable, and such that $\mathbf{X}$ is covered by open d-subspaces $\mathbf{Y} \subset \mathbf{X}$ with equivalences $\mathbf{Y} \simeq U \times_{g,W,h} V$, where U, V, W are manifolds with $\dim U + \dim V - \dim W = n$, and $g : U \to W$, $h : V \to W$ are smooth maps, and $U \times_{g,W,h} V$ is the fibre product in the 2-category **dSpa**. (The 2-category structure is *essential* to define the fibre product here.)
Write **dMan** for the full 2-subcategory of d-manifolds in **dSpa**.
Alternatively, we can write the local models as $\mathbf{Y} \simeq V \times_{0,E,s} V$, where V is a manifold, $E \to V$ a vector bundle, $s : V \to E$ a smooth section, and $n = \dim V - \mathrm{rank}\, E$. Then (V, E, s) is a *Kuranishi neighbourhood* on $\mathbf{X}$ (compare with Kuranishi spaces).
We call such $V \times_{0,E,s} V$ *affine d-manifolds*.

2.6. D-orbifolds, d-manifolds with corners

In a similar way, I define 2-categories of *d-stacks* **dSta**, which are a Deligne–Mumford stack version of d-spaces locally modelled on quotients $[\mathbf{X}/G]$ for $\mathbf{X}$ a d-space and G a finite group, and *d-orbifolds* $\mathbf{dOrb} \subset \mathbf{dSta}$. D-orbifolds $\mathbf{X}$ are locally modelled by Kuranishi neighbourhoods (V, E, s) with V an orbifold, $E \to V$ a vector bundle and $s : V \to E$ a smooth section (that is, $\mathbf{X}$ is locally equivalent to a fibre product $V \times_{0,E,s} V$ in **dSta**).

I also define 2-categories $\mathbf{dSpa^b}, \mathbf{dSpa^c}, \mathbf{dMan^b}, \mathbf{dMan^c}, \mathbf{dSta^b}, \mathbf{dSta^c}, \mathbf{dOrb^b}, \mathbf{dOrb^c}$ of d-spaces, d-manifolds, d-stacks and d-orbifolds *with boundary*, and *with corners*.

Many moduli spaces of J-holomorphic curves will be d-orbifolds with corners. Doing 'things with corners' properly, especially in the derived context, is more complicated than you would expect. I will say more about this in Lecture 4.

2.7. Why should **dMan** be a 2-category?

Here is one reason why any class of 'derived manifolds' should be (at least) a 2-category. One property we want of **dMan** (or of Kuranishi spaces, etc.) is that it contains manifolds **Man** as a subcategory, and if X, Y, Z are manifolds and $g : X \to Z$, $h : Y \to Z$ are smooth then a fibre product $\mathbf{W} = X \times_{g,Z,h} Y$ should exist in **dMan**, characterized by a universal property in **dMan**, and should be a d-manifold of 'virtual dimension'

$$\operatorname{vdim} \mathbf{W} = \dim X + \dim Y - \dim Z.$$

Note that g, h need not be transverse, and $\operatorname{vdim} \mathbf{W}$ may be negative. Consider the case $X = Y = *$, the point, $Z = \mathbb{R}$, and $g, h : * \mapsto 0$. If **dMan** were an ordinary category then as $*$ is a terminal object, the unique fibre product $* \times_{0,\mathbb{R},0} *$ would be $*$. But this has virtual dimension 0, not -1. So **dMan** must be some kind of higher category.

Why is a 2-category enough?

Usually in derived algebraic geometry, one considers an ∞-category of objects (derived stacks, etc.). But we work in a 2-category, a truncation of Spivak's ∞-category of derived manifolds.
Here are two reasons why this truncation does not lose important information. Firstly, d-manifolds correspond to *quasi-smooth* derived schemes $\mathbf{X}$, whose cotangent complexes $\mathbb{L}_{\mathbf{X}}$ lie in degrees $[-1,0]$. So $\mathbb{L}_{\mathbf{X}}$ lies in a 2-category of complexes, not an ∞-category. Note that $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$ is étale in **dMan** iff $\Omega_{\mathbf{f}}:\mathbf{f}^*(\mathbb{L}_{\mathbf{Y}})\to\mathbb{L}_{\mathbf{X}}$ is an equivalence.
Secondly, the existence of *partitions of unity* in differential geometry means that our structure sheaves $\mathcal{O}_{\mathbf{X}}$ are 'fine' or 'soft', which simplifies behaviour. Partitions of unity are also essential in gluing by equivalences in **dMan**. Our '2-category style derived geometry' would not work well in a conventional algebro-geometric context, rather than a differential-geometric one.

3. 'Standard model' d-manifolds

Let V be a manifold, $E\to V$ a vector bundle, and $s:V\to E$ a smooth section. Then we can define an explicit affine d-manifold $\mathbf{S}_{V,E,s}$ in a 2-Cartesian diagram in **dMan**:

$$\begin{array}{ccc} \mathbf{S}_{V,E,s} & \xrightarrow{\quad\boldsymbol{\pi}\quad} & \mathbf{V} \\ \downarrow{\scriptstyle\boldsymbol{\pi}} & \;\;\eta\Uparrow & \downarrow{\scriptstyle\mathbf{0}} \\ \mathbf{V} & \xrightarrow{\quad\mathbf{s}\quad} & \mathbf{E}. \end{array}$$

We have $\mathbf{S}_{V,E,s}=\operatorname{Spec}\mathfrak{C}^\bullet$, for $\mathfrak{C}^\bullet$ as in the example in §2.3.
We call $\mathbf{S}_{V,E,s}$ a 'standard model' d-manifold. It is similar to *Kuranishi neighbourhoods* in Fukaya–Oh–Ohta-Ono's Kuranishi spaces. It has dimension $\operatorname{vdim}\mathbf{S}_{V,E,s}=\dim V-\operatorname{rank}E$. Every affine d-manifold is equivalent to some $\mathbf{S}_{V,E,s}$, and every d-manifold is locally equivalent to some $\mathbf{S}_{V,E,s}$.

'Standard model' 1-morphisms

Let V, W be manifolds, $E \to V$, $F \to W$ vector bundles, and $s : V \to E$, $t : W \to F$ smooth sections, so we have d-manifolds $\mathbf{S}_{V,E,s}$, $\mathbf{S}_{W,F,t}$. Suppose $f : V \to W$ is smooth, and $\hat{f} : E \to f^*(F)$ is a morphism of vector bundles on V satisfying $\hat{f} \circ s = f^*(t) + O(s^2)$ in $C^\infty(f^*(F))$. Then we define a 'standard model' 1-morphism $\mathbf{S}_{f,\hat{f}} : \mathbf{S}_{V,E,s} \to \mathbf{S}_{W,F,t}$. Two 1-morphisms $\mathbf{S}_{f,\hat{f}}, \mathbf{S}_{g,\hat{g}}$ are equal iff $g = f + O(s^2)$ and $\hat{g} = \hat{f} + O(s)$.

Theorem

Every 1-morphism $\mathbf{g} : \mathbf{S}_{V,E,s} \to \mathbf{S}_{W,F,t}$ *in* **dMan** *is of the form* $\mathbf{S}_{f,\hat{f}}$, *possibly after making* V *smaller.*

Theorem

A 'standard model' 1-morphism $\mathbf{S}_{f,\hat{f}} : \mathbf{S}_{V,E,s} \to \mathbf{S}_{W,F,t}$ *is étale (a local equivalence) in* **dMan** *iff for each* $v \in V$ *with* $s(v) = 0$ *and* $w = f(v) \in W$, *the following sequence is exact:*

$$0 \longrightarrow T_v V \xrightarrow{\mathrm{d}s(v) \oplus \mathrm{d}f(v)} E_v \oplus T_w W \xrightarrow{\hat{f}(v) \oplus -\mathrm{d}t(w)} F_w \longrightarrow 0.$$

$\mathbf{S}_{f,\hat{f}}$ *is an equivalence iff also* $f|_{s^{-1}(0)} : s^{-1}(0) \to t^{-1}(0)$ *is a bijection.*

Example

In Kuranishi spaces, a 'coordinate change' $(f, \hat{f}) : (V, E, s) \to (W, F, t)$ is embeddings $f : V \hookrightarrow W$ and $\hat{f} : E \hookrightarrow f^*(F)$ with $\hat{f} \circ s = f^*(t)$, $f^*(TW)/TV \cong f^*(F)/E$. The theorem shows $\mathbf{S}_{f,\hat{f}}$ is étale, or an equivalence.

'Standard model' 2-morphisms

Let $\mathbf{S}_{V,E,s}, \mathbf{S}_{W,F,t}$ be 'standard model' d-manifolds, and $\mathbf{S}_{f,\hat{f}}, \mathbf{S}_{g,\hat{g}} : \mathbf{S}_{V,E,s} \to \mathbf{S}_{W,F,t}$ 'standard model' 1-morphisms. Suppose $\Lambda : E \to f^*(TW)$ is a morphism of vector bundles on V, with $g = f + \Lambda \cdot s + O(s^2)$ and $\hat{g} = \hat{f} + \Lambda \cdot f^*(\mathrm{d}t) + O(s)$. Then we can define a 'standard model' 2-morphism $\mathbf{S}_\Lambda : \mathbf{S}_{f,\hat{f}} \Rightarrow \mathbf{S}_{g,\hat{g}}$. Every 2-morphism $\eta : \mathbf{S}_{f,\hat{f}} \Rightarrow \mathbf{S}_{g,\hat{g}}$ is $\mathbf{S}_\Lambda$ for some Λ. Also $\mathbf{S}_\Lambda = \mathbf{S}_{\Lambda'}$ iff $\Lambda' = \Lambda + O(s)$.

These 'standard models' give a very explicit geometric picture of objects, 1- and 2-morphisms in **dMan**. The $O(s), O(s^2)$ notation tells you how much information about V, E, s the d-manifolds and morphisms remember.
One should use these ideas to relate d-manifolds/d-orbifolds and Kuranishi spaces, and to see how to make a new, 2-categorical definition of Kuranishi spaces.

4. Differential geometry of d-manifolds
4.1. Cotangent complexes of d-manifolds

If $\mathbf{X}$ is a d-manifold, its cotangent complex $\mathbb{L}_{\mathbf{X}}^\bullet$ is *perfect*, that is, $\mathbb{L}_{\mathbf{X}}^\bullet$ is equivalent locally on $\underline{X}$ in the 2-category $\mathrm{qcoh}^{[-1,0]}(\underline{X})$ of 2-term complexes of quasicoherent sheaves on $\underline{X}$ to a complex of vector bundles $\mathcal{E}^{-1} \to \mathcal{E}^0$, and $\mathrm{rank}\,\mathcal{E}^0 - \mathrm{rank}\,\mathcal{E}^{-1} = \mathrm{vdim}\,\mathbf{X}$. For $x \in \mathbf{X}$, define the *cotangent space* $T_x^*\mathbf{X} = H^0(\mathbb{L}_{\mathbf{X}}|_x)$ and the *obstruction space* $O_x\mathbf{X} = H^{-1}(\mathbb{L}_{\mathbf{X}}|_x)$, with $\dim T_x^*\mathbf{X} - \dim O_x\mathbf{X} = \mathrm{vdim}\,\mathbf{X}$. A 1-morphism of d-manifolds $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ induces a 1-morphism $\mathrm{d}\mathbf{f} : \underline{f}^*(\mathbb{L}_{\mathbf{Y}}^\bullet) \to \mathbb{L}_{\mathbf{X}}^\bullet$ in $\mathrm{qcoh}^{[-1,0]}(\underline{X})$.

Theorem

*A 1-morphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ in **dMan** is étale if and only if $\mathrm{d}\mathbf{f} : \underline{f}^*(\mathbb{L}_{\mathbf{Y}}^\bullet) \to \mathbb{L}_{\mathbf{X}}^\bullet$ is an equivalence in $\mathrm{qcoh}^{[-1,0]}(\underline{X})$, if and only if $H^0(\mathrm{d}\mathbf{f}|_x) : T_{f(x)}^*Y \to T_x^*X$ and $H^{-1}(\mathrm{d}\mathbf{f}|_x) : O_{f(x)}^*Y \to O_x^*X$ are isomorphisms for all $x \in \mathbf{X}$.*

4.2. D-transversality and fibre products

Let $g : X \to Z$, $h : Y \to Z$ be smooth maps of manifolds. Then g, h are *transverse* if for all $x \in X$, $y \in Y$ with $g(x) = h(y) = z$ in Z, the map $\mathrm{d}g|_x \oplus \mathrm{d}h|_y : T_z^*Z \to T_x^*X \oplus T_x^*Y$ is injective. If g, h are transverse then a fibre product $X \times_{g,Z,h} Y$ exists in **Man**.

Similarly, we call 1-morphisms of d-manifolds $\mathbf{g} : \mathbf{X} \to \mathbf{Z}$, $\mathbf{h} : \mathbf{Y} \to \mathbf{Z}$ *d-transverse* if for all $x \in \mathbf{X}$, $y \in \mathbf{Y}$ with $\mathbf{g}(x) = \mathbf{h}(y) = z$ in $\mathbf{Z}$, the map $H^{-1}(\mathrm{d}\mathbf{g}|_x) \oplus H^{-1}(\mathrm{d}\mathbf{h}|_y) : O_z^*\mathbf{Z} \to O_x^*\mathbf{X} \oplus O_y^*\mathbf{Y}$ is injective.
Note that d-transversality is *much weaker* than transversality of manifolds, and often holds automatically.

Theorem

Let $\mathbf{g} : \mathbf{X} \to \mathbf{Z}$ *and* $\mathbf{h} : \mathbf{Y} \to \mathbf{Z}$ *be d-transverse* 1*-morphisms in* **dMan**. *Then a fibre product* $\mathbf{W} = \mathbf{X} \times_{\mathbf{g},\mathbf{Z},\mathbf{h}} \mathbf{Y}$ *exists in* **dMan**.

If $\mathbf{Z}$ is a manifold, $O_z^*\mathbf{Z} = 0$ and d-transversality is trivial, giving:

Corollary

All fibre products of the form $\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y}$ *with* $\mathbf{X}, \mathbf{Y}$ *d-manifolds and* $\mathbf{Z}$ *a manifold exist in the* 2*-category* **dMan**.

The same holds in **dOrb**. This is a very useful property of d-manifolds and d-orbifolds. For example, moduli spaces $\bar{\boldsymbol{\mathcal{M}}}_k(\gamma)$ of J-holomorphic discs in homology class γ in a symplectic manifold with boundary in a Lagrangian L and k boundary marked points are d-orbifolds with corners satisfying

$$\partial\bar{\boldsymbol{\mathcal{M}}}_k(\gamma) = \textstyle\coprod_{i+j=k,\ \alpha+\beta=\gamma} \bar{\boldsymbol{\mathcal{M}}}_{i+1}(\alpha) \times_L \bar{\boldsymbol{\mathcal{M}}}_{j+1}(\beta),$$

where the d-orbifold fibre products over the manifold L exist.

4.3. Gluing by equivalences

A 1-morphism $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$ in **dMan** is an *equivalence* if there exist $\mathbf{g}:\mathbf{Y}\to\mathbf{X}$ and 2-morphisms $\eta:\mathbf{g}\circ\mathbf{f}\Rightarrow\mathbf{id_X}$ and $\zeta:\mathbf{f}\circ\mathbf{g}\Rightarrow\mathbf{id_Y}$.

Theorem

Let $\mathbf{X},\mathbf{Y}$ be d-manifolds, $\emptyset\neq\mathbf{U}\subseteq\mathbf{X}$, $\emptyset\neq\mathbf{V}\subseteq\mathbf{Y}$ open d-submanifolds, and $\mathbf{f}:\mathbf{U}\to\mathbf{V}$ an equivalence. Suppose the topological space $Z=X\cup_{U=V}Y$ made by gluing X,Y using f is Hausdorff. Then there exists a d-manifold $\mathbf{Z}$, unique up to equivalence, open $\hat{\mathbf{X}},\hat{\mathbf{Y}}$ $\subseteq\mathbf{Z}$ with $\mathbf{Z}=\hat{\mathbf{X}}\cup\hat{\mathbf{Y}}$, equivalences $\mathbf{g}:\mathbf{X}\to\hat{\mathbf{X}}$ and $\mathbf{h}:\mathbf{Y}\to\hat{\mathbf{Y}}$, and a 2-morphism $\eta:\mathbf{g}|_{\mathbf{U}}\Rightarrow\mathbf{h}\circ\mathbf{f}$.

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Equivalence is the natural notion of when two objects in **dMan** are 'the same'. In the theorem $\mathbf{Z}$ is a *pushout* $\mathbf{X}\amalg_{\mathbf{id_U},\mathbf{U},\mathbf{f}}\mathbf{Y}$ in **dMan**.

The theorem generalizes to gluing families of d-manifolds $\mathbf{X}_i:i\in I$ by equivalences on double overlaps $\mathbf{X}_i\cap\mathbf{X}_j$, with (weak) conditions on triple overlaps $\mathbf{X}_i\cap\mathbf{X}_j\cap\mathbf{X}_k$.

We can take the $\mathbf{X}_i$ to be 'standard model' d-manifolds $\mathbf{S}_{V_i,E_i,s_i}$, and the equivalences on overlaps $\mathbf{X}_i\cap\mathbf{X}_j$ to be 1-morphisms $\mathbf{S}_{e_{ij},\hat{e}_{ij}}$.

This is very useful for proving existence of d-manifold structures on moduli spaces.