The d-orbifold programme. Lecture 3 of 5: D-orbifold structures on moduli spaces. D-orbifolds as representable 2-functors

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For the first part of the talk, see preliminary version of book at `http://people.maths.ox.ac.uk/~joyce/dmanifolds.html`.
The second part is work in progress, no papers yet.

1. Introduction

Almost any moduli space used in any enumerative invariant problem over $\mathbb{R}$ or $\mathbb{C}$ has a d-manifold or d-orbifold structure, natural up to equivalence. This lecture explains how to prove this. We discuss three methods to prove the existence of a d-manifold or d-orbifold structure on a moduli space:

(i) Directly using analysis.
(ii) By combining results from the literature on existence of other structures (e.g. Kuranishi spaces, polyfolds, $\mathbb{C}$-schemes with obstruction theories) with a truncation functor from these structures to d-manifolds or d-orbifolds.
(iii) By using 'representable 2-functors', a differential-geometric version of Grothendieck's moduli functor approach in algebraic geometry.

We deal with (i),(ii) quickly, and spend most of the time on (iii).

1.1. D-manifolds and nonlinear elliptic equations

Theorem

Let V be a Banach manifold, $E \to V$ a Banach vector bundle, and $s : V \to E$ a smooth Fredholm section, with constant Fredholm index $n \in \mathbb{Z}$. Then there is a d-manifold $\mathbf{X}$, unique up to equivalence in **dMan**, *with topological space $X = s^{-1}(0)$ and $\mathrm{vdim}\,\mathbf{X} = n$.*

Nonlinear elliptic equations on compact manifolds induce nonlinear Fredholm maps on Hölder or Sobolev spaces of sections. Thus:

Corollary

Let $\mathfrak{M}$ be a moduli space of solutions of a nonlinear elliptic equation on a compact manifold, with fixed topological invariants. Then $\mathfrak{M}$ extends to a d-manifold.

(Note that this does **not** include problems involving dividing by a gauge group, such as moduli of J-holomorphic curves.)

1.2. Kuranishi spaces

Kuranishi spaces (both without boundary, and with corners) appear in the work of Fukaya–Oh–Ohta–Ono as the geometric structure on moduli spaces of J-holomorphic curves in symplectic geometry. They do not define morphisms between Kuranishi spaces, so Kuranishi spaces are not a category. But they do define morphisms $\mathbf{f} : \boldsymbol{\mathcal{X}} \to Z$ from Kuranishi spaces $\boldsymbol{\mathcal{X}}$ to manifolds or orbifolds Z, and 'fibre products' $\boldsymbol{\mathcal{X}} \times_Z \boldsymbol{\mathcal{Y}}$ of Kuranishi spaces over manifolds or orbifolds (these are an ad hoc construction, they don't satisfy a universal property).
I began this project to find a better definition of Kuranishi space, with well-behaved morphisms. D-manifolds and d-orbifolds are the result.

Theorem

(a) *Suppose* $\mathbf{X}$ *is a d-orbifold. Then (after many choices) one can construct a Kuranishi space* $\mathcal{X}'$ *with the same topological space and dimension.*
(b) *Let* $\mathcal{X}'$ *be a Kuranishi space. Then one can construct a d-orbifold* $\mathbf{X}''$*, unique up to equivalence in* **dOrb**, *with the same topological space and dimension.*
(c) *Doing* **(a)** *then* **(b)**, $\mathbf{X}$ *and* $\mathbf{X}''$ *are equivalent in* **dOrb**.
(d) *The constructions of* **(a)**,**(b)** *identify orientations, morphisms* $\mathbf{f}:\mathbf{X}\to Y$ *to orbifolds* Y*, and 'fibre products' over orbifolds, for d-orbifolds and Kuranishi spaces.*

Roughly speaking, the theorem says that d-orbifolds **dOrb** and Kuranishi spaces are equivalent categories, except that Kuranishi spaces are not a category. The moral is (I claim): *the 'correct' way to define Kuranishi spaces is as d-orbifolds.*
Combining the theorem with constructions of Kuranishi structures on moduli spaces, gives d-orbifold structures on moduli spaces.

1.3. Polyfolds

Polyfolds, due to Hofer, Wysocki and Zehnder, are a rival theory to Kuranishi spaces. They do form a category. Polyfolds remember much more information than Kuranishi spaces or d-orbifolds, so the truncation functor goes only one way.

Theorem

There is a functor $\Pi_{\mathbf{PolFS}}^{\mathbf{dOrb^c}}:\mathbf{PolFS}\to\mathrm{Ho}(\mathbf{dOrb})$, *where* **PolFS** *is a category whose objects are triples* $(\mathcal{V},\mathcal{E},s)$ *of a polyfold* $\mathcal{V}$*, a fillable strong polyfold bundle* $\mathcal{E}$ *over* V*, and an sc-smooth Fredholm section* s *of* E *with constant Fredholm index.*

Here $\mathrm{Ho}(\mathbf{dOrb})$ is the homotopy category of the 2-category **dOrb**.
Combining the theorem with constructions of polyfold structures on moduli spaces, gives d-orbifold structures on moduli spaces.

1.4. $\mathbb{C}$-schemes and $\mathbb{C}$-stacks with obstruction theories

In algebraic geometry, the standard method of forming virtual cycles is to use a proper scheme or Deligne–Mumford stack equipped with a *perfect obstruction theory* (Behrend–Fantechi). They are used to define algebraic Gromov–Witten invariants, Donaldson–Thomas invariants of Calabi–Yau 3-folds, ...

Theorem

There is a functor $\Pi_{\mathbf{SchObs}}^{\mathbf{dMan}} : \mathbf{Sch}_{\mathbb{C}}\mathbf{Obs} \to \mathrm{Ho}(\mathbf{dMan})$, *where* $\mathbf{Sch}_{\mathbb{C}}\mathbf{Obs}$ *is a category whose objects are triples* $(X, E^\bullet, \phi)$, *for* X *a separated, second countable* $\mathbb{C}$*-scheme and* $\phi : E^\bullet \to \mathbb{L}_X$ *a perfect obstruction theory on* X *with constant virtual dimension.*
The analogue holds for $\Pi_{\mathbf{StaObs}}^{\mathbf{dOrb}} : \mathbf{Sta}_{\mathbb{C}}\mathbf{Obs} \to \mathrm{Ho}(\mathbf{dOrb})$, *replacing* $\mathbb{C}$*-schemes by Deligne–Mumford* $\mathbb{C}$*-stacks, and d-manifolds by d-orbifolds.*

So, many complex algebraic moduli spaces have d-manifold or d-orbifold structures.

2. D-orbifolds as representable 2-functors, moduli spaces

Disclaimer: the rest of this lecture is work in progress (or not yet begun). The ideas are sketchy and incomplete, but I'm fairly confident they will work eventually.

In this lecture we discuss only moduli spaces and d-orbifolds *without boundary*, and moduli spaces of *nonsingular* J-holomorphic curves, without nodes. The issues of boundaries, and singular curves, will be discussed in Lecture 4.

2.1 Classical representable functors

Recall the Grothendieck approach to moduli spaces in algebraic geometry, using *moduli functors*. Write $\mathbf{Sch}_{\mathbb{C}}$ for the category of $\mathbb{C}$-schemes, and $\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}}$ for the subcategory of affine $\mathbb{C}$-schemes. Any $\mathbb{C}$-scheme X defines a functor $\mathrm{Hom}(-,X):\mathbf{Sch}_{\mathbb{C}}^{\mathbf{op}}\to\mathbf{Sets}$ mapping each $\mathbb{C}$-scheme S to the set $\mathrm{Hom}(S,X)$, where $\mathbf{Sch}_{\mathbb{C}}^{\mathbf{op}}$ is the *opposite category* to $\mathbf{Sch}_{\mathbb{C}}$ (reverse directions of morphisms). By the Yoneda Lemma, the $\mathbb{C}$-scheme X is determined up to isomorphism by the functor $\mathrm{Hom}(-,X)$ up to natural isomorphism. This is still true if we restrict to $\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}}$. Thus, given a functor $F:(\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}})^{\mathbf{op}}\to\mathbf{Sets}$, we can ask if there exists a $\mathbb{C}$-scheme X (necessarily unique up to canonical isomorphism) with $F\cong\mathrm{Hom}(-,X)$. If so, we call F a *representable functor*.

Classical stacks

To extend this from $\mathbb{C}$-schemes to Deligne–Mumford or Artin $\mathbb{C}$-stacks, we consider functors $F:(\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}})^{\mathbf{op}}\to\mathbf{Groupoids}$, where a groupoid is a category all of whose morphisms are isomorphisms. (We can regard a set as a category all of whose morphisms are identities, so replacing **Sets** by **Groupoids** is a generalization.)

A *stack* is a functor $F:(\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}})^{\mathbf{op}}\to\mathbf{Groupoids}$ satisfying a sheaf-type condition: if S is an affine $\mathbb{C}$-scheme and $\{S_i:i\in I\}$ an open cover of S (in some algebraic topology) then we should be able to reconstruct $F(S)$ from $F(S_i)$, $F(S_i\cap S_j)$, $F(S_i\cap S_j\cap S_k)$, $i,j,k\in I$, and the functors between them.

A Deligne–Mumford or Artin $\mathbb{C}$-stack is a stack $F:(\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}})^{\mathbf{op}}\to\mathbf{Groupoids}$ satisfying extra geometric conditions.

Grothendieck's moduli schemes

Suppose we have an algebro-geometric moduli problem (e.g. vector bundles on a smooth projective $\mathbb{C}$-scheme Y) for which we want to form a moduli scheme. Grothendieck tells us that we should define a *moduli functor* $F : (\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}})^{\mathbf{op}} \to \mathbf{Sets}$, such that for each affine $\mathbb{C}$-scheme S, $F(S)$ is the set of isomorphism classes of families of the relevant objects over S (e.g. vector bundles over $Y \times S$). Then we should try to prove F is a representable functor, using some criteria for representability. If it is, $F \cong \mathrm{Hom}(-, \mathcal{M})$, where $\mathcal{M}$ is the (*fine*) *moduli scheme*.

To form a *moduli stack*, we define $F : (\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}})^{\mathbf{op}} \to \mathbf{Groupoids}$, so that for each affine $\mathbb{C}$-scheme S, $F(S)$ is the groupoid of families of objects over S, with morphisms isomorphisms of families, and try to show F satisfies the criteria to be an Artin stack.

2.2. D-orbifolds as representable 2-functors

D-orbifolds **dOrb** are a 2-category with all 2-morphisms invertible. Thus, if $\mathbf{S}, \mathbf{X} \in \mathbf{dOrb}$ then $\mathbf{Hom}(\mathbf{S}, \mathbf{X})$ is a groupoid, and $\mathbf{Hom}(-, \mathbf{X}) : \mathbf{dOrb}^{\mathbf{op}} \to \mathbf{Groupoids}$ is a 2-functor, which determines $\mathbf{X}$ up to equivalence in **dOrb**. This is still true if we restrict to affine d-manifolds $\mathbf{dMan}^{\mathbf{aff}} \subset \mathbf{dOrb}$. Thus, we can consider 2-functors $F : (\mathbf{dMan}^{\mathbf{aff}})^{\mathbf{op}} \to \mathbf{Groupoids}$, and ask whether there exists a d-orbifold $\mathbf{X}$ (unique up to equivalence) with $F \simeq \mathbf{Hom}(-, \mathbf{X})$. If so, we call F a *representable 2-functor*.

Why use $(\mathbf{dMan}^{\mathbf{aff}})^{\mathbf{op}}$ as the domain of the functor? A d-orbifold $\mathbf{X}$ also induces a functors $\mathrm{Hom}(-, \mathbf{X}) : \mathcal{C}^{\mathbf{op}} \to \mathbf{Groupoids}$ for $\mathcal{C} = \mathbf{Man}, \mathbf{Orb}, \mathbf{C^\infty Sch}, \mathbf{C^\infty Sta}, \mathbf{dMan}, \mathbf{dOrb}, \mathbf{dSpa}, \mathbf{dSta}, \ldots$. We want $\mathcal{C}$ large enough that $\mathbf{dOrb} \hookrightarrow \mathbf{Funct}(\mathcal{C}^{\mathbf{op}}, \mathbf{Groupoids})$ is an embedding, but otherwise as small as possible, as we must prove things for all objects in $\mathcal{C}$, so a smaller $\mathcal{C}$ saves work.

Criteria for representable 2-functors

Let $F : (\mathbf{dMan^{aff}})^{\mathbf{op}} \to \mathbf{Groupoids}$ be a functor. When is F representable (that is, $F \simeq \mathrm{Hom}(-, \mathbf{X})$ for some d-orbifold $\mathbf{X}$)?

It is good to have usable *criteria for representability*, such that if one can show the criteria hold in an example, then we know F is representable (even without constructing the d-orbifold $\mathbf{X}$).

I expect there are nice criteria of the form:

(A) F satisfies a sheaf-type condition, i.e. F is a *stack*;

(B) the 'coarse topological space' $\mathcal{M} = F(\text{point})/\text{isos of } F$ is Hausdorff and second countable, and each point x of $\mathcal{M}$ has finite stabilizer group $\mathrm{Aut}(x)$; and

(C) F admits a 'Kuranishi neighbourhood' of dimension $n \in \mathbb{Z}$ near each $x \in \mathcal{M}$, a local model with a universal property.

Functors satisfying (A) (*stacks*) are a kind of geometric space, even if they are not d-orbifolds. They have points, and a topology, and one can work locally on them.

2.3. Moduli 2-functors in differential geometry

Suppose we are given a moduli problem in differential geometry (e.g. J-holomorphic curves in a symplectic manifold) and we want to form a moduli space $\boldsymbol{\mathcal{M}}$ as a d-orbifold. I propose that we should define a *moduli 2-functor* $F : (\mathbf{dMan^{aff}})^{\mathbf{op}} \to \mathbf{Groupoids}$, such that for each affine d-manifold $\mathbf{S}$, $F(\mathbf{S})$ is the category of families of the relevant objects over $\mathbf{S}$. Then we should try to prove F satisfies (A)–(C), and so is represented by a d-orbifold $\boldsymbol{\mathcal{M}}$; here (A),(B) will usually be easy, and (C) the difficult part.

If F is represented by $\boldsymbol{\mathcal{M}}$, then there will automatically exist a *universal family* of objects over $\boldsymbol{\mathcal{M}}$.

Example: moduli functors of J-holomorphic curves

Let (M,ω) be a symplectic manifold, and J an almost complex structure on M. Suppose we want to construct $F:(\mathbf{dMan^{aff}})^{\mathbf{op}}\rightarrow\mathbf{Groupoids}$ representing the moduli space of J-holomorphic maps $u:\Sigma\rightarrow M$, where (Σ,j) is a nonsingular genus g Riemann surface, and $[u(\Sigma)]=\beta\in H_2(M,\mathbb{Z})$.
Then, for each affine d-manifold $\mathbf{S}$, we must construct a groupoid $F(\mathbf{S})$ of families of J-holomorphic maps $u:\Sigma\rightarrow M$ over the base $\mathbf{S}$. There is a natural way to do this:

- Objects of $F(\mathbf{S})$ are quadruples $(\mathbf{X},\boldsymbol{\pi},\mathbf{u},j)$, where $\mathbf{X}$ is a d-manifold with $\mathrm{vdim}\,\mathbf{X}=\mathrm{vdim}\,\mathbf{S}+2$, $\boldsymbol{\pi}:\mathbf{X}\rightarrow\mathbf{S}$ a proper submersion of d-manifolds with $\pi^{-1}(s)$ a genus g surface for all $s\in\mathbf{S}$, $\mathbf{u}:\mathbf{X}\rightarrow M$ is a 1-morphism with $[\mathbf{u}(\pi^{-1}(S))]=\beta$ for all $s\in\mathbf{S}$, and $j:\mathbb{T}_{\boldsymbol{\pi}}\rightarrow\mathbb{T}_{\boldsymbol{\pi}}$ is bundle linear with $\mathbf{j}^2=-\mathrm{id}$ and $\mathbf{u}^*(J)\circ\mathrm{d}\mathbf{u}=\mathrm{d}\mathbf{u}\circ j$, for $\mathbb{T}_{\boldsymbol{\pi}}$ the relative tangent bundle of $\boldsymbol{\pi}$.

- Morphisms $[\mathbf{i},\eta,\zeta]:(\mathbf{X},\boldsymbol{\pi},\mathbf{u},j)\rightarrow(\mathbf{X}',\boldsymbol{\pi}',\mathbf{u}',j')$ in $F(\mathbf{S})$ are $\sim$-equivalence classes $[\mathbf{i},\eta,\zeta]$ of triples $(\mathbf{i},\eta,\zeta)$, where $\mathbf{i}:\mathbf{X}\rightarrow\mathbf{X}'$ is an equivalence in $\mathbf{dMan}$, and $\eta:\boldsymbol{\pi}\Rightarrow\boldsymbol{\pi}'\circ\mathbf{i}$, $\zeta:\mathbf{u}\Rightarrow\mathbf{u}'\circ\mathbf{i}$ are 2-morphisms, and $H^0(\mathrm{d}\mathbf{i})$ identifies j,j', and $(\mathbf{i},\eta,\zeta)\sim(\tilde{\boldsymbol{\imath}},\tilde\eta,\tilde\zeta)$ if there exists a 2-morphism $\alpha:\mathbf{i}\Rightarrow\tilde{\boldsymbol{\imath}}$ with $\tilde\eta=(\mathrm{id}_{\boldsymbol{\pi}'}*\alpha)\odot\eta$ and $\tilde\zeta=(\mathrm{id}_{\mathbf{u}'}*\alpha)\odot\zeta$.
- If $\mathbf{f}:\mathbf{T}\rightarrow\mathbf{S}$ is a 1-morphism in $\mathbf{dMan^{aff}}$, the functor $F(\mathbf{f}):F(\mathbf{S})\rightarrow F(\mathbf{T})$ acts by $F(\mathbf{f}):(\mathbf{X},\boldsymbol{\pi},\mathbf{u},j)\mapsto(\mathbf{X}\times_{\boldsymbol{\pi},\mathbf{S},\mathbf{f}}\mathbf{T},\boldsymbol{\pi}_{\mathbf{T}},\mathbf{u}\circ\boldsymbol{\pi}_{\mathbf{X}},\boldsymbol{\pi}_{\mathbf{X}}^*(j))$ on objects and in a natural way on morphisms, with $\mathbf{X}\times_{\boldsymbol{\pi},\mathbf{S},\mathbf{f}}\mathbf{T}$ the fibre product in $\mathbf{dMan}$.
- If $\mathbf{f},\mathbf{g}:\mathbf{T}\rightarrow\mathbf{S}$ are 1-morphisms and $\theta:\mathbf{f}\Rightarrow\mathbf{g}$ a 2-morphism in $\mathbf{dMan^{aff}}$, then $F(\theta):F(\mathbf{f})\Rightarrow F(\mathbf{g})$ is a natural isomorphism of functors, $F(\theta):(\mathbf{X},\boldsymbol{\pi},\mathbf{u},j)\mapsto[\mathbf{i},\eta,\zeta]$ for $(\mathbf{X},\boldsymbol{\pi},\mathbf{u},j)$ in $F(\mathbf{S})$, where $[\mathbf{i},\eta,\zeta]:(\mathbf{X}\times_{\boldsymbol{\pi},\mathbf{S},\mathbf{f}}\mathbf{T},\boldsymbol{\pi}_{\mathbf{T}},\mathbf{u}\circ\boldsymbol{\pi}_{\mathbf{X}},\boldsymbol{\pi}_{\mathbf{X}}^*(j))\rightarrow(\mathbf{X}\times_{\boldsymbol{\pi},\mathbf{S},\mathbf{g}}\mathbf{T},\boldsymbol{\pi}_{\mathbf{T}},\mathbf{u}\circ\boldsymbol{\pi}_{\mathbf{X}},\boldsymbol{\pi}_{\mathbf{X}}^*(j))$ in $F(\mathbf{T})$, with $\mathbf{i}:\mathbf{X}\times_{\boldsymbol{\pi},\mathbf{S},\mathbf{f}}\mathbf{T}\rightarrow\mathbf{X}\times_{\boldsymbol{\pi},\mathbf{S},\mathbf{g}}\mathbf{T}$ induced by $\theta:\mathbf{f}\Rightarrow\mathbf{g}$.

Conjecture

The moduli functor $F : (\mathbf{dMan}^{\mathbf{aff}})^{\mathbf{op}} \to \mathbf{Groupoids}$ *above is represented by a d-orbifold.*

Some remarks:

- I may have got the treatment of almost complex structures in the definition of F wrong — this is a first guess.
- I expect to be able to prove the conjecture (perhaps after correcting the definition). The proof won't be specific to J-holomorphic curves — there should be a standard method for proving representability of moduli functors of solutions of nonlinear elliptic equations with gauge symmetries, which would also work for many other classes of moduli problems.
- Proving the Conjecture will involve verifying the representability criteria (A)–(C) above for F.

- The definition of F involves fibre products $\mathbf{X} \times_{\boldsymbol{\pi},\mathbf{S},\mathbf{f}} \mathbf{T}$ in **dMan**, which exist as $\boldsymbol{\pi} : \mathbf{X} \to \mathbf{S}$ is a submersion. Existence of suitable fibre products is *crucial* for the representable 2-functor approach. This becomes complicated when boundaries and corners are involved – see Lecture 4.
- Current definitions of differential-geometric moduli spaces (e.g. Kuranishi spaces, polyfolds) are generally very long, complicated ad hoc constructions, with no obvious naturality. In contrast, if we allow differential geometry over d-manifolds, my approach gives you a short, natural definition of the moduli functor F (only 2 slides above give a nearly complete definition!), followed by a long proof that F is representable. The effort moves from a construction to a theorem.
- Can write $\mathbf{X}, \mathbf{S}$ as 'standard model' d-manifolds (Lecture 2), and $\boldsymbol{\pi}, \mathbf{f}, \eta, \zeta, \ldots$ as 'standard model' 1- and 2-morphisms. Thus, can express F in terms of Kuranishi neighbourhoods and classical differential geometry.

- The definition of F involves *only finite-dimensional families of smooth objects*, with no analysis, Banach spaces, etc. (But the proof of (C) will involve analysis and Banach spaces.) This enables us to sidestep some analytic problems.
- In some problems, there will be several moduli spaces, with morphisms between them. E.g. if we include *marked points* in our J-holomorphic curves (do this by modifying objects $(\mathbf{X},\boldsymbol{\pi},\mathbf{u},j)$ in $F(\mathbf{S})$ to include morphisms $\mathbf{z}_1,\ldots,\mathbf{z}_k:\mathbf{S}\to\mathbf{X}$ with $\boldsymbol{\pi}\circ\mathbf{z}_i\simeq\mathbf{id}_{\mathbf{S}}$), then we can have 'forgetful functors' between moduli spaces forgetting some of the marked points. Such forgetful functors appear as 2-*natural transformations* $\Theta:F\Rightarrow G$ between moduli functors $F,G:(\mathbf{dMan^{aff}})^{\mathbf{op}}\to\mathbf{Groupoids}$. If F,G are representable, they induce 1-morphisms between the d-orbifolds.

3. Avoiding sc-smoothness

Consider the moduli space $\mathcal{M}$ of J-holomorphic maps $u:(\Sigma,j)\to(M,J)$, where $(\Sigma,j)\cong\mathbb{CP}^1$, but we do not fix an isomorphism $(\Sigma,j)\cong\mathbb{CP}^1$. As a set we have

$$\mathcal{M}\cong\bigl\{u\in\mathrm{Map}_{C^\infty}(\mathbb{CP}^1,M):J\circ\mathrm{d}u=\mathrm{d}u\circ j\bigr\}/\,\mathrm{Aut}(\mathbb{CP}^1),$$

where $\mathrm{Aut}(\mathbb{CP}^1)$ is the finite-dimensional Lie group $\mathrm{PSL}(2,\mathbb{C})$, and $\mathrm{Map}_{C^\infty}(\mathbb{CP}^1,M)$ is some kind of 'infinite-dimensional manifold', on which $\mathrm{Aut}(\mathbb{CP}^1)$ acts 'smoothly'.
To do the analysis, one usually replaces $\mathrm{Map}_{C^\infty}(\mathbb{CP}^1,M)$ by Hölder space $\mathrm{Map}_{C^{k,\alpha}}(\mathbb{CP}^1,M)$ or Sobolev space $\mathrm{Map}_{L^p_k}(\mathbb{CP}^1,M)$ versions, which are Banach manifolds. However, there is a problem: the actions of $\mathrm{PSL}(2,\mathbb{C})$ on $\mathrm{Map}_{C^{k,\alpha}}(\mathbb{CP}^1,M)$ and $\mathrm{Map}_{L^p_k}(\mathbb{CP}^1,M)$ are *only continuous, not differentiable*.
In the theory of polyfolds, the idea of *sc-smoothness* has been developed to deal with this — roughly, one considers the spaces $\mathrm{Map}_{C^{k,\alpha}}(\mathbb{CP}^1,M)$ for all $k=0,1,2\ldots$ together.

Some recent discussion of Kuranishi spaces has suggested that not including sc-smoothness in the picture is a big issue.
I take the opposite view: I would say that in both Kuranishi spaces, and representable 2-functors for d-orbifolds, we are really *only concerned with finite-dimensional families of smooth objects*, and the non-smoothness of reparametrization actions is an artefact of the Banach space set-up. If you have the right point of view, it should be a minor issue which can be worked around, and sc-smoothness is unnecessary.

I will explain how, in the d-orbifold and representable 2-functors approach, I believe one can avoid the problem with non-smoothness of reparametrization actions entirely. The key point is to minimize the parts of the proof in which one uses Banach spaces.

The next part is somewhat speculative.

Let $F : (\mathbf{dMan^{aff}})^{\mathbf{op}} \to \mathbf{Groupoids}$ be a moduli functor of genus g J-holomorphic curves in a symplectic manifold (M, J), as in §2.3, so that for each $\mathbf{S} \in \mathbf{dMan^{aff}}$, objects of $F(\mathbf{S})$ are quadruples $(\mathbf{X}, \boldsymbol{\pi}, \mathbf{u}, j)$ for $\boldsymbol{\pi} : \mathbf{X} \to \mathbf{S}$ a family of Riemann surfaces over $\mathbf{S}$ with family of almost complex structures j, and $\mathbf{u} : \mathbf{X} \to M$ is a family of J-holomorphic maps.
Write $G : (\mathbf{dMan^{aff}})^{\mathbf{op}} \to \mathbf{Groupoids}$ for the moduli functor of genus g Riemann surfaces without maps to M, so that objects of $G(\mathbf{S})$ are triples $(\mathbf{X}, \boldsymbol{\pi}, j)$, and let $\Theta : F \Rightarrow G$ be the forgetful 2-natural transformation mapping $\Theta : (\mathbf{X}, \boldsymbol{\pi}, \mathbf{u}, j) \mapsto (\mathbf{X}, \boldsymbol{\pi}, j)$.

Note: G is a moduli functor of *prestable* curves, not stable curves (we do not insist automorphism groups are finite), and Θ does not involve stabilizing complex structures.

To prove F is representable (that is, the moduli space is a d-orbifold), we have to verify criteria (A) F is a stack, (B) on the topological space of F, and (C) on existence of Kuranishi neighbourhoods for F. Here (A),(B) should be straightforward. The place where one needs to use analysis and Banach spaces is (C), and this is where non-smoothness of the reparametrization action should enter as an issue.

Now: I claim that G is not a d-orbifold (its automorphism groups may not be finite), but something else: a *smooth C^∞-Artin stack* (this needs proof). That, G is a purely classical, non-derived object, a smooth Artin stack in the C^∞ world, locally modelled on quotients $[Z/H]$ for Z a smooth manifold and H a Lie group.

Objects G which are 'smooth' in this sense should have the nice property that any 1-morphism $\mathbf{S} \to G$ for $\mathbf{S}$ an affine d-manifold factors through $\mathbf{S} \to V$ for V a manifold. So you can understand G using only 1-morphisms $V \to G$ for V a manifold.

Criteria for representability relative to $\Theta : F \Rightarrow G$

I expect that when $F, G : (\mathbf{dMan^{aff}})^{\mathbf{op}} \to \mathbf{Groupoids}$ and we have a 2-natural transformation $\Theta : F \Rightarrow G$ for G a smooth C^∞-Artin stack, then we can prove alternative criteria for F to be a representable 2-functor: F is representable if it satisfies (A),(B) in 2.2, but instead of (C) satisfies:

(C)′ for every manifold B and 2-natural transformation $\Phi : \mathrm{Hom}(-, B) \Rightarrow G$, the 2-functor $F \times_{\Theta,G,\Phi} \mathrm{Hom}(-, B)$ satisfies (C) (i.e. has Kuranishi neighbourhoods).

Note that $F \times_{\Theta,G,\Phi} \mathrm{Hom}(-, B)$ will automatically satisfy (A),(B). Also, it should be sufficient to verify (C)′ for a single atlas $\Phi : \mathrm{Hom}(-, B) \Rightarrow G$ for G, rather than for all B, Φ.

What all this means in terms of J-holomorphic curves:

- The manifold B and transformation $\Phi : \mathrm{Hom}(-, B) \Rightarrow G$ correspond to a family of genus g Riemann surfaces $\pi : C \to B$, $j : T(C/B) \to T(C/B)$ over the base B. This is a purely classical object in ordinary differential geometry.
- Given such $\pi : C \to B$, j, the 2-functor $F \times_{\Theta,G,\Phi} \mathrm{Hom}(-, B)$ represents J-holomorphic maps to (M, J) *from Riemann surfaces in the particular family* $\pi : C \to B$, rather than from arbitrary genus g Riemann surfaces.
- to prove $F \times_{\Theta,G,\Phi} \mathrm{Hom}(-, B)$ has Kuranishi neighbourhoods, we can use Hölder or Sobolev spaces and elliptic operators *without dividing by* $\mathrm{Aut}(\Sigma)$, for Σ the Riemann surface.
- The groups $\mathrm{Aut}(\Sigma)$ live in the stabilizer groups of G. When we lift along $\Phi : B \Rightarrow G$, for B a manifold with trivial stabilizer groups, dividing by $\mathrm{Aut}(\Sigma)$ goes away.